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AN A POSTERIORI ERROR ESTIMATOR FOR *HP*-ADAPTIVE DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC EIGENVALUE PROBLEMS

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Abstract

In this paper we present a residual-based *a posteriori* error estimator for *hp*-adaptive discontinuous Galerkin (DG) methods for elliptic eigenvalue problems. In particular we use as a model problem the Laplace eigenvalue problem on bounded domains in \mathbb{R}^d , $d = 2, 3$, with homogeneous Dirichlet boundary conditions. Analogous error estimators can be easily obtained for more complicated elliptic eigenvalue problems. We prove the reliability and efficiency of the residual based error estimator and use numerical experiments to show that, under an *hp*-adaptation strategy driven by the error estimator, exponential convergence can be achieved, even for non-smooth eigenfunctions.

1 Introduction

Eigenvalue problems appear naturally in many physical situations, for example, when studying acoustics and vibration analysis, the Schrödinger equation, nuclear reactor criticality and the linear stability analysis of steady solutions to nonlinear differential equations. A popular numerical method for the solution of the eigenvalue problem is by a finite element method (FEM), see Boffi [1] for an up to date review. As with any numerical approach, it is important to be able to quantify the error made by way of an *a*

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a posteriori error estimate, which can then also be used to drive an adaptive mesh/polynomial enrichment process. Although *a posteriori* error analysis is a mature subject for source problems, for eigenvalues it is still very much in its infancy; for the conforming FEM we refer the reader to [24, 25, 23] in the case of residual based error estimates and to [21] for a goal oriented approach; for the discontinuous Galerkin finite element method (DGFEM) see [20] where the goal oriented approach is applied in the context of linear stability analysis for the incompressible Navier–Stokes equations. To the authors’ knowledge, the work here represents a first attempt at residual based *a posteriori* error estimation for a DGFEM applied to an eigenvalue problem.

As ever, before we can tackle more difficult problems we must understand how to deal with a simple model problem, in our case the Laplace eigenvalue with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \subset \mathbb{R}^d, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where $d = 2, 3$. Here, Ω is a bounded polygonal domain with boundary $\Gamma = \partial\Omega$.

The standard weak formulation of (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$A(u, v) \equiv \int_{\Omega} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} u \, v \, dx \equiv \lambda b(u, v) \quad \forall v \in H_0^1(\Omega), \quad (1.2)$$

where the space $H_0^1(\Omega)$ is the standard space of functions with gradient in $L^2(\Omega)$ and with zero trace on Γ .

Discontinuous Galerkin methods offer advantages in the context of *hp*-adaptivity over standard conforming FEMs. For example they provide increased flexibility in mesh design (irregular grids are admissible) and the freedom to choose the elemental polynomial degrees without the need to enforce continuity between elements. In this article we develop a residual based *a posteriori* error estimator for the *hp*-symmetric interior penalty discontinuous Galerkin method (SIPG) discretisation (see [2]) of the Laplace eigenvalue problem (1.1). Following the techniques developed in [11, 8] for source problems (which in turn are based on [10]), our approach enables us to show reliability and efficiency of our error estimator. In essence, the proofs require recasting the DGFEM in a non-consistent manner and decomposing the DG solution into a suitable conforming and a nonconforming part. The projection operator from DG space to conforming space and the corresponding *hp*-stability estimates used are those in [8]. We show that the

error in both the eigenvalues and the eigenfunctions can be bounded above and below in terms of a computable residual based term and an uncomputable term, however, we show that, at least for eigenfunctions $u \in H^2(\Omega)$ the uncomputable term is of higher order than the residual term and can thus be ignored. In order to show the higher order nature of this term we require *hp*- *a priori* error estimates for both the eigenvalue and eigenvector, in both the energy norm and L_2 norm. As such, we first extend the *h*- *a priori* estimates from [7, 22] to the *hp*-setting. For eigenfunctions with lower regularity we show by using numerical experiments that the uncomputable term can also be regarded as higher order.

The paper is structured as follows. In the next section we introduce the SIPG discretisation for the model problem after first defining some appropriate functions spaces and trace operators. We then prove *a priori* error estimates in appropriately defined norms in Section 3. The *a posteriori* error estimator is stated in Section 4 and its reliability and efficiency shown, up to higher order terms. In Section 5 we present a number of numerical experiments to validate our theoretical results. The first experiment is on a square domain, while the second is on an L-shaped domain with non-smooth eigenfunctions; in both cases exponential rates of convergence are attained under the *hp*-adaptation strategy.

2 Discontinuous Galerkin discretization

In this section, we introduce the *hp*-version SIPG finite element method for the discretization of (1.1).

