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Packing Bipartite Graphs with Covers of Complete Bipartite Graphs ^{*}

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Abstract. For a set $\mathcal{S}$ of graphs, a perfect $\mathcal{S}$ -packing ($\mathcal{S}$ -factor) of a graph G is a set of mutually vertex-disjoint subgraphs of G that each are isomorphic to a member of $\mathcal{S}$ and that together contain all vertices of G . If G allows a covering (locally bijective homomorphism) to a graph H , then G is an H -cover. For some fixed H let $\mathcal{S}(H)$ consist of all connected H -covers. Let $K_{k,\ell}$ be the complete bipartite graph with partition classes of size k and ℓ , respectively. For all fixed $k, \ell \geq 1$, we determine the computational complexity of the problem that tests whether a given bipartite graph has a perfect $\mathcal{S}(K_{k,\ell})$ -packing. Our technique is partially based on exploring a close relationship to pseudo-coverings. A pseudo-covering from a graph G to a graph H is a homomorphism from G to H that becomes a covering to H when restricted to a spanning subgraph of G . We settle the computational complexity of the problem that asks whether a graph allows a pseudo-covering to $K_{k,\ell}$ for all fixed $k, \ell \geq 1$.

1 Introduction

Throughout the paper we consider undirected graphs with no loops and no multiple edges. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be some fixed set of mutually vertex-disjoint graphs. A set of (not necessarily vertex-induced) mutually vertex-disjoint subgraphs of G , each isomorphic to a member of $\mathcal{S}$, is called an $\mathcal{S}$ -packing. Packings naturally generalize matchings (the case in which $\mathcal{S}$ only contains edges). They arise in many applications, both practical ones such as exam scheduling [12], and theoretical ones such as the study of degree constraint graphs (cf. the survey of Hell [11]). If $\mathcal{S}$ consists of a single subgraph S , we write S -packing instead of $\mathcal{S}$ -packing. The problem of finding an S -packing of a graph G that packs the maximum number of vertices of G is NP-hard for

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all fixed connected graphs S on at least three vertices, as shown by Hell and Kirkpatrick [13].

A packing of a graph is *perfect* if every vertex of the graph belongs to one of the subgraphs of the packing. Perfect packings are also called *factors* and from now on we call a perfect $\mathcal{S}$ -packing an $\mathcal{S}$ -*factor*. We call the corresponding decision problem the $\mathcal{S}$ -FACTOR problem. For a survey on graph factors we refer to the monograph of Plummer [19].

Our Focus. We study a relaxation of $K_{k,\ell}$ -factors, where $K_{k,\ell}$ denotes the *biclique* (complete connected bipartite graph) with partition classes of size k and ℓ , respectively. In order to explain this relaxation we first need to introduce some new terminology.

A *homomorphism* from a graph G to a graph H is a vertex mapping $f : V_G \rightarrow V_H$ satisfying the property that $f(u)f(v)$ belongs to E_H whenever the edge uv belongs to E_G . If for every $u \in V_G$ the restriction of f to the neighborhood of u , i.e., the mapping $f_u : N_G(u) \rightarrow N_H(f(u))$, is bijective then we say that f is a *locally bijective* homomorphism or a *covering* [2, 16]. The graph G is then called an H -*cover* and we write $G \xrightarrow{B} H$. Locally bijective homomorphisms have applications in distributed computing [1] and in constructing highly transitive regular graphs [3]. For a specified graph H , we let $\mathcal{S}(H)$ consist of all connected H -covers. In this paper we study $\mathcal{S}(K_{k,\ell})$ -factors of bipartite graphs.

Fig. 1. Examples: (a) a $K_{2,3}$. (b) a bipartite $K_{2,3}$ -cover. (c) a bipartite $K_{2,3}$ -pseudo-cover that is no $K_{2,3}$ -cover and that has no $K_{2,3}$ -factor. (d) a bipartite graph with a $K_{2,3}$ -factor that is not a $K_{2,3}$ -pseudo-cover. (e) a bipartite graph with an $\mathcal{S}(K_{2,3})$ -factor but with no $K_{2,3}$ -factor and that is not a $K_{2,3}$ -pseudo-cover.

Our Motivation. Since a $K_{1,1}$ -factor is a perfect matching, $K_{1,1}$ -FACTOR is polynomial-time solvable. The $K_{k,\ell}$ -FACTOR problem is known to be NP-complete for all other $k, \ell \geq 1$, due to the aforementioned result of Hell and Kirkpatrick [13]. These results have some consequences for our relaxation. In order to explain this, we make the following observation, which holds because only a tree has a unique cover (namely the tree itself) and the graph $K_{k,\ell}$ is a tree if $k = 1$ or $\ell = 1$.

Observation 1 $\mathcal{S}(K_{k,\ell}) = \{K_{k,\ell}\}$ if and only if $\min\{k, \ell\} = 1$.

Because $\mathcal{S}(K_{1,\ell}) = \{K_{1,\ell}\}$ by Observation 1, the above results immediately imply that $\mathcal{S}(K_{1,\ell})$ -FACTOR is only polynomial-time solvable if $\ell = 1$; it is NP-complete otherwise. What about our relaxation for $k, \ell \geq 2$? Note that, for these values of k, ℓ , the size of the set $\mathcal{S}(K_{k,\ell})$ is unbounded. The only result known so far is for $k = \ell = 2$; Hell, Kirkpatrick, Kratochvíl and Kříž [14] showed that $\mathcal{S}(K_{2,2})$ -FACTOR is NP-complete for general graphs, as part of their computational complexity classification of finding restricted 2-factors; we explain the reason why an $\mathcal{S}(K_{2,2})$ -factor is a restricted 2-factor later.

For bipartite graphs, the following is known. Firstly, Monnot and Toulouse [18] researched path factors in bipartite graphs and showed that the $K_{2,1}$ -FACTOR problem stays NP-complete when restricted to the class of bipartite graphs. Secondly, we observed that as a matter of fact the proof of the NP-completeness result for $\mathcal{S}(K_{2,2})$ -FACTOR in [14] is even a proof for bipartite graphs.

