# Durham Research Online

**Deposited in DRO:**

14 December 2012

**Version of attached file:**

Accepted Version

**Peer-review status of attached file:**

Peer-reviewed

**Citation for published item:**

Miles, R. and Ward, T. (2008) 'Uniform periodic point growth in entropy rank one.', Proceedings of the American Mathematical Society., 136 (01). pp. 359-365.

**Further information on publisher's website:**

**Publisher's copyright statement:**

**Additional information:**

# UNIFORM PERIODIC POINT GROWTH
# IN ENTROPY RANK ONE

RICHARD MILES AND THOMAS WARD

ABSTRACT. We show that algebraic dynamical systems with entropy rank one
have uniformly exponentially many periodic points in all directions.

## 1. INTRODUCTION

Let $\alpha$ be an action of $\mathbb{Z}^d$ by continuous automorphisms of a compact metrizable abelian group $X$ (such a system is called an algebraic $\mathbb{Z}^d$-action). For a continuous map $\beta : X \to X$ write $h(\beta)$ for the topological entropy and

$$\mathrm{F}(\beta) = \{x \in X \mid \beta x = x\}$$

for the set of fixed points. The action $\alpha$ is said to have *entropy rank one* if, for each $\mathbf{n} \in \mathbb{Z}^d$, $h(\alpha^{\mathbf{n}}) < \infty$.

If $\alpha$ is a mixing entropy rank one action and the topological dimension $\dim(X)$ is finite, $\mathrm{F}(\alpha^{\mathbf{n}})$ is finite for all $\mathbf{n} \in \mathbb{Z}^d$. Our purpose here is to show that under natural conditions $|\mathrm{F}(\alpha^{\mathbf{n}})|$ exhibits uniform exponential growth. Write $\mathbf{n}_j \to \infty$ for a sequence in $\mathbb{Z}^d$ if for any finite set $F \subset \mathbb{Z}^d$ there is some $J$ for which $j > J$ implies that $\mathbf{n}_j \notin F$. Equivalently, this means $\|\mathbf{n}_j\| \to \infty$ as $j \to \infty$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{Z}^d$. The Noetherian condition mentioned in Theorem 1.1 is explained in Section 2.

**Theorem 1.1.** *Let $\alpha$ be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group $X$. Then there exist constants $C_1, C_2 \geq 0$ such that*

$$\limsup_{\mathbf{n} \to \infty} \frac{1}{\|\mathbf{n}\|} \log |\mathrm{F}(\alpha^{\mathbf{n}})| = C_1 < \infty$$

*and*

$$\liminf_{\mathbf{n} \to \infty} \frac{1}{\|\mathbf{n}\|} \log |\mathrm{F}(\alpha^{\mathbf{n}})| = C_2.$$

*If $\dim(X) > 0$ and the action is Noetherian then $C_2 > 0$.*

An affirmative answer to Lehmer's problem (see Remark 2.3) would render the finite dimension assumption in Theorem 1.1 redundant. The assumption is also not required for a mixing Noetherian entropy rank one algebraic action; in that setting it is a consequence of the Noetherian condition. Essentially, the only non-trivial conclusion in the theorem is that $C_2 > 0$.

The action $\alpha$ is *expansive* if there exists a neighbourhood $U$ of $0_X$ such that

$$\bigcap_{\mathbf{n} \in \mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0_X\}.$$

For $d = 1$ and $\alpha$ expansive, the growth rate of periodic points exists, so $C_1 = C_2$ in Theorem 1.1. The constant coincides with the entropy and is non-zero provided $X$ is infinite.

The $\mathbb{Z}^2$-action on a zero-dimensional group in Example 1.3 is expansive yet has $C_2 = 0$. Expansive actions on connected groups are more indicative of the import of a positive value for $C_2$. In terms of expansive subdynamics (see [3] and [6]), there are sequences $\mathbf{n} \to \infty$ converging to non-expansive lines; along such sequences $|\mathrm{F}(\alpha^{\mathbf{n}})|$ is much smaller than the same expression with $\mathbf{n}$ of similar Euclidean size and far from non-expansive directions. Nonetheless, there is a uniform exponential growth in all directions.