Throughout, we assume that the computational domain Ω can be partitioned into a shape-regular mesh \mathcal{T} , *i.e.* there exists a constant C_{reg} such that for any element K

$$h_K \leq C_{\text{reg}} \rho_K , \quad (2.3)$$

where h_K is the diameter of the element and ρ_K is the diameter of the biggest ball inscribed in K . Also we assume that the elements are affine quadrilaterals or hexahedra. We store the elemental diameters in the mesh size vector $\underline{h} = \{ h_K : K \in \mathcal{T} \}$. Let us also denote by h the maximum of all h_K in the mesh. In order to be able to deal with irregular meshes we need to define the faces of a mesh \mathcal{T} . We refer to F as an interior mesh face of \mathcal{T} if $F = \partial K \cap \partial K'$ for two neighboring elements $K, K' \in \mathcal{T}$ whose intersection has a positive surface measure. The set of all interior mesh faces is denoted by $\mathcal{F}_I(\mathcal{T})$. Analogously, if the intersection $F = \partial K \cap \Gamma$ of the boundary of an element $K \in \mathcal{T}$ and Γ is of positive surface measure, we refer to F

as a boundary mesh face of \mathcal{T} . The set of all boundary mesh faces of \mathcal{T} is denoted by $\mathcal{F}_B(\mathcal{T})$ and we set $\mathcal{F}(\mathcal{T}) = \mathcal{F}_I(\mathcal{T}) \cup \mathcal{F}_B(\mathcal{T})$. The diameter of a face F is denoted by h_F . We allow for 1-irregularly refined meshes \mathcal{T} defined as follows. Let K be an element of \mathcal{T} and F an elemental face in $\mathcal{F}(K)$. Then F may contain at most one hanging node located in the center of F and at most one hanging node in the middle of each elemental edge of F .

Next, let us define the jumps and averages of piecewise smooth functions across faces of the mesh \mathcal{T} . To that end, let the interior face $F \in \mathcal{F}_I(\mathcal{T})$ be shared by two neighboring elements K and K^e where the superscript e stands for “exterior”. For a piecewise smooth function v , we denote by $v|_F$ the trace on F taken from inside K , and by $v^e|_F$ the one taken from inside K^e . The average and jump of v across the face F are then defined as

$$\{\{v\}\} = \frac{1}{2}(v|_F + v^e|_F), \quad \llbracket v \rrbracket = v|_F \underline{n}_K + v^e|_F \underline{n}_{K^e}.$$

Here, \underline{n}_K and \underline{n}_{K^e} denote the unit outward normal vectors on the boundary of elements K and K^e , respectively. Similarly, if \underline{q} is piecewise smooth vector field, its average and (normal) jump across F are given by

$$\{\{\underline{q}\}\} = \frac{1}{2}(\underline{q}|_F + \underline{q}^e|_F), \quad \llbracket \underline{q} \rrbracket = \underline{q}|_F \cdot \underline{n}_K + \underline{q}^e|_F \cdot \underline{n}_{K^e}.$$

On a boundary face $F \in \mathcal{F}_B(\mathcal{T})$, we accordingly set $\{\{\underline{q}\}\} = \underline{q}$ and $\llbracket v \rrbracket = v\underline{n}$, with \underline{n} denoting the unit outward normal vector on Γ . The other trace operators will not be used on boundary faces and are thereby left undefined.

In order to define the hp -version finite element space on \mathcal{T} , we begin by introducing polynomial spaces on elements and faces. To that end, let $K \in \mathcal{T}$ be an element. We set

$$\mathcal{Q}_p(K) = \{v : K \rightarrow \mathbb{R} : v \circ T_K \in \mathcal{Q}_p(\widehat{K})\}, \quad (2.4)$$

with $\mathcal{Q}_p(\widehat{K})$ denoting the set of tensor product polynomials on the reference element \widehat{K} of degree less than or equal to p in each coordinate direction on \widehat{K} . In addition, if $F \in \mathcal{F}(K)$ is a face of K and \widehat{F} the corresponding face on the reference element \widehat{K} , we define

$$\mathcal{Q}_p(F) = \{v : F \rightarrow \mathbb{R} : v \circ T_K|_F \in \mathcal{Q}_p(\widehat{F})\}, \quad (2.5)$$

where $\mathcal{Q}_p(\widehat{F})$ denotes the set of tensor product polynomials on \widehat{F} of degree less than or equal to p in each coordinate direction on \widehat{F} . Then, we assign

a polynomial degree $p_K \geq 1$ with each element K of the mesh \mathcal{T} . We then introduce the degree vector $\underline{p} = \{p_K : K \in \mathcal{T}\}$. We assume that \underline{p} is of bounded local variation, that is, there is a constant $\varrho \geq 1$, independent of the mesh \mathcal{T} sequence under consideration, such that

$$\varrho^{-1} \leq p_K/p_{K'} \leq \varrho \quad (2.6)$$

for any pair of neighboring elements $K, K' \in \mathcal{T}$. For a mesh face $F \in \mathcal{F}(\mathcal{T})$, we introduce the face polynomial degree p_F by

$$p_F = \begin{cases} \max\{p_K, p_{K^e}\}, & \text{if } F = \partial K \cap \partial K^e \in \mathcal{F}_I(\mathcal{T}), \\ p_K, & \text{if } F = \partial K \cap \Gamma \in \mathcal{F}_B(\mathcal{T}). \end{cases} \quad (2.7)$$