Our interest in bipartite graphs stems from a close relationship of $\mathcal{S}(K_{k,\ell})$ -factors of bipartite graphs and so-called $K_{k,\ell}$ -pseudo-covers, which originate from topological graph theory and have applications in the area of distributed computing [4, 5]. A homomorphism f from a graph G to a graph H is a *pseudo-covering* from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H . In that case G is called an *H-pseudo-cover* and we write $G \xrightarrow{P} H$. The computational complexity classification of the H -PSEUDO-COVER problem, which is to test for a fixed graph H (i.e., not being part of the input) whether $G \xrightarrow{P} H$ for some given G is still open, and our paper can also be seen as a first investigation into this question. We explain the exact relationship between factors and pseudo-coverings in detail later on; we refer to Figure 1 for some examples that illustrate the notions introduced.

Our Results and Paper Organization.

Section 2 contains additional terminology, notations and some basic observations. In Section 3 we pinpoint the relationship between factors and pseudo-coverings. In Section 4 we completely classify the computational complexity of the $\mathcal{S}(K_{k,\ell})$ -FACTOR problem for bipartite graphs. Recall that $\mathcal{S}(K_{1,1})$ -FACTOR is polynomial-time solvable on general graphs. We first prove that $\mathcal{S}(K_{1,\ell})$ -FACTOR is NP-complete on bipartite graphs for all fixed $\ell \geq 2$. By applying our result of Section 3, we then show that NP-completeness of every remaining case can be shown by proving NP-completeness of the corresponding $K_{k,\ell}$ -PSEUDO-COVER problem. We classify the complexity of $K_{k,\ell}$ -PSEUDO-COVER in Section 5. We show that it is indeed NP-complete on bipartite graphs for all fixed

pairs $k, \ell \geq 2$ by adapting the hardness construction of Hell, Kirkpatrick, Kratochvíl and Kříž [14] for restricted 2-factors. In contrast to $\mathcal{S}(K_{k,\ell})$ -FACTOR, we show that $K_{k,\ell}$ -PSEUDO-COVER is polynomial-time solvable for all $k, \ell \geq 1$ with $\min\{k, \ell\} = 1$. In Section 6 we further discuss the relationships between pseudo-coverings and locally constrained homomorphisms, such as the aforementioned coverings. We shall see that as a matter of fact the NP-completeness result for $K_{k,\ell}$ -PSEUDO-COVER for fixed $k, \ell \geq 3$ also follows from a result of Kratochvíl, Proskurowski and Telle [15] who proved that $K_{k,\ell}$ -COVER is NP-complete for $k, \ell \geq 3$. This problem is to test whether $G \xrightarrow{B} K_{k,\ell}$ for a given graph G . However, the same authors [15] showed that $K_{k,\ell}$ -COVER is polynomial-time solvable when $k = 2$ or $\ell = 2$. Hence, for those pairs (k, ℓ) we can only use our hardness proof in Section 5.

2 Preliminaries

From now on let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_\ell\}$ denote the partition classes of $K_{k,\ell}$. If $k = 1$ then we say that x_1 is the *center* of $K_{1,\ell}$. If $\ell = 1$ and $k \geq 2$, then y_1 is called the center. We denote the degree of a vertex u in a graph G by $\deg_G(u)$.

Recall that a homomorphism f from a graph G to a graph H is a pseudo-covering from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H . We would like to stress that this is *not* the same as saying that f is a vertex mapping from V_G to V_H such that f restricted to some spanning subgraph G' of G becomes a covering. The reason is that in the latter setting it may well happen that f is not a homomorphism from G to H . For instance, f might map two adjacent vertices of G to the same vertex of H . However, there is an alternative definition which turns out to be very useful for us. In order to present it we need the following notations.

We let $f^{-1}(x)$ denote the set $\{u \in V_G \mid f(u) = x\}$. For a subset $S \subseteq V_G$, $G[S]$ denotes the *induced subgraph* of G by S , i.e., the graph with vertex set S and edges uv whenever $uv \in E_G$. For $xy \in E_H$ with $x \neq y$, we write $G[x, y] = G[f^{-1}(x) \cup f^{-1}(y)]$. Because f is a homomorphism, $G[x, y]$ is a bipartite graph with partition classes $f^{-1}(x)$ and $f^{-1}(y)$. We can now state the alternative definition of pseudo-coverings.

Proposition 1 ([4]). *A homomorphism f from a graph G to a graph H is a pseudo-covering if and only if $G[x, y]$ contains a perfect matching for all $x, y \in V_H$. Consequently, $|f^{-1}(x)| = |f^{-1}(y)|$ for all $x, y \in V_H$.*

Let f be a pseudo-covering from a graph G to a graph H . We then sometimes call the vertices of H *colors* of vertices of G . Due to Proposition 1, $G[x, y]$ must contain a perfect matching M_{xy} . Let $uv \in M_{xy}$ for $xy \in E_H$. Then we say that v is a *matched neighbor* of u , and we call the set of matched neighbors of u the *matched neighborhood* of u .

3 How Factors Relate to Pseudo-Covers

Our next result shows how $\mathcal{S}(K_{k,\ell})$ -factors relate to $K_{k,\ell}$ -pseudo-covers.

Theorem 1. *Let G be a graph on n vertices. Then G is a $K_{k,\ell}$ -pseudo-cover if and only if G has an $\mathcal{S}(K_{k,\ell})$ -factor and G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$.*

Proof. First suppose that $G = (V, E)$ is a $K_{k,\ell}$ -pseudo-cover. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then f is a homomorphism from G to $K_{k,\ell}$, which is a bipartite graph. Consequently, G must be bipartite as well. Let A and B denote the partition classes of G . Then we may assume without loss of generality that $f(A) = X$ and $f(B) = Y$. Due to Proposition 1 we then find that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. By the same proposition we find that each $G[x_i, y_j]$ contains a perfect matching M_{ij} . We define the spanning subgraph $G' = (V, \bigcup_{ij} M_{ij})$ of G and observe that every component in G' is a $K_{k,\ell}$ -cover. Hence G has an $\mathcal{S}(K_{k,\ell})$ -factor.