**Example 1.2.** Consider the $\mathbb{Z}^2$-action $\alpha$ dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto 2r$ and $r \mapsto 3r$ on $\mathbb{Z}[\frac{1}{6}]$. Figure 1 shows the map

$$\mathbf{n} \mapsto |\mathrm{F}(\alpha^{\mathbf{n}})| \in \mathbb{N}.$$

Notice that $|\mathrm{F}(\alpha^{-\mathbf{n}})| = |\mathrm{F}(\alpha^{\mathbf{n}})|$, so only the region $n_2 \geq 0$ is shown, with $\infty$ denoting the lattice point $(0,0)$ corresponding to the identity map $\alpha^{(0,0)}$. In an

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 211 | 227 | 235 | 239 | 241 | 121 | 485 | 971 | 1943 | 3887 | *7775* |
| 49 | 65 | 73 | 77 | 79 | 5 | 161 | 323 | 647 | *1295* | 2591 |
| **5** | 11 | 19 | 23 | 25 | 13 | 53 | 107 | *215* | 431 | 863 |
| 23 | **7** | **1** | 5 | 7 | 1 | 17 | *35* | 71 | 143 | 287 |
| 29 | 13 | **5** | 1 | 1 | 1 | *5* | 11 | 23 | 47 | 95 |
| 31 | 5 | 7 | **1** | **1** | $\infty$ | 1 | 1 | 7 | 5 | 31 |

FIGURE 1. Periodic point counts for $\times 2, \times 3$.

expansive direction like $(1,1)$ we have $|\mathrm{F}(\alpha^{n(1,1)})| = 6^n - 1$ and the exponential growth rate along the sequence in italics is clear. For lattice points close to the non-expansive line $2^x 3^y = 1$ we find (for example) that $|\mathrm{F}(\alpha^{(-5,3)})| = 5$, and it is not immediately clear that there is exponential growth along the sequence shown in bold; Theorem 1.1 asserts that there is.

**Example 1.3.** Consider the expansive $\mathbb{Z}^2$-action $\alpha$ on $X$ dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto u_1 r$ and $r \mapsto u_2 r$ on

$$\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]/\langle 2, 1 + u_1 + u_2 \rangle$$

(this is an example of the type introduced by Ledrappier [12]; $X$ is a zero-dimensional group). As shown in [4], $|\mathrm{F}(\alpha^{(n,0)})| = 2^{n - 2^{\mathrm{ord}_2(n)}}$. In particular,

$$\lim_{n \to \infty} \frac{1}{\|(2^n, 0)\|} \log |\mathrm{F}(\alpha^{(2^n,0)})| = 0,$$

showing that some assumption on the topological dimension on $X$ is needed to have $C_2 > 0$ in Theorem 1.1. If the map dual to $r \mapsto u_2 r$ is ignored, then this example is a non-expansive (and non-Noetherian) $\mathbb{Z}$-action on a zero-dimensional group with $C_2 = 0$.

**Example 1.4.** The Noetherian condition is needed to have $C_2 > 0$ even for $d = 1$ on connected groups. For example, the automorphism $\alpha$ dual to the map $r \mapsto 2r$ on $\mathbb{Q}$ has $|\operatorname{F}(\alpha^n)| = 1$ for all $n$. For rings between $\mathbb{Z}$ and $\mathbb{Q}$ a variety of exotic periodic point behavior is possible (see [20] or [21] for the details).

The full statement of Theorem 1.1 does however apply to Noetherian non-expansive systems. Thus, for example, the genuinely partially hyperbolic systems like that described by Damjanović and Katok [5, Ex. 7.3] satisfy the hypotheses.

## 2. Proof of Theorem 1.1

The case $d = 1$ is covered by [4] and [16], so we may assume $d \geq 2$. Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [11].

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \ldots, u_d$ with integer coefficients. If $\alpha$ is an algebraic $\mathbb{Z}^d$-action of a compact abelian group $X$, the character group $M = \widehat{X}$ has the structure of a discrete countable $R_d$-module, obtained by first identifying the dual automorphism $\widehat{\alpha}^{\mathbf{n}}$ with multiplication by the monomial $u^{\mathbf{n}} = u_1^{n_1} \ldots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, any countable $R_d$-module $M$ has an associated algebraic $\mathbb{Z}^d$-action obtained by dualizing the action induced by multiplying by monomials on $M$. The action $\alpha = \alpha_M$ is described as *Noetherian* if the module $M$ is Noetherian. A full account of this correspondence and the resulting theory is given in Schmidt's monograph [19].

Entropy rank one actions are described in [7] and developments concerning their periodic points may be found in the papers [16] and [18]. By a slight abuse of notation, write $h(\cdot)$ for the topological entropy of maps and for the extension of the entropy function to all of $\mathbb{R}^d$ in the sense explained below.