For a partition \mathcal{T} of Ω and a polynomial degree vector \underline{p} on \mathcal{T} , we define the hp -version DG finite element space by

$$S_{\underline{p}}(\mathcal{T}) = \{v \in L^2(\Omega) : v|_K \in \mathcal{Q}_{p_K}(K), K \in \mathcal{T}\}. \quad (2.8)$$

Let us also denote by p the minimum of all p_K in the mesh.

We need several norms in the analysis. The standard L^2 norm is denoted by $\|\cdot\|_{0,\Omega}$ and the standard H^1 norm is denoted by $\|\cdot\|_{1,\Omega}$. We shall also need the following DG norms already used in [16, 17, 18]:

Definition 2.1 (DG norm) For any $u \in \mathcal{S}(h) := S_{\underline{p}}(\mathcal{T}) + H^1(\Omega)$

$$\|u\|_{\mathcal{T}}^2 := \int_{\Omega} (\nabla u)^2 dx + \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \frac{h_F}{\gamma p_F^2} \{\{\underline{n} \cdot \nabla u\}\}^2 ds + \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \int_F \llbracket u \rrbracket^2 ds.$$

Definition 2.2 (Energy norm) For any $u \in \mathcal{S}(h)$

$$\|u\|_{\mathbb{E},\mathcal{T}}^2 = \sum_{K \in \mathcal{T}} \|\nabla u\|_{0,K}^2 + \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \|\llbracket u \rrbracket\|_{0,F}^2. \quad (2.9)$$

Finally, we denote with $\|\cdot\|_{s,\Omega}$ the norm of the Sobolev space $H^s(\Omega)$, with $s \geq 1$ and when we need to restrict a norm to a subpart \mathcal{B} of the domain Ω , we will state this explicitly, for example by $\|\cdot\|_{0,\mathcal{B}}$, $\|\cdot\|_{1,\mathcal{B}}$, etc.

All the analysis in this work has been developed for the SIPG method [2, 3] which is known to be a stable and consistent method for sufficiently large values of γ , see below. The SIPG discrete version of the eigenvalue problem (1.2) is: find $(\lambda_{hp}, u_{hp}) \in \mathbb{R} \times S_{\underline{p}}(\mathcal{T})$ such that

$$A_{hp}(u_{hp}, v_{hp}) = \lambda_{hp} b(u_{hp}, v_{hp}) \quad \forall v_{hp} \in S_{\underline{p}}(\mathcal{T}), \quad (2.10)$$

and with $\|u_{hp}\|_{0,\Omega} = 1$. The bilinear form $A_{hp}(u, v)$ is given by

$$\begin{aligned} A_{hp}(u, v) &= \sum_{K \in \mathcal{T}} \int_K \nabla u \cdot \nabla v \, dx - \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \left(\{\{\nabla u\}\} \cdot \llbracket v \rrbracket + \{\{\nabla v\}\} \cdot \llbracket u \rrbracket \right) ds \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \int_F \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds, \end{aligned} \quad (2.11)$$

where the gradient operator ∇ is defined elementwise and the parameter $\gamma > 0$ is the interior penalty parameter.

To be able to carry out the *a posteriori* analysis, we must perform a non-consistent reformulation of the DG discretization (2.10). To this end, we introduce the following lifting operator already used in [13, 2]. For any v belonging to $\mathcal{S}(h)$, we define $\mathcal{L}(v) \in [S_{\underline{p}}(\mathcal{T})]^d$ by

$$\int_{\Omega} \mathcal{L}(v) \cdot \mathbf{q}_{hp} \, dx = \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \llbracket v \rrbracket \cdot \{\{\mathbf{q}_{hp}\}\} \, ds, \quad \forall \mathbf{q}_{hp} \in S_{\underline{p}}(\mathcal{T})^d. \quad (2.12)$$

Now, the following extended bilinear form $\tilde{A}_{hp}(u, v)$ can be introduced:

$$\begin{aligned} \tilde{A}_{hp}(u, v) &= \sum_{K \in \mathcal{T}} \int_K \nabla u \cdot \nabla v \, dx - \sum_{K \in \mathcal{T}} \int_K \mathcal{L}(u) \cdot \nabla v + \mathcal{L}(v) \cdot \nabla u \, dx \\ &\quad + \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \int_F \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds, \end{aligned} \quad (2.13)$$

and the corresponding discrete problem is to find $(\lambda_{hp}, u_{hp}) \in \mathbb{R} \times S_{\underline{p}}(\mathcal{T})$ such that

$$\tilde{A}_{hp}(u_{hp}, v_{hp}) = \lambda_{hp} b(u_{hp}, v_{hp}), \quad \forall v_{hp} \in S_{\underline{p}}(\mathcal{T}). \quad (2.14)$$