Now suppose that G has an $\mathcal{S}(K_{k,\ell})$ -factor $\{F_1, \dots, F_p\}$. Also suppose that G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. Since $\{F_1, \dots, F_p\}$ is an $\mathcal{S}(K_{k,\ell})$ -factor, there exists a covering f_i from F_i to $K_{k,\ell}$ for $i = 1, \dots, p$. Let f be the mapping defined on V such that $f(u) = f_i(u)$ for all $u \in V$. Let A_X be the set of vertices of A that are mapped to a vertex in X and let A_Y be the set of vertices of A that are mapped to a vertex in Y . We define subsets B_X and B_Y of B in the same way. This leads to the following equalities:

$$\begin{aligned} |A_X| + |A_Y| &= \frac{kn}{k+\ell} \\ |B_X| + |B_Y| &= \frac{\ell n}{k+\ell} \\ |A_Y| &= \frac{\ell}{k} |B_X| \\ |B_Y| &= \frac{\ell}{k} |A_X|. \end{aligned}$$

Suppose that $\ell \neq k$. Then this set of equalities has a unique solution, namely, $|A_X| = \frac{kn}{k+\ell} = |A|$, $|A_Y| = |B_X| = 0$, and $|B_Y| = \frac{\ell n}{k+\ell} = |B|$. Hence, we find that f maps all vertices of A to vertices of X and all vertices of B to Y . This means that f is a homomorphism from G to $K_{k,\ell}$ that becomes a covering when restricted to the spanning subgraph obtained by taken the disjoint union of the subgraphs $\{F_1, \dots, F_p\}$. In other words, f is a pseudo-covering from G to $K_{k,\ell}$, as desired.

Suppose that $\ell = k$. In this case we have that $|V_{F_i} \cap A| = |V_{F_i} \cap B|$ for $i = 1, \dots, p$, and since each F_i is connected by definition, either $f(V_{F_i} \cap A) = X$ and $f(V_{F_i} \cap B) = Y$, or $f(V_{F_i} \cap A) = Y$ and $f(V_{F_i} \cap B) = X$. In the second case, we can exchange the roles of X and Y and find another covering f_i from F_i such that $f(V_{F_i} \cap A) = X$ and $f(V_{F_i} \cap B) = Y$. Hence, we can assume without loss of generality that each f_i maps $V_{F_i} \cap A$ to X and $V_{F_i} \cap B$ to Y ; so, $|A_X| = |A| = |B_Y| = |B|$ and $|A_Y| = |B_X| = 0$. This completes the proof of Theorem 1. $\square$

4 Classifying the $\mathcal{S}(K_{k,\ell})$ -Factor Problem

Here is the main theorem of this section.

Theorem 2. *The $\mathcal{S}(K_{k,\ell})$ -FACTOR problem is solvable in polynomial time for $k = \ell = 1$. Otherwise it is NP-complete, even for the class of bipartite graphs.*

Proof. We may assume without loss of generality that $k \leq \ell$. First we consider the case when $k = \ell = 1$. Due to Observation 1, the $\mathcal{S}(K_{1,1})$ -FACTOR problem is equivalent to the problem of finding a perfect matching, which can be solved in polynomial time. We deal with the case when $k = 1$ and $\ell \geq 2$ in Proposition 2. Finally, for all $k \geq 2$ and all $\ell \geq 2$, we show in Proposition 3 that if the $K_{k,\ell}$ -PSEUDO-COVER problem is NP-complete, then so is the $\mathcal{S}(K_{k,\ell})$ -FACTOR problem for the class of bipartite graphs. Then the result for this case follows from Theorem 4, in which we show that $K_{k,\ell}$ -PSEUDO-COVER is NP-complete for all $k \geq 2$ and all $\ell \geq 2$. $\square$

The proof of Theorem 2 is conditional upon proving Propositions 2 and 3, and Theorem 4. We prove Theorem 4 in Section 5, and show Propositions 2 and 3 in this section.

Proposition 2 deals with the case $k = 1$ and $\ell \geq 2$. Recall that for general graphs the NP-completeness of this case immediately follows from Observation 1 and the aforementioned result of Hell and Kirkpatrick [13]. However, we consider bipartite graphs. For this purpose, a result by Monnot and Toulouse [18] is of importance for us. Here, P_k denotes a path on k vertices.

Theorem 3 ([18]). *For any fixed $k \geq 3$, the P_k -FACTOR problem is NP-complete for the class of bipartite graphs.*

We use Theorem 3 to prove Proposition 2.

Proposition 2. *For any fixed $\ell \geq 2$, $\mathcal{S}(K_{1,\ell})$ -FACTOR and $K_{1,\ell}$ -FACTOR are NP-complete, even for the class of bipartite graphs.*

Proof. By Observation 1, $\mathcal{S}(K_{1,\ell}) = \{K_{1,\ell}\}$ for all $\ell \geq 2$. Hence we may restrict ourselves to $K_{1,\ell}$ -FACTOR. Clearly, $K_{1,\ell}$ -FACTOR is in NP for all $\ell \geq 2$. Note that $P_3 = K_{1,2}$. Hence the case $\ell = 2$ follows from Theorem 3.

Let $\ell = 3$. We prove that $K_{1,3}$ -FACTOR is NP-complete by reduction from $K_{1,2}$ -FACTOR. Let $G = (V, E)$ be a bipartite graph with partition classes A and B . We will construct a bipartite graph G' from G such that G has an $K_{1,2}$ -factor if and only if G' has a $K_{1,3}$ -factor.

First we make a key observation, namely that all $K_{1,2}$ -factors of G (if there are any) have the same number α of centers in A and the same number β of centers in B . This is so, because the following two equalities

$$\begin{aligned}\alpha + 2\beta &= |A| \\ \beta + 2\alpha &= |B|\end{aligned}$$

that count the number of vertices in A and B , respectively, have a unique solution. In order to obtain G' we do as follows. Let $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_q\}$. First we consider the vertices in A . For $i = 1, \dots, p$, we introduce

- a new vertex s_i with edge $s_i a_i$
- a new vertex t_i with edge $s_i t_i$
- three new vertices u_i^1, u_i^2, u_i^3 with edges $t_i u_i^1, t_i u_i^2, t_i u_i^3$
- a new vertex w_i with edges $u_i^1 w_i, u_i^2 w_i, u_i^3 w_i$.