**Proposition 2.1.** *Let $\alpha_M$ be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group. Then*

(1) *The set of associated primes of the associated $R_d$-module, $\operatorname{Asc}(M)$ is finite. For each $\mathfrak{p} \in \operatorname{Asc}(M)$, the domain $R_d/\mathfrak{p}$ has Krull dimension 1 and its field of fractions $\mathbb{K}(\mathfrak{p})$ is a global field.*

(2) *There exist positive integers $m(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Asc}(M)$, such that for every non-zero $\mathbf{n} \in \mathbb{Z}^d$,*

$$(2.1) \qquad h(\alpha_M^{\mathbf{n}}) = \sum_{\mathfrak{p} \in \operatorname{Asc}(M)} m(\mathfrak{p}) h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})$$

*and*

$$(2.2) \qquad |\operatorname{F}(\alpha_M^{\mathbf{n}})| \leqslant \prod_{\mathfrak{p} \in \operatorname{Asc}(M)} |\operatorname{F}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})|^{m(\mathfrak{p})},$$

*with equality if $M$ is Noetherian.*

(3) *For each $\mathfrak{p} \in \operatorname{Asc}(M)$, there is a finite set of places $\mathcal{S}(\mathfrak{p})$ of $\mathbb{K}(\mathfrak{p})$ such that*

$$(2.3) \qquad h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) = \sum_{v \in \mathcal{S}(\mathfrak{p})} \max\{\mathbf{l}_v \cdot \mathbf{n}, 0\} > 0,$$

*and*

$$(2.4) \qquad |\operatorname{F}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})| = \prod_{v \in \mathcal{S}(\mathfrak{p})} |\xi_{\mathfrak{p}}^{\mathbf{n}} - 1|_v,$$

where $\xi_{\mathfrak{p}} = (\overline{u}_1, \ldots, \overline{u}_d)$ *denotes the image of u in* $(R_d/\mathfrak{p})^d$ *and* $\mathbf{l}_v = (\log|\overline{u}_1|_v, \ldots, \log|\overline{u}_d|_v)$.

*Proof.* Let $\mathbf{n} \in \mathbb{Z}^d$ be non-zero. Since $\alpha_M$ is mixing, [19, Prop. 6.6] shows that for each $\mathfrak{p} \in \mathrm{Asc}(M)$, $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ is ergodic, so $h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) > 0$. It follows from [7] that for each $\mathfrak{p} \in \mathrm{Asc}(M)$, $R_d/\mathfrak{p}$ has Krull dimension 1 and $\mathbb{K}(\mathfrak{p})$ is a global field. If $\mathrm{char}(R_d/\mathfrak{p}) > 0$ then $h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) \geq \log 2$. Via Yuzvinskiĭ's formula (see [19, Th. 14.1] or [23]), this implies there are only finitely many such $\mathfrak{p} \in \mathrm{Asc}(M)$, as $h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) < \infty$.

Also, there can be only finitely many $\mathfrak{p} \in \mathrm{Asc}(M)$ with $\mathrm{char}(R_d/\mathfrak{p}) = 0$, as

$$\dim(X) \geqslant |\{\mathfrak{p} \in \mathrm{Asc}(M) : \mathrm{char}(R_d/\mathfrak{p}) = 0\}|.$$

This establishes (1).

The method of [7, Lem. 8.2] shows that in any prime filtration of a Noetherian submodule of $M$, each prime $\mathfrak{p} \in \mathrm{Asc}(M)$ appears with a maximum multiplicity

$$m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})}(M \otimes_{R_d} \mathbb{K}(\mathfrak{p})),$$

which is finite by similar reasoning to the proof of (1). By adopting an algorithm for obtaining a prime filtration which selects the associated primes of $M$ first, one obtains a Noetherian submodule $L \subset M$ such that each prime $\mathfrak{p} \in \mathrm{Asc}(M)$ appears with multiplicity $m(\mathfrak{p})$ in a filtration of $L$. Furthermore, if $L \neq M$, then each $\mathfrak{q} \in \mathrm{Asc}(M/L)$ is maximal and $R_d/\mathfrak{q}$ is a finite field. Hence, Yuzvinskiĭ's formula shows that $h(\alpha_{M/L}^{\mathbf{n}}) = 0$ and $h(\alpha_M^{\mathbf{n}}) = h(\alpha_L^{\mathbf{n}})$; the formula (2.1) then follows from [15, Lem. 4.3].