Remark 2.3 *It is clear that $\tilde{A}_{hp}(\cdot, \cdot) \equiv A_{hp}(\cdot, \cdot)$ on $S_{\underline{p}}(\mathcal{T}) \times S_{\underline{p}}(\mathcal{T})$ and $\tilde{A}_{hp}(\cdot, \cdot) \equiv A(\cdot, \cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$.*

It is straightforward to see that the energy norm related to problem (1.2) and the standard norm of $H_0^1(\Omega)$ are equivalent, *i.e.*,

$$\exists c_a, C_a > 0: \quad c_a \|u\|_{1,\Omega} \leq A(u, u)^{1/2} \leq C_a \|u\|_{1,\Omega}, \quad \text{for all } u \in H_0^1(\Omega),$$

which also implies that the bilinear form $A(\cdot, \cdot)$ is coercive.

Remark 2.4 *The coercivity of the bilinear form $A(\cdot, \cdot)$ implies that the spectrum is positive, because for any eigenpair (λ, u) , with $\|u\|_{0,\Omega} = 1$, we have:*

$$0 < c_a^2 \|u\|_{1,\Omega}^2 \leq A(u, u) = \lambda b(u, u) = \lambda.$$

Another easy-to-prove property for both the bilinear forms $A(\cdot, \cdot)$ and $b(\cdot, \cdot)$ is continuity, *i.e.*,

$$\begin{aligned} \exists C_a > 0 & : A(u, v) \leq C_a \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \text{for all } u, v \in H_0^1(\Omega), \\ \exists C_b > 0 & : b(u, v) \leq C_b \|u\|_{0,\Omega} \|v\|_{0,\Omega}, \quad \text{for all } u, v \in L^2(\Omega). \end{aligned}$$

It has already been proved in [3, Theorem 3.3, Theorem 3.5] that the bilinear form $A_{hp}(\cdot, \cdot)$ is continuous in $H^2(\mathcal{T}) := \{v \in L^2(\Omega) : |v|_K \in H^2(K), \forall K \in \mathcal{T}\}$ for $\gamma > 0$, *i.e.*,

$$|A_{hp}(u, v)| \leq C_A \|u\|_{\mathcal{T}} \|v\|_{\mathcal{T}}, \quad \text{for all } u, v \in H^2(\mathcal{T}),$$

with a constant $C_A > 0$ independent of h and p , and that it is also coercive in $S_p(\mathcal{T})$ for sufficiently large γ , *i.e.*,

$$A_{hp}(u, u) \geq c_A \|u\|_{\mathcal{T}}^2, \quad \text{for all } u \in S_p(\mathcal{T}),$$

with a constant $c_A > 0$ independent of h and p .

Similarly, it has been proved in [13, Lemma 4.3, Lemma 4.4] that the bilinear form $\tilde{A}_{hp}(\cdot, \cdot)$ is continuous on $\mathcal{S}(h)$, *i.e.*,

$$|\tilde{A}_{hp}(u, v)| \leq C_{\tilde{A}} \|u\|_{\mathbb{E},\mathcal{T}} \|v\|_{\mathbb{E},\mathcal{T}}, \quad (2.15)$$

with a constant $C_{\tilde{A}} > 0$ independent of h and p , and that it is also coercive in $H_0^1(\Omega)$, *i.e.*,

$$\tilde{A}_{hp}(u, u) = \|u\|_{\mathbb{E},\mathcal{T}}^2.$$

3 *A priori* analysis

In this section we present standard *a priori* results for the SIPG method applied to eigenvalue problems. Throughout the section we assume that Ω is convex so all eigenfunctions of (1.1) are in $H^s(\Omega)$, with $s \geq 2$.

We start with a very simple result that shows every computed eigenvalue λ_{hp} is positive. It follows naturally that for any eigenfunction $\|u_{hp}\|_{0,\Omega} = 1$ we have

$$0 < c_A \|u_{hp}\|_{\mathcal{T}}^2 \leq A_{hp}(u_{hp}, u_{hp}) = \lambda_{hp} b(u_{hp}, u_{hp}) = \lambda_{hp},$$

since the only $v \in S_{\underline{p}}(\mathcal{T})$ such that $\|v\|_{\mathcal{T}} = 0$ is $v \equiv 0$. Together with Remark 2.4 we can conclude that all eigenvalues of (1.2) and all $N = \dim S_{\underline{p}}(\mathcal{T})$ eigenvalues of (2.10) are positive and so we can order them as $0 < \lambda_1 \leq \lambda_2 \dots$ and $0 < \lambda_{1, hp} \leq \lambda_{2, hp} \dots \leq \lambda_{N, hp}$, where they have been counted with their multiplicity. In view of Remark 2.3 we have that the discrete eigenvalues from both (2.10) and (2.14) coincide.