Finally we add $2p + \alpha$ new vertices $x_1, \dots, x_{2p+\alpha}$ and add edges such that the subgraph induced by the w -vertices and the x -vertices is complete bipartite. We denote the set of s -vertices by S , the set of t -vertices by T , the set of u -vertices by U , the set of w -vertices by W , and the set of x -vertices by X . We repeat the above process with respect to B . For clarity we denote the new vertices with respect to B by s', t', u', w', x' , and corresponding sets by S', T', U', W', X' , respectively. This yields the graph G' which is bipartite with partition classes $A \cup S' \cup T \cup U' \cup W \cup X'$ and $B \cup S \cup T' \cup U \cup W' \cup X$. Also see Figure 2.

Fig. 2. The graph G' .

We are now ready to prove our claim that G has a $K_{1,2}$ -factor if and only if G' has a $K_{1,3}$ -factor.

Suppose that G has a $K_{1,2}$ -factor. We first extend the three-vertex stars in this factor to four-vertex stars by adding the edge $a_i s_i$ for every star center a_i and the edge $b_i s'_i$ for every star center b_i . As we argued above, A contains α centers and B contains β centers. This means that we can add:

- $p - \alpha$ stars with center in T , one leaf in S and two leaves in U ;
- α stars with center in T and three leaves in U ;
- $p - \alpha$ stars with center in W , one leaf in U and two leaves in X ;
- α stars with center in W and three leaves in X .

This is possible because $|S| = p$, $|T| = p$, $|U| = 3p$, $|W| = p$ and $|X| = 2(p - \alpha) + 3\alpha = 2p + \alpha$. With respect to B we can proceed in the same way. Hence, we obtained a $K_{1,3}$ -factor of G' .

Suppose that G' has a $K_{1,3}$ -factor. Let γ be the number of star centers in A that belong to stars with one leaf in S and two leaves in B . Let δ be the number of star centers in B that belong to stars with one leaf in S' and two leaves in A . We first show that $\gamma \geq \alpha$.

In order to obtain a contradiction, suppose that $\gamma < \alpha$. Because every s -vertex (resp. u -vertex) has degree two, no vertex in S (resp. U) is a star center. Let p_1 be the number of star centers in T that belong to stars with a leaf in S (and two leafs in U) and let p_2 be the number of star centers in T that belong to stars with all three leafs in U . By our construction, every star center in W belongs to a star that either has one leaf in U and two leafs in X , or else has three leafs in X . Let q_1 be the number of star centers in W of the first type, and let q_2 be the number of star centers in W of the second type. Finally, let r be the number of star centers in X (centers of stars with all leafs in W). Then by using counting arguments in combination with the equalities $|S| = |T| = |W| = p$, $|U| = 3p$ and $|X| = 2p + \alpha$, we derive the following equalities:

$$\begin{aligned}\gamma + p_1 &= p \\ p_1 + p_2 &= p \\ 2p_1 + 3p_2 + q_1 &= 3p \\ q_1 + q_2 + 3r &= p \\ 2q_1 + 3q_2 + r &= 2p + \alpha\end{aligned}$$

The last two equalities imply that $q_2 = \alpha + 5r$. Equality $\gamma + p_1 = p$ and our assumption $\gamma < \alpha$ implies that $p_1 > p - \alpha$. Equalities $p_1 + p_2 = p$ and $2p_1 + 3p_2 + q_1 = 3p$ lead to $p_1 = q_1$. Hence, we find that $q_1 > p - \alpha$. Substituting $q_1 > p - \alpha$ and $q_2 = \alpha + 5r$ into equality $q_1 + q_2 + 3r = p$ yields $8r < 0$ and this is not possible. Hence $\gamma \geq \alpha$.

By the same reasoning as above we find that $\delta \geq \beta$ holds. This has the following consequence. Let γ^* denote the number of star centers in A that belong to stars with three leaves in B and let δ^* denote the number of star centers in B that belong to stars with three leaves in A . Then we find that

$$p = \gamma + 2\delta + \gamma^* + 3\delta^* \geq \alpha + 2\beta + \gamma^* + 3\delta^*.$$

Recall that $\alpha + 2\beta = p$. If we substitute this in the above equation, we find that $p \geq p + \gamma^* + 3\delta^*$. Hence $\gamma = \alpha$, $\delta = \beta$ and $\gamma^* = \delta^* = 0$. This means that the restriction of the $K_{1,3}$ -factor to G is a $K_{1,2}$ -factor of G , which is what we had to show.

For $\ell \geq 4$ we can proceed in a similar way as for the case $\ell = 3$ (or use induction). This completes the proof of Proposition 2. $\square$

Here is Proposition 3, which allows us to consider the $K_{k,\ell}$ -PSEUDO-COVER problem for all $k \geq 2$ and all $\ell \geq 2$.

Proposition 3. *Fix arbitrary integers $k, \ell \geq 2$. If the $K_{k,\ell}$ -PSEUDO-COVER problem is NP-complete, then so is the $\mathcal{S}(K_{k,\ell})$ -FACTOR problem for the class of bipartite graphs.*

Proof. Let $k, \ell \geq 2$. Let $G = (V, E)$ be an input graph on n vertices of the $K_{k,\ell}$ -PSEUDO-COVER problem. By Theorem 1, we may assume without loss of

generality that G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. Then, by Theorem 1, we find that $G \xrightarrow{P} K_{k,\ell}$ holds if and only if G has an $\mathcal{S}(K_{k,\ell})$ -factor. This finishes the proof of Proposition 3. $\square$

5 Classifying the $K_{k,\ell}$ -Pseudo-Cover Problem

Here is the main theorem of this section.

Theorem 4. *The $K_{k,\ell}$ -PSEUDO-COVER problem can be solved in polynomial time for any fixed k, ℓ with $\min\{k, \ell\} = 1$. Otherwise it is NP-complete.*

Proof. When $\min\{k, \ell\} = 1$ we use Proposition 4. When $\min\{k, \ell\} \geq 2$, we use Proposition 5. $\square$

The proof of Theorem 2 is conditional upon proving Propositions 4 and 5. The remainder of this section is devoted to these two propositions. We start with Proposition 4.