If $M$ is Noetherian, equality in (2.2) is given by [16, Th. 3.2]. If $M$ is not Noetherian, (2.2) follows from [16, Th. 3.2] applied to $L$, together with the inequality

$$|\mathrm{F}(\alpha_M^{\mathbf{n}})| \leqslant |\mathrm{F}(\alpha_L^{\mathbf{n}})|,$$

which is established using a similar method to the proof of [17, Lem. 2.6].

Finally, the entropy formula (2.3) follows from [7, Prop. 8.5] and the periodic point counting formula (2.4) is [16, Lem 3.1]. $\qquad\square$

*Proof of Theorem 1.1.* Let $M = \widehat{X}$ denote the dual $R_d$-module and let $\mathfrak{p} \in \mathrm{Asc}(M)$. For a fixed $\mathfrak{p} \in \mathrm{Asc}(M)$ and any non-zero $\mathbf{n} \in \mathbb{Z}^d$, set

$$(2.5) \qquad f(\mathbf{n}) = \frac{1}{\|\mathbf{n}\|} \log|\mathrm{F}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})|.$$

Let $h : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ denote the directional entropy function for the action $\alpha_{R_d/\mathfrak{p}}$. This is the function obtained by extending the entropy formula (2.3) to values of $\mathbb{R}^d$ (see [7, Sec. 8] for further details).

For any vector $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$, write $\widehat{\mathbf{v}} \in \mathsf{S}_{d-1}$ for the unique vector of unit length with the property that $\mathbf{v} = \lambda \widehat{\mathbf{v}}$ for some scalar $\lambda > 0$. From (2.3) and (2.4) it follows that

$$(2.6) \qquad f(\mathbf{n}) = g(\mathbf{n}) + h(\widehat{\mathbf{n}})$$

where

$$g(\mathbf{n}) = \frac{1}{\|\mathbf{n}\|} \sum_{v \in \mathcal{S}(\mathfrak{p})} \log|1 - \phi_v(\mathbf{n})|_v$$

and

$$\phi_v(\mathbf{n}) = \begin{cases} \xi_{\mathfrak{p}}^{-\mathbf{n}} & \text{if } |\xi_{\mathfrak{p}}^{\mathbf{n}}|_v > 1; \\ \xi_{\mathfrak{p}}^{\mathbf{n}} & \text{if } |\xi_{\mathfrak{p}}^{\mathbf{n}}|_v \leqslant 1. \end{cases}$$

Notice that $\phi_v(\mathbf{n}) \neq 1$ if $\mathbf{n} \neq 0$ by the assumption that the action is mixing.

To establish the lower bound $C_2 > 0$ in Theorem 1.1, first note that equality in (2.2) gives the expression

$$\frac{1}{\|\mathbf{n}\|} \log |\mathrm{F}(\alpha_M^{\mathbf{n}})|$$

as a finite sum of terms of the form (2.5) and, crucially, the assumption $\dim(X) > 0$ means at least one of these terms arises from a prime $\mathfrak{p}$ such that $\mathbb{K}(\mathfrak{p})$ is an algebraic number field (rather than a function field of positive characteristic). For the lower bound, it is therefore enough by (2.2) to consider only the asymptotic behavior of $f$ with the assumption that it arises from such a prime.

We need to know that $g$ does not make an asymptotic contribution. It is clear that $g$ cannot be too large, since

$$\prod_{v \in \mathcal{S}(\mathfrak{p})} |1 - \phi_v(\mathbf{n})|_v \leq 2^{|\mathcal{S}(\mathfrak{p})|}.$$

On the other hand, it cannot be large and negative for the following reason. If $v$ is an infinite (Archimedean) place, then Baker's theorem [1] can be used to find constants $A, B \geq 0$ such that

$$|1 - \phi_v(\mathbf{n})|_v \geq \frac{A}{\max\{n_i\}^B}$$

(see [4] or [9] for similar arguments; roughly speaking, the issue is to bound the proximity of $\xi_{\mathfrak{p}}^{\mathbf{n}}$ and 1 in terms of $\mathbf{n}$). If $v$ is a finite place, then Yu's $p$-adic bounds for linear forms in logarithms [22] give a similar lower bound (note that $d \geq 2$ by assumption). Since $\mathcal{S}(\mathfrak{p})$ is finite, this shows that

$$(2.7) \qquad g(\mathbf{n}) \to 0 \text{ as } \mathbf{n} \to \infty.$$

Assume, for a contradiction, that $C_2 = 0$. Then there is a sequence $\mathbf{n}_j \to \infty$ as $j \to \infty$ with the property that