3.1 Non-pollution and completeness results

The results of non-pollution and completeness of the spectrum for the hp -case are simple extensions of the analogous results for the h -case only, which are already present in the literature (see [7]). For brevity, we shall not present the complete proofs, but just discuss how the results can be extended to the hp -case.

Theorem 3.1 (Non-pollution of the spectrum) *Let $B \subset \mathbb{C}$ be an open set containing the spectrum of problem (1.2). Then, for sufficiently large $N = \dim S_{\underline{p}}(\mathcal{T})$, B contains the spectrum of (2.10).*

The result of non-pollution of the spectrum can be proved following the same arguments as in the proof of [7, Theorem 4.1], with the only difference that in the hp -case we have from [3, Theorem 4.1] the following estimate to be used in [7, Lemma 4.3]: for all $f \in L^2(\Omega)$

$$\|(T - T_h)f\|_{\mathbb{E}, \mathcal{T}} \leq C \frac{h^{\min\{p+1, s\}-1}}{p^{s-3/2}} \|f\|_{0, \Omega}, \quad (3.16)$$

where T and T_h are respectively the continuous and the discrete solution operators.

The next two results can be shown in the hp -case by simply extending [7, Property 2] using [3, Theorem 4.1], *i.e.*, for any $f \in L^2(\Omega)$, let us denote with $w \in H^2(\Omega)$ the solution of $-\Delta w = f$ on Ω , and with w_{hp} its DG approximated solution, then

$$\|w - w_{hp}\|_{\mathbb{E}, \mathcal{T}} \leq C \frac{h^{\min\{p+1, s\}-1}}{p^{s-3/2}} \|w\|_{2, \Omega},$$

and also

$$\lim_{N \rightarrow \infty} \inf_{w_{hp} \in S_{\underline{p}}(\mathcal{T}_N)} \|w - w_{hp}\|_{\mathbb{E}, \mathcal{T}_N} = 0 \quad \forall w \in H^2(\Omega).$$

Theorem 3.2 (Completeness of the spectrum) *For any eigenvalue λ of (1.2), there is an eigenvalue λ_{hp} of (2.10) such that*

$$\lim_{N \rightarrow \infty} \lambda - \lambda_{hp} = 0 .$$

Theorem 3.3 (Non-pollution and completeness of the eigenspaces) *When $N \rightarrow \infty$, the eigenspaces of (2.10) converge to the eigenspaces of (1.2).*

The distance of an approximate eigenfunction from the true eigenspace is a crucial quantity in the convergence analysis for eigenvalue problems especially in the case of non-simple eigenvalues.

Definition 3.4 *Given a function $v \in L^2(\Omega)$ and a finite dimensional subspace $\mathcal{P} \subset L^2(\Omega)$, we define:*

$$\text{dist}(v, \mathcal{P})_{0, \Omega} := \min_{w \in \mathcal{P}} \|v - w\|_{0, \Omega} . \quad (3.17)$$

Similarly, given a function $v \in S_{\underline{p}}(\mathcal{T})$ and a finite dimensional subspace $\mathcal{P} \subset H_0^1(\Omega)$, we define:

$$\text{dist}(v, \mathcal{P})_{\mathbb{E}, \mathcal{T}} := \min_{w \in \mathcal{P}} \|v - w\|_{\mathbb{E}, \mathcal{T}} . \quad (3.18)$$

Now let λ_j be any eigenvalue of problem (1.1) and let $M(\lambda_j)$ denote the span of all corresponding eigenfunctions according to (1.1), moreover let $M_1(\lambda_j) = \{u \in M(\lambda_j) : \|u\|_{\mathbb{E}, \mathcal{T}} = 1\}$. Also let us denote for an eigenvalue λ_j of multiplicity R the space $M_{hp}(\lambda_j)$ spanned by all computed eigenfunctions $u_{j+i, hp}$, $i = 0, \dots, R - 1$ such that $\lambda_{j+i, hp}$ is an approximation of λ_j for all i .

In order to make further progress we need an assumption on the regularity of solutions of elliptic problems defined by the bilinear form $A_{hp}(\cdot, \cdot)$.

Assumption 3.5 *We assume that there exists a constant $C_{\text{ell}} > 0$ with the following property. For $f \in L^2(\Omega)$, if $v \in H_0^1(\Omega)$ solves the problem $A_{hp}(v, w) = b(f, w)$ for all $w \in H_0^1(\Omega)$, then*

$$\|v\|_{2, \Omega} \leq C_{\text{ell}} \|f\|_{0, \Omega} , \quad (3.19)$$

where $v \in H^2(\Omega)$.

Similar assumptions can be found in [6, 7, 2].

3.2 Identity results

The focus of this subsection is Lemma 3.8 which links together the two quantities of interest in our convergence analysis, namely the error in the eigenvalues and the error in the eigenfunctions.