Proposition 4. *The $K_{k,\ell}$ -PSEUDO-COVER problem can be solved in polynomial time for any fixed k, ℓ with $\min\{k, \ell\} = 1$.*

Proof. Let $k = 1$, $\ell \geq 1$, and G be a graph. We show that deciding whether G is a $K_{1,\ell}$ -pseudo-cover comes down to solving the problem of finding a perfect matching in a graph of size at most $\ell|V_G|$. Because the latter can be done in polynomial time, this means that we have proven the proposition.

If $\ell = 1$, then deciding whether G is a $K_{1,\ell}$ -pseudo-cover is readily seen to be equivalent to finding a perfect matching in G .

Now suppose that $\ell \geq 2$. We first check in polynomial time whether G is bipartite with partition classes A and B , such that $|A| = \frac{n}{1+\ell}$ and $|B| = \frac{\ell n}{1+\ell}$. If not, then Theorem 1 tells us that G is a no-instance. Otherwise we continue as follows. Because $k = 1$ and $\ell \geq 2$, we can distinguish between A and B . We replace each vertex $a \in A$ by ℓ copies $a^1, \dots, a^\ell$ and make each a^i adjacent to all neighbors of a . This leads to a bipartite graph G' , the partition classes of which have the same size. We claim that G is a $K_{1,\ell}$ -pseudo-cover if and only if G' has a perfect matching.

First suppose that G is a $K_{1,\ell}$ -pseudo-cover. Then there exists a pseudo-covering f from G to $K_{1,\ell}$. Because $k = 1$ and $\ell \geq 2$, we find that $f(a) = x_1$ for all $a \in A$ and $f(B) = Y$. Consider a vertex $a \in A$. Let $b_1, \dots, b_\ell$ be its matched neighbors. In G' we select the edges $a^i b_i$ for $i = 1, \dots, \ell$. After having done this for all vertices in A , we obtain a perfect matching of G' .

Now suppose that G' has a perfect matching. We define a mapping f by $f(a) = x_1$ for all $a \in A$ and $f(b) = y_i$ if and only if $a^i b$ is a matching edge in G' , where a^i is the i th copy of a . Then f is a pseudo-covering from G to $K_{1,\ell}$. Hence, G is a $K_{1,\ell}$ -pseudo-cover. This completes the proof of Proposition 4. $\square$

We now prove that $K_{k,\ell}$ -PSEUDO-COVER is NP-complete for all $k, \ell \geq 2$ (Proposition 5). Our proof is inspired by the proof of Hell, Kirkpatrick, Kratochvíl, and Kříž [14]. They consider the problem of testing if a graph has an $\mathcal{S}_L$ -factor for any set $\mathcal{S}_L$ of cycles, the length of which belongs to some specified set L . This is useful for our purposes because of the following. If $L = \{4, 8, 12, \dots\}$, then an $\mathcal{S}_L$ -factor of a bipartite graph G with partition classes A and B of size $\frac{n}{2}$ is an $\mathcal{S}(K_{2,2})$ -factor of G that is also a $K_{2,2}$ -pseudo-cover of G by Theorem 1. However, for $k = \ell \geq 3$, this is not longer true, and when $k \neq \ell$ the problem is not even “symmetric” anymore. Below we show how to deal with these issues. We refer to Section 6 for an alternative proof for the case $k, \ell \geq 3$. However, our construction for $k, \ell \geq 2$ does not become simpler when we restrict ourselves to $k, \ell \geq 2$ with $k = 2$ or $\ell = 2$. Therefore, we decided to present our NP-completeness result for all k, ℓ with $k, \ell \geq 2$.

Recall that we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_\ell\}$. We first state a number of useful lemmas. Hereby, we use the alternative definition in terms of perfect matchings, as provided by Proposition 1, when we argue on pseudo-coverings.

Let $G_1(k, \ell)$ be the graph in Figure 3. It contains a vertex a with $\ell - 1$ neighbors $b_1, \dots, b_{\ell-1}$ and a vertex d with $k - 1$ neighbors $c_1, \dots, c_{k-1}$. For any $i \in [1, \ell - 1]$, $j \in [1, k - 1]$, it contains an edge $b_i c_j$. Finally, it contains a vertex e which is only adjacent to d .

Fig. 3. The graph $G_1(k, \ell)$.

Lemma 2. *Let $G_1(k, \ell)$ be an induced subgraph of a bipartite graph G such that only a and e have neighbors outside $G_1(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then $f(a) = f(e)$. Moreover, a has only one matched neighbor outside $G_1(k, \ell)$ and this matched neighbor has color $f(d)$, where d is the only matched neighbor of e inside $G_1(k, \ell)$.*

Proof. Due to their degrees, all edges incident to the b -vertices and the c -vertices must be in a perfect matching. Since $\deg_G(d) = k$, all the edges incident to d must be in a perfect matching. Hence, we find $|f(\{a, c_1, \dots, c_{k-1}\})| = k$ and $|f(\{d, b_1, \dots, b_{\ell-1}\})| = \ell$. This means that $f(a)$ is the only color missing in the neighborhood of d . Consequently, $f(e) = f(a)$. Moreover, $f(d)$ is not a color of a b -vertex. Hence, $f(d)$ must be the color of the matched neighbor of a outside $G_1(k, \ell)$. $\square$

Lemma 3. *Let G be a bipartite graph that contains $G_1(k, \ell)$ as an induced subgraph, such that only a and e have neighbors outside $G_1(k, \ell)$ and such that a and e have no common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_1(k, \ell)$ and by adding a new vertex u that is adjacent to every vertex of G that is a neighbor of a or e outside $G_1(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k, \ell}$, such that $f(u) \in X$ and such that u has exactly one neighbor v of a in its matched neighborhood. Then G is a $K_{k, \ell}$ -pseudo-cover.*

Proof. We may assume without loss of generality that $f(u) = x_k$ and $f(v) = y_\ell$. We modify f as follows. Let $f(a) = f(e) = x_k$ and $f(d) = y_\ell$. Let $f(b_j) = y_j$ for $j = 1, \dots, \ell - 1$ and $f(c_i) = x_i$ for all $i = 1, \dots, k - 1$. In this way we find a pseudo-covering from G to $K_{k, \ell}$. $\square$

Let $G_2(k, \ell)$ be the graph in Figure 4. It contains k vertices $u_1, \dots, u_k$. It also contains $(k-1)k$ vertices $v_{h,i}$ for $h = 1, \dots, k-1, i = 1, \dots, k$, and $(k-1)(\ell-1)$ vertices $w_{i,j}$ for $i = 1, \dots, k-1, j = 1, \dots, \ell-1$. For $h = 1, \dots, k-1, i = 1, \dots, k, j = 1, \dots, \ell-1$, $G_2(k, \ell)$ contains an edge $u_i v_{h,i}$ and an edge $v_{h,i} w_{h,j}$.