$$(2.8) \qquad \lim_{j \to \infty} f(\mathbf{n}_j) = 0.$$

It follows from (2.6) and (2.7) that

$$(2.9) \qquad h(\widehat{\mathbf{n}_j}) \to 0 \text{ as } j \to \infty.$$

Now the entropy function $h$ restricted to the unit sphere $\mathsf{S}_{d-1}$ is a continuous function on a compact set, so (2.9) implies that there is some $\mathbf{v} \in \mathsf{S}_{d-1}$ for which

$$(2.10) \qquad h(\mathbf{v}) = 0.$$

If the ray through $\mathbf{v}$ happens to meet a point $\mathbf{m} \in \mathbb{Z}^d$ then we have an immediate contradiction: the automorphism $\alpha_{R_d/\mathfrak{p}}^{\mathbf{m}}$ would have zero entropy, contradicting (2.3). In order to show that any point $\mathbf{v}$ with (2.10) gives a contradiction we need to make a quantitative version of that argument.

The explicit formula (2.3) for $h$ means there is a list of vectors

$$\mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathbb{R}^d$$

with the property that for any $\mathbf{u} \in \mathbb{R}^d$,

$$h(\mathbf{u}) = \mathbf{u} \cdot \mathbf{a}_k,$$

for some $k$, $1 \leqslant k \leqslant r$. It follows that for any $\mathbf{n} \in \mathbb{Z}^d$ and $\mathbf{u} \in \mathbb{R}^d$,

$$|h(\mathbf{n}) - h(\mathbf{u})| \leqslant \max_{1 \leqslant k \leqslant r} \max_{1 \leqslant i \leqslant d} \{|a_{k,i}|\}\|\mathbf{n} - \mathbf{u}\|,$$

where $\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,d})$. In particular, since $h(\lambda \mathbf{v}) = 0$ for all $\lambda > 0$ we may find a sequence $(\mathbf{m}_j)$ of vectors in $\mathbb{Z}^d$ with $\|\mathbf{m}_j - \lambda_j \mathbf{v}\| \to 0$ as $j \to \infty$ (for some sequence $(\lambda_j)$ of scalars) and hence have

(2.11) $$h(\mathbf{m}_j) \to 0 \text{ as } j \to \infty.$$

On the other hand, the dual group of $R_d/\mathfrak{p}$ is connected and finite-dimensional. Hence, by Yuzvinskiĭ's formula, for any $\mathbf{n} \neq 0$,

$$h(\mathbf{n}) \geqslant \mathsf{m}(P) > 0$$

where $\mathsf{m}(P) = \int_0^1 \log |P(e^{2\pi i s})| \mathrm{d}s$ denotes the logarithmic Mahler measure of some polynomial $P$ of degree no greater than $\dim_{\mathbb{Q}}(\mathbb{K}(\mathfrak{p}))$. It follows that there is a constant $C > 0$ (depending only on $\mathfrak{p}$) for which

$$h(\mathbf{n}) > C > 0 \text{ for any } \mathbf{n} \neq 0$$

(the existence of a lower bound for the non-zero logarithmic Mahler measure of polynomials of bounded degree is well-known; see [2] or [8] for the background). This lower bound contradicts (2.11), so (2.8) is impossible. Thus $C_2 > 0$.

The upper bound is clear: the inequality (2.2) together with the explicit formula (2.4) gives a uniform constant $C_1$ with the property that

$$|\mathrm{F}(\alpha^{\mathbf{n}})| \leq C_1^{\max_{1 \leqslant i \leqslant d}\{n_i\}}.$$

$\square$

**Remark 2.2.** Baker's theorem provides the key estimates in many dynamical problems; see [4] and [9] for examples. As pointed out by Lind [14], in order to establish the logarithmic growth rate of periodic points for a quasihyperbolic toral automorphism (a typical application), sometimes all that is needed is a weaker and earlier result due to Gel'fond [10]. Here we need something closer to the full weight of the theorems of Baker and Yu, because we are in higher rank.