Definition 3.6 (Residual of a linear problem) *Let define the residual for a linear problem $-\Delta u = f$, with $f \in L^2(\Omega)$, as*

$$\mathcal{R}(u, v) := \tilde{A}_{hp}(u, v) - b(f, v) , \quad (3.20)$$

where $u \in H^s(\Omega)$, with $s \geq 2$, is the solution of the linear problem and $v \in \mathcal{S}(h)$.

We extend Definition 3.6 to the eigenvalue case allowing $f = \lambda_j u_j$, so for any eigenpair (λ_j, u_j) of problem (1.1):

$$\mathcal{R}(u_j, v) := \tilde{A}_{hp}(u_j, v) - \lambda_j b(u_j, v) , \quad (3.21)$$

where $v \in \mathcal{S}(h)$.

Let us also recall a useful result for linear problems, which is analogous to the result in [14, 15]; we omit the details of the proof for brevity.

Lemma 3.7 *Let $u \in H^s(\Omega)$, with $s \geq 2$, be the continuous solution of $\tilde{A}_{hp}(u, v) = b(f, v)$, with $f \in L^2(\Omega)$, then for all $v \in \mathcal{S}(h)$ there exists a constant $C > 0$ independent of h and p such that*

$$\mathcal{R}(u, v) \leq C \frac{h^{\min\{p+1, s-1\}}}{p^{s-1}} \|u\|_{s, \Omega} \|v\|_{\mathbb{E}, \mathcal{T}} .$$

Lemma 3.8 (Identity result for the extended form) *Let (λ_l, u_l) be a true eigenpair of problem (1.2) with $\|u_l\|_{0, \Omega} = 1$ and let $(\lambda_{j, hp}, u_{j, hp})$ be a computed eigenpair of problem (2.14) with $\|u_{j, hp}\|_{0, \Omega} = 1$. Then we have:*

$$\tilde{A}_{hp}(u_l - u_{j, hp}, u_l - u_{j, hp}) = \lambda_l \|u_l - u_{j, hp}\|_{0, \Omega}^2 + \lambda_{j, hp} - \lambda_l + 2\mathcal{R}(u_l, u_j - u_{j, hp}).$$

Proof. Using the linearity of the bilinear form $\tilde{A}_{hp}(\cdot, \cdot)$ and using (1.2) , (2.14), we have

$$\tilde{A}_{hp}(u_l - u_{j, hp}, u_l - u_{j, hp}) = \lambda_l + \lambda_{j, hp} - 2\tilde{A}_{hp}(u_l, u_{j, hp}) + 2\lambda_l b(u_l, u_{j, hp}) - 2\lambda_l b(u_l, u_{j, hp}) . \quad (3.22)$$

Furthermore, by analogous arguments we obtain

$$\|u_l - u_{j, hp}\|_{0, \Omega}^2 = 2 - 2b(u_l, u_{j, hp}). \quad (3.23)$$

Substituting (3.23) into (3.22) we obtain

$$\tilde{A}_{hp}(u_l - u_{j,hp}, u_l - u_{j,hp}) = \lambda_l \|u_l - u_{j,hp}\|_{0,\Omega}^2 + \lambda_{j,hp} - \lambda_l - 2\tilde{A}_{hp}(u_l, u_{j,hp}) + 2\lambda_l b(u_l, u_{j,hp}).$$

Finally, noticing that $\tilde{A}_{hp}(u_l, u_j) = \lambda_l b(u_l, u_j)$ and using (3.21) we obtain the result. \blacksquare

3.3 Convergence results

The proof of the next lemma is analogous to the proof of Theorem 3.3 in [14], so it is omitted for brevity.

Lemma 3.9 *For all $f \in L^2(\Omega)$, such that $Tf \in H^s(\Omega)$, with $s \geq 2$, we have that*

$$\|(T - T_h)f\|_{0,\Omega} \leq C \frac{h^{\min(p+1,s)}}{p^{s-1/2}} \|Tf\|_{s,\Omega}.$$

Theorem 3.10 *Suppose that Ω is a convex domain and suppose $1 \leq j \leq \dim S_p(\mathcal{T})$. Let λ_j be an eigenvalue of (1.1) with corresponding eigenspace $M(\lambda_j)$ of dimension $R \geq 1$ and let $(\lambda_{j,hp}, u_{j,hp})$ be an eigenpair of (2.10). Then, for a sufficiently rich DG finite element space*

$$(i) \quad |\lambda_j - \lambda_{j,hp}| \leq C_1^2 \frac{h^{2(\min\{p+1,s\}-1)}}{p^{2s-3}}. \quad (3.24)$$

$$(ii) \quad \text{dist}(u_{j,hp}, M_1(\lambda_j))_{E,\mathcal{T}} \leq C_1 \frac{h^{\min\{p+1,s\}-1}}{p^{s-3/2}}. \quad (3.25)$$

$$(iii) \quad \text{dist}(u_{j,hp}, M_1(\lambda_j))_{0,\Omega} \leq C_2 \frac{h^{\min\{p+1,s\}}}{p^{s-1/2}}. \quad (3.26)$$

The constants C_1, C_2 depend on the spectral information $\{(\lambda_\ell, u_\ell) : \ell = 1, \dots, j\}$, the separation constant ρ , the constants $C_{\text{ell}}, C_{\text{reg}}$ in Assumption 3.5 and in (2.3), respectively.