Fig. 4. The graph $G_2(k, \ell)$ from Lemma 4.

Lemma 4. *Let G be a bipartite graph that has $G_2(k, \ell)$ as an induced subgraph such that only u -vertices have neighbors outside $G_2(k, \ell)$. Let f be a pseudo-covering from G to $K_{k, \ell}$. Then each u_i has exactly one matched neighbor t_i outside $G_2(k, \ell)$. Moreover, $|f(\{u_1, \dots, u_k\})| = 1$ and $|f(\{t_1, \dots, t_k\})| = k$.*

Proof. Because all v -vertices have degree ℓ and all w -vertices have degree k , all edges of $G_2(k, \ell)$ must be in perfect matchings. If $k \neq \ell$, this means that every v -vertex must get an x -color, whereas every u -vertex and every w -vertex must get

a y -color. Moreover, if $k = \ell$, then we may assume this without loss of generality. As all v -vertices have degree ℓ , the vertices in any $\{u_i, w_{h,1}, \dots, w_{h,\ell-1}\}$ have different x -colors. Moreover, the way we defined the edges between the u -vertices and the v -vertices implies that every u -vertex must have the same y -color, i.e., $|f(\{u_1, \dots, u_k\})| = 1$. Because all edges of $G_2(k, \ell)$ are perfect matching edges and every u -vertex has degree $k - 1$ in $G_2(k, \ell)$, we find that every u_i has exactly one matched neighbor t_i outside $G_2(k, \ell)$. In the (matched) neighborhood of $\{u_1, u_2, \dots, u_k\}$ in $G_2(k, \ell)$, each color x_i appears exactly $k - 1$ times. Consequently, in the matched neighborhood of $\{u_1, u_2, \dots, u_k\}$ outside $G_2(k, \ell)$, each x_i appears once and thus $|f(\{t_1, \dots, t_k\})| = k$.

Lemma 5. *Let G be a bipartite graph that has $G_2(k, \ell)$ as an induced subgraph, such that only u -vertices have neighbors outside $G_2(k, \ell)$ and such that no two u -vertices have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_2(k, \ell)$ and by adding a new vertex s that is adjacent to every vertex of G that is a neighbor of some u -vertex outside $G_2(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$, such that $f(s) \in Y$ and such that s has exactly one neighbor t_i of every u_i in its matched neighborhood. Then G is a $K_{k,\ell}$ -pseudo-cover.*

Proof. We may assume without loss of generality that $f(s) = y_\ell$ and $f(t_i) = x_i$ for $i = 1, \dots, k$. We modify f as follows. For $i = 1, \dots, k$, we let $f(u_i) = y_1$. For $i = 1, \dots, k - 1$ and $j = 2, \dots, \ell$ we let $f(w_{i,j}) = y_j$. For $h = 1, \dots, k - 1$ and $i = 1, \dots, k$, we let $f(v_{h,i}) = x_{h+i}$ if $h + i \leq k$ and $f(v_{h,i}) = x_{h+i-k}$ otherwise. In this way we find a pseudo-covering from $G_2(k, \ell)$ to $K_{k,\ell}$. $\square$

Let $G_3(k, \ell)$ be the graph defined in Figure 5. It contains k copies of $G_1(k, \ell)$, where we denote the a -vertex and e -vertex of the i th copy by a_i and e_i , respectively. It also contains a copy of $G_2(k, \ell)$ with edges $e_i u_i$ and $a_i u_{i+1}$ for $i = 1, \dots, k$ (where $u_{k+1} = u_1$). The construction is completed by adding a vertex p adjacent to all a -vertices and by adding vertices $q, r_1, \dots, r_{\ell-2}$ that are adjacent to all e -vertices. Here we assume that there is no r -vertex in case $\ell = 2$.

Lemma 6. *Let G be a bipartite graph that has $G_3(k, \ell)$ as an induced subgraph, such that only p and q have neighbors outside $G_3(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every a_i is a matched neighbor of p and no e_i is a matched neighbor of q , or else every e_i is a matched neighbor of q and no a_i is a matched neighbor of p .*

Proof. We first show the claim below.

Claim. Either every $e_i u_i$ is in a perfect matching and no $a_i u_{i+1}$ is in a perfect matching, or every $a_i u_{i+1}$ is in a perfect matching and no $e_i u_i$ is in a perfect matching.

We prove this claim as follows. Every u_i is missing exactly one color in its matched neighborhood in $G_2(k, \ell)$ by Lemma 4. This means that, for any i , either $a_{i-1} u_i$ is in a perfect matching, or else $e_i u_i$ is in a perfect matching. We

Fig. 5. The graph $G_3(k, \ell)$.

show that in the first case $e_{i-1}u_{i-1}$ is not in a perfect matching, and that in the second case a_iu_{i+1} is not in a perfect matching.

Suppose that $a_{i-1}u_i$ is in a perfect matching. By Lemma 4, u_{i-1} and u_i have the same color. By Lemma 2, d_{i-1} is a matched neighbor of e_{i-1} with $f(d_{i-1}) = f(u_{i-1})$. Hence, $e_{i-1}u_{i-1}$ is not in a perfect matching. Suppose that e_iu_i is in a perfect matching. Then by the same reasoning, a_iu_{i+1} is not in a perfect matching.