**Remark 2.3.** Lehmer's problem [13] asks if there is a uniform lower bound for all positive Mahler measures. As shown by Lind (see [15]) this is equivalent to a uniform lower bound for the topological entropy of any mixing compact group automorphism. If 'Lehmer's conjecture' (that there is such a bound and that it is attained by the expected polynomial) holds, then the topological entropy of any compact group automorphism with positive entropy is at least

$$0.162 \cdots = \mathsf{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1).$$

This does not imply a uniform bound for $C_2$ in Theorem 1.1 because $C_2$ is influenced by the geometry of the acting group as well as the collection of maps in its image. For example, the $\mathbb{Z}^2$-action $\alpha$ corresponding to the module $R_2/\langle u_1 - 2, u_1^k u_2 - 3\rangle$ has the property that $h(\alpha^{(k,1)}) = \log 3$, so the corresponding constant $C_2$ cannot exceed $\log 3/\sqrt{1 + k^2}$.

## References

[1] A. Baker. *Transcendental number theory*, Cambridge University Press, London, 1975.

[2] P. E. Blanksby and H. L. Montgomery, Algebraic integers near the unit circle, *Acta Arith.*, **18** (1971), 355–369.

[3] Mike Boyle and Douglas Lind, Expansive subdynamics, *Trans. Amer. Math. Soc.* **349** (1997), no. 1, 55–102.

[4] Vijay Chothi, Graham Everest, and Thomas Ward, $S$-integer dynamical systems: periodic points, *J. Reine Angew. Math.* **489** (1997), 99–132.

[5] Danijela Damjanović and Anatole Katok, Local rigidity of partially hyperbolic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$. I. KAM method and actions on the torus, Preprint.

[6] Manfred Einsiedler, Douglas Lind, Richard Miles, and Thomas Ward, Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions, *Ergodic Theory Dynam. Systems* **21** (2001), no. 6, 1695–1729.

[7] Manfred Einsiedler and Douglas Lind, Algebraic $\mathbb{Z}^d$-actions of entropy rank one, *Trans. Amer. Math. Soc.* **356** (2004), no. 5, 1799–1831.

[8] Graham Everest and Thomas Ward, *Heights of polynomials and entropy in algebraic dynamics*, Springer–Verlag London, Ltd., London, 1999.

[9] Graham Everest, Richard Miles, Shaun Stevens and Thomas Ward, Orbit-counting in non-hyperbolic dynamical systems, *J. Reine Angew. Math.*, to appear.

[10] A. O. Gel′fond, *Transcendental and algebraic numbers*, Dover Publications, Inc., New York 1960. Translated from the first Russian edition by Leo F. Boron.

[11] Bruce Kitchens and Klaus Schmidt, Automorphisms of compact groups, *Ergodic Theory Dynam. Systems* **9** (1989), no. 4, 691–735.

[12] François Ledrappier, Un champ markovien peut être d'entropie nulle et mélangeant, *C. R. Acad. Sci. Paris Sér. A-B* **287** (1978), no. 7, A561–A563.

[13] D. H. Lehmer, Factorization of certain cyclotomic functions, *Ann. of Math.* (2) **34** (1933), no. 3, 461–479.

[14] Douglas Lind, Dynamical properties of quasihyperbolic toral automorphisms, *Ergodic Theory Dynam. Systems* **2** (1982), no. 1, 49–68.

[15] Douglas Lind, Klaus Schmidt, and Thomas Ward, Mahler measure and entropy for commuting automorphisms of compact groups, *Invent. Math.* **101** (1990), no. 3, 593–629.

[16] Richard Miles, Zeta functions for elements of entropy rank one actions, *Ergodic Theory Dynam. Systems*, to appear.

[17] Richard Miles, Periodic points of endomorphisms on solenoids and related groups, preprint, 2006.

[18] Richard Miles and Thomas Ward, Periodic point data detects subdynamics in entropy rank one, *Ergodic Theory Dynam. Systems* **26** (2006), no. 6, 1913–1930.

[19] Klaus Schmidt, *Dynamical systems of algebraic origin*, Progress in Mathematics, vol. 128, Birkhäuser Verlag, Basel, 1995.

[20] Thomas Ward. An uncountable family of group automorphisms, and a typical member. *Bull. London Math. Soc.* **29** (1997) no. 5, 577–584.

[21] Thomas Ward. Almost all $S$-integer dynamical systems have many periodic points. *Ergodic Theory Dynam. Systems*, **18** (1998), no. 2, 471–486.

[22] Kunrui Yu. Linear forms in $p$-adic logarithms. II. *Compositio Math.*, **74** (1990), no. 1, 15–113.

[23] S. A. Yuzvinskiĭ, Calculation of the entropy of a group-endomorphism, *Sibirsk. Mat. Ž.*, **8**, (1967), 230–239.

School of Mathematics, University of East Anglia, Norwich, NR4 7TJ, UK