Proof.

In order to prove (i) we recall equation (3.18) from [6], *i.e.*,

$$|\lambda_j - \lambda_{j,hp}| \leq \sup_{0 \leq i \leq R} |\lambda_j - \lambda_{j+i,hp}| \leq C(\delta_h(M(\lambda_j), S_p(\mathcal{T})))^2,$$

where

$$\delta_h(M(\lambda_j), S_{\underline{p}}(\mathcal{T})) := \sup_{\substack{u \in M(\lambda_j) \\ \|u\|_{E, \mathcal{T}}=1}} \inf_{v_{hp} \in S_{\underline{p}}(\mathcal{T})} \|u - v_{hp}\|_{E, \mathcal{T}} .$$

Then the result comes from [3, Theorem 4.1].

In order to prove (ii) we use the arguments in [7]. In particular we have that if λ_j is an eigenvalue of (1.1), then it is straightforward to see that $\mu_j = \lambda_j^{-1}$ is an eigenvalue of T . Let Γ be a circle in the complex plane centered at μ_j which does not enclose any other point of $\sigma(T)$. As in [7, Sections 5-6], using the spectral projections

$$E = \frac{1}{2\pi i} \int_{\Gamma} (z - T)^{-1} dz , \quad E_h = \frac{1}{2\pi i} \int_{\Gamma} (z - T_h)^{-1} dz ,$$

we have

$$\begin{aligned} \text{dist}(u_{j, hp}, M_1(\lambda_j))_{E, \mathcal{T}} &\leq \sup_{\substack{u_{hp} \in M_{hp}(\lambda_j) \\ \|u_{hp}\|_{E, \mathcal{T}}=1}} \inf_{v \in M(\lambda_j)} \|v - u_{hp}\|_{E, \mathcal{T}} \\ &= \sup_{\substack{u_{hp} \in S_{\underline{p}}(\mathcal{T}) \\ \|u_{hp}\|_{E, \mathcal{T}}=1}} \inf_{v \in L^2(\Omega)} \|Ev - E_h u_{hp}\|_{E, \mathcal{T}} . \end{aligned}$$

Then taking $v = u_{hp}$ we have

$$\begin{aligned} \text{dist}(u_{j, hp}, M_1(\lambda_j))_{E, \mathcal{T}} &\leq \sup_{\substack{u_{hp} \in S_{\underline{p}}(\mathcal{T}) \\ \|u_{hp}\|_{E, \mathcal{T}}=1}} \|Eu_{hp} - E_h u_{hp}\|_{E, \mathcal{T}} \\ &\leq \|E - E_h\|_{\mathcal{L}(S_{\underline{p}}(\mathcal{T}), S_{\underline{p}}(\mathcal{T}))} \leq \|E - E_h\|_{\mathcal{L}(L^2(\Omega), S_{\underline{p}}(\mathcal{T}))} , \end{aligned}$$

where the norm of an operator is defined as:

$$\|P\|_{\mathcal{L}(A, B)} := \sup_{\substack{v \in A \\ \|v\|_A=1}} \|Pv\|_B .$$

Using an argument similar to [6, Theorem 3.11], we have that

$$\|E - E_h\|_{\mathcal{L}(L^2(\Omega), S_{\underline{p}}(\mathcal{T}))} \leq C \|T - T_{hp}\|_{\mathcal{L}(L^2(\Omega), S_{\underline{p}}(\mathcal{T}))} .$$

To conclude the proof we use (3.16).

In order to prove (iii), we use an argument similar to Lemma 3.5 in [30]:

$$\text{dist}(u_{j, hp}, M_1(\lambda_j))_{0, \Omega} \leq \|u - u_{j, hp}\|_{0, \Omega} = \|u - E_h u\|_{0, \Omega} ,$$

where $u \in M(\lambda_j)$ is the eigenvector such that $u_{j,hp} = E_h u$. We know that such a u exists because for a sufficiently rich finite element space $E_h : M(\lambda_j) \rightarrow M_{hp}(\lambda_j)$ is one-to-one, see [4, 5].