Suppose that e_1u_1 is in a perfect matching. Then a_1u_2 is not in a perfect matching, and consequently e_2u_2 is in a perfect matching, and so on, until we deduce that every e_iu_i is in a perfect matching and no a_iu_{i+1} is in a perfect matching. Suppose that e_1u_1 is not in a perfect matching. Then by the same reasoning we can show the opposite. This proves the claim.

Note that every $e_i r_j$ must be in a perfect matching due to the degree of r_j . Thus, every e_i has exactly one matched neighbor in $\{q, u_i\}$. Moreover, each a_i has exactly one matched neighbor in $\{p, u_{i+1}\}$. Applying the claim then yields the desired result. $\square$

Lemma 7. *Let G be a graph that has $G_3(k, \ell)$ as an induced subgraph such that only p and q have neighbors outside $G_3(k, \ell)$ and such that p and q do not have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_3(k, \ell)$ and by adding a new vertex r^* that is adjacent to every vertex of G that is a neighbor of p or q outside $G_3(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k, \ell}$ such that $f(r^*) \in Y$ and such that either all vertices in the matched*

neighborhood of r^* in G' are all neighbors of p in G , or else are all neighbors of q in G . Then G is a $K_{k,\ell}$ -pseudo-cover.

Proof. We may assume without loss of generality that $f(r^*) = y_\ell$. We show how to modify f . Let $f(p) = f(q) = y_\ell$. Let $f(a_i) = f(e_i) = x_i$ for $1 \leq i \leq k$. Let $f(r_i) = y_{i+1}$ for $1 \leq i \leq \ell - 2$. Let $f(u_i) = y_1$ for $1 \leq i \leq k$.

First suppose that the matched neighborhood of r^* in G' is in the neighborhood of p in G . We define perfect matching edges as follows: the matched neighbor of each a_i outside the i th copy of $G_1(k, \ell)$ is u_{i+1} ; the matched neighbors of each e_i outside the i th copy of $G_1(k, \ell)$ are q and the r -vertices. By Lemmas 3 and 5, we can extend f to all other vertices of $G_3(k, \ell)$. Hence, we find that G is a $K_{k,\ell}$ -pseudo-cover.

Now suppose that the matched neighborhood of r^* in G' is in the neighborhood of q in G . We define perfect matching edges as follows: the matched neighbor of each a_i outside the i th copy of $G_1(k, \ell)$ is p ; the matched neighbors of each e_i outside the i th copy of $G_1(k, \ell)$ are u_i and the r -vertices. By Lemmas 3 and 5, we can extend f to all other vertices of $G_3(k, \ell)$. Hence, also in this case, G is a $K_{k,\ell}$ -pseudo-cover. $\square$

Let $G_4(k, \ell)$ be the graph in Figure 6. It is constructed as follows. We take k copies of $G_3(\ell, k)$. We denote the p -vertex and the q -vertex of the i th copy by $p_{1,i}$ and $q_{1,i}$, respectively. We take ℓ copies of $G_3(k, \ell)$. We denote the p -vertex and the q -vertex of the j th copy by $p_{2,j}$ and $q_{2,j}$, respectively. We add an edge between any $p_{1,i}$ and $p_{2,j}$.

Fig. 6. The graph $G_4(k, \ell)$.

Lemma 8. Let G be a bipartite graph that has $G_4(k, \ell)$ as an induced subgraph such that only the q -vertices have neighbors outside $G_4(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every $p_{1,i}p_{2,j}$ is in a perfect matching and all matched neighbors of every q -vertex are in $G_4(k, \ell)$, or else no edge $p_{1,i}p_{2,j}$

is in a perfect matching and all matched neighbors of every q -vertex are outside $G_4(k, \ell)$.

Proof. Suppose that there is an edge $p_{1,i}p_{2,j}$ in a perfect matching. Then, $p_{1,i}$ and $p_{2,j}$ have a matched neighbor outside their corresponding copy of $G_3(\ell, k)$ and $G_3(k, \ell)$, respectively. Hence, by Lemma 6, all matched neighbors of $q_{1,i}$ and $q_{2,j}$ are inside $G_4(k, \ell)$ and all edges $p_{1,i}p_{2,j'}$ and $p_{1,i'}p_{2,j}$ are in perfect matchings. We apply Lemma 6 a number of times and are done. If no edge $p_{1,i}p_{2,j}$ is in a perfect matching, then by Lemma 6, all matched neighbors of every q -vertex are outside $G_4(k, \ell)$. $\square$

We are now ready to show Proposition 5, where we present our NP-completeness reduction.

Proposition 5. *The $K_{k,\ell}$ -PSEUDO-COVER problem is NP-complete for any fixed k, ℓ with $k, \ell \geq 2$.*

Proof. We reduce from the problem $(k + \ell)$ -DIMENSIONAL MATCHING, which is NP-complete as $k + \ell \geq 3$ (see [10]). In this problem, we are given $k + \ell$ mutually disjoint sets $Q_{1,1}, \dots, Q_{1,k}, Q_{2,1}, \dots, Q_{2,\ell}$, all of equal size m , and a set H of hyperedges $h \in \prod_{i=1}^k Q_{1,i} \times \prod_{j=1}^\ell Q_{2,j}$. The question is whether H contains a $(k + \ell)$ -dimensional matching, i.e., a subset $M \subseteq H$ of size $|M| = m$ such that for any distinct pairs $(q_{1,1}, \dots, q_{1,k}, q_{2,1}, \dots, q_{2,\ell})$ and $(q'_{1,1}, \dots, q'_{1,k}, q'_{2,1}, \dots, q'_{2,\ell})$ in M we have $q_{1,i} \neq q'_{1,i}$ for $i = 1, \dots, k$ and $q_{2,j} \neq q'_{2,j}$ for $j = 1, \dots, \ell$.

Given such an instance, we construct a bipartite graph G with partition classes V_1 and V_2 . First we put all elements in $Q_{1,1} \cup \dots \cup Q_{1,k}$ in V_1 , and all elements in $Q_{2,1} \cup \dots \cup Q_{2,\ell}$ in V_2 . Then we introduce an extra copy of $G_4(k, \ell)$ for each hyperedge $h = (q_{1,1}, \dots, q_{1,k}, q_{2,1}, \dots, q_{2,\ell})$ by adding the missing vertices and edges of this copy to G . We observe that indeed G is bipartite. We also observe that G has polynomial size.

We claim that $((Q_{1,1}, \dots, Q_{1,k}, Q_{2,1}, \dots, Q_{2,\ell}), H)$ admits a $(k + \ell)$ -dimensional matching M if and only if G is a $K_{k,\ell}$ -pseudo-cover.

Suppose that $((Q_{1,1}, \dots, Q_{1,k}, Q_{2,1}, \dots, Q_{2,\ell}), H)$ admits a $(k + \ell)$ -dimensional matching M . We define a homomorphism f from G to $K_{k,\ell}$ as follows. For each hyperedge $h = (q_{1,1}, \dots, q_{1,k}, q_{2,1}, \dots, q_{2,\ell})$, we let $f(p_{1,i}) = f(q_{1,i}) = x_i$ for $i = 1, \dots, k$ and $f(p_{2,j}) = f(q_{2,j}) = y_j$ for $j = 1, \dots, \ell$.

For all $h \in M$, we let every q -vertex of h has all its matched neighbors in the copy of $G_4(k, \ell)$ that corresponds to h , and we define the matched neighbors of every p -vertex of h by choosing the edges $p_{1,i}p_{2,j}$ as matching edges. Since M is a $(k + \ell)$ -dimensional matching, the matched neighbors of every p -vertex and every q -vertex are now defined. We note that the restriction of f to the union S of the p -vertices of all the hyperedges is a pseudo-covering from $G[S]$ to $K_{k,\ell}$. Then, by repeatedly applying Lemma 7, we find that G is a $K_{k,\ell}$ -pseudo-cover.

Conversely, suppose that f is a pseudo-covering from G to $K_{k,\ell}$. By Lemma 8, every q -vertex has all its matched neighbors in exactly one copy of $G_4(k, \ell)$ that corresponds to a hyperedge h such that the matched neighbor of every q -vertex in h is as a matter of fact in that copy $G_4(k, \ell)$. We now define M to be the set

of all such hyperedges. Then M is a $(k + \ell)$ -dimensional matching: any q -vertex appears in exactly one hyperedge of M . $\square$

6 Further Research on Pseudo-coverings

Pseudo-coverings are closely related to the so-called locally constrained homomorphisms, which are homomorphisms with some extra restrictions on the neighborhood of each vertex. In Section 1 we already defined a covering which is also called a locally bijective homomorphism. There are two other types of such homomorphisms. First, a homomorphism from a graph G to a graph H is called *locally injective* or a *partial covering* if for every $u \in V_G$ the restriction of f to the neighborhood of u , i.e., the mapping $f_u : N_G(u) \rightarrow N_H(f(u))$, is injective. Second, a homomorphism from a graph G to a graph H is called *locally surjective* or a *role assignment* if the mapping $f_u : N_G(u) \rightarrow N_H(f(u))$ is surjective for every $u \in V_G$. See [7] for a survey.

The following observation is insightful. Recall that $G[x, y]$ denotes the induced bipartite subgraph of a graph G with partition classes $f^{-1}(x)$ and $f^{-1}(y)$ for some homomorphism f from G to a graph H .

Observation 9 ([9]) *Let f be a homomorphism from a graph G to a graph H . For every edge xy of H ,*

- *f is locally bijective if and only if $G[x, y]$ is 1-regular (i.e., a perfect matching) for all $xy \in E_H$;*
- *f is locally injective if and only if $G[x, y]$ has maximum degree at most one (i.e., a matching) for all $xy \in E_H$;*
- *f is locally surjective if and only if $G[x, y]$ has minimum degree at least one for all $xy \in E_H$.*

By definition, every covering is a pseudo-covering. We observe that this is in line with Proposition 1 and Observation 9. Moreover, by these results, we find that every pseudo-covering is a locally surjective homomorphism. This leads to the following result.

Proposition 6. *For any fixed graph H , if H -COVER is NP-complete, then so is H -PSEUDO-COVER.*

Proof. Let H be a graph for which H -COVER is NP-complete. Let G be an instance of H -COVER. It is folklore that G and H must have the same degree refinement matrix in case $G \xrightarrow{B} H$ holds. We refer to e.g. Kristiansen and Telle [17] for the definition of a degree refinement matrix and how to compute this matrix in polynomial time. For us, it is only relevant that we may assume without loss of generality that G and H have the same degree refinement matrix. We claim that in that case $G \xrightarrow{B} H$ if and only if $G \xrightarrow{P} H$ holds.

Suppose that $G \xrightarrow{B} H$. Then by definition we have $G \xrightarrow{P} H$.

Suppose that $G \xrightarrow{P} H$. By Proposition 1 and Observation 9 we find that $G \xrightarrow{S} H$ holds. Kristiansen and Telle [17] showed that $G \xrightarrow{S} H$ implies $G \xrightarrow{B} H$ whenever G and H have the same degree refinement matrix. $\square$

Due to Proposition 6, the NP-completeness of $K_{k,\ell}$ -PSEUDO-COVER for $k, \ell \geq 3$ also follows from the NP-completeness of $K_{k,\ell}$ -COVER for these values of k, ℓ . The latter is shown by Kratochvíl, Proskurowski and Telle [15]. However, these authors show in the same paper [15] that $K_{k,\ell}$ -COVER is solvable in polynomial time for the cases k, ℓ with $\min\{k, \ell\} \leq 2$. Hence for these cases we have to rely on our proof in Section 5.

Another consequence of Proposition 6 is that H -PSEUDO-COVER is NP-complete for all k -regular graphs H for any $k \geq 3$ due to a hardness result for the corresponding H -COVER [6]. However, a complete complexity classification of H -PSEUDO-COVER is still open, just as dichotomy results for H -PARTIAL COVER and H -COVER are not known, whereas for the locally surjective case a complete complexity classification has been given [8]. So far, we could obtain some partial results but a complete classification of the complexity of H -PSEUDO-COVER seems already difficult for trees (we found many polynomial-time solvable and NP-complete cases).

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