$$\begin{aligned} \|u - E_h u\|_{0,\Omega} &= \|[E - E_h]u\|_{0,\Omega} = \frac{1}{2\pi} \left\| \int_{\Gamma_j} (z - T_h)^{-1} (T - T_h) \frac{u}{z - \mu} dz \right\|_{0,\Omega} \\ &\leq C \sup_{z \in \Gamma_j} \|(z - T_h)^{-1}\|_{\mathcal{L}(L^2, L^2)} \|(T - T_h)u\|_{0,\Omega} \end{aligned}$$

where $\|(z - T_h)^{-1}\|_{\mathcal{L}(L^2, L^2)}$ is bounded because from standard DG analysis $\|T - T_h\|_{\mathcal{L}(L^2, L^2)}$ tends to zero. Then the result follows from Lemma 3.9. \blacksquare

4 Energy norm *a posteriori* error estimates

The main results in this section are reliability for eigenfunctions and eigenvalues (Theorem 4.5 and Theorem 4.6) and efficiency (Theorem 4.13) for the residual error estimator introduced below. The reliability ensures that, up to a constant and to asymptotic high order terms, the error estimator η_j gives rise to an *a posteriori* upper bound for errors in both eigenvalues and eigenfunctions, on the other hand, the efficiency ensures that, up to a constant and to asymptotic high order terms, the true error bounds the error estimator η_j from above. Together these two results ensures that the computable quantity η_j is linearly proportional to the true error, up to higher order terms. So it is safe to assume that the true error decays on a sequence of meshes where the *a posteriori* error η_j decays, too. The main results in this section holds also for non convex domains Ω .

As in [11, 8], we shall make use of an auxiliary 1-irregular mesh $\tilde{\mathcal{T}}$ of affine quadrilaterals. We construct the auxiliary mesh $\tilde{\mathcal{T}}$ refining the mesh \mathcal{T} such that no-hanging nodes in \mathcal{T} are hanging nodes in $\tilde{\mathcal{T}}$ as well.

We then introduce the following auxiliary DG finite element space on the mesh $\tilde{\mathcal{T}}$:

$$S_{\tilde{p}}(\tilde{\mathcal{T}}) = \{v \in L^2(\Omega) : v|_{\tilde{K}} \circ T_{\tilde{K}} \in \mathcal{Q}_{p_{\tilde{K}}}(\tilde{K}), \tilde{K} \in \tilde{\mathcal{T}}\},$$

where the auxiliary polynomial degree vector \tilde{p} is defined by $p_{\tilde{K}} = p_K$ for all children $\tilde{K} \in \tilde{\mathcal{T}}$ of an element $K \in \mathcal{T}$.

The next theorem, which comes from [11, 8], defines an averaging operator for the auxiliary mesh $\tilde{\mathcal{T}}$.

Theorem 4.1 *There exists an averaging operator $I_{hp} : S_{\underline{p}}(\mathcal{T}) \rightarrow S_{\underline{p}}^c(\tilde{\mathcal{T}})$, where*

$$S_{\underline{p}}^c(\tilde{\mathcal{T}}) = S_{\underline{p}}(\tilde{\mathcal{T}}) \cap H_0^1(\Omega) , \quad (4.27)$$

that satisfies

$$\sum_{\tilde{K} \in \tilde{\mathcal{T}}} \|\nabla(v - I_{hp}v)\|_{L^2(\tilde{K})}^2 \lesssim \sum_{F \in \mathcal{F}(\mathcal{T})} p_F^2 h_F^{-1} \|\llbracket v \rrbracket\|_{L^2(F)}^2, \quad (4.28)$$

$$\sum_{\tilde{K} \in \tilde{\mathcal{T}}} \|v - I_{hp}v\|_{L^2(\tilde{K})}^2 \lesssim \sum_{F \in \mathcal{F}(\mathcal{T})} p_F^{-2} h_F \|\llbracket v \rrbracket\|_{L^2(F)}^2. \quad (4.29)$$

In the sequel, we shall use the symbols \lesssim and \gtrsim to denote bounds that are valid up to positive constants independent of \underline{h} and \underline{p} .

4.1 Residual-based error estimator

Let $(\lambda_{j, hp}, u_{j, hp})$ be a computed eigenpair of (2.10). For each element $K \in \mathcal{T}$, we introduce the following local error indicator $\eta_{j, K}$ which is given by the sum of the three terms:

$$\eta_{j, K}^2 = \eta_{j, R_K}^2 + \eta_{j, F_K}^2 + \eta_{j, J_K}^2 , \quad (4.30)$$

where the first term η_{j, R_K} is the residual in the interior of the element K :

$$\eta_{j, R_K}^2 = p_K^{-2} h_K^2 \|\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}\|_{0, K}^2 ,$$

the second term η_{j, F_K} is the residual on the faces of K in the interior of the domain Ω :

$$\eta_{j, F_K}^2 = \frac{1}{2} \sum_{F \in \mathcal{F}_I(K)} p_F^{-1} h_F \|\llbracket \nabla u_{j, hp} \rrbracket\|_{0, F}^2 ,$$

and finally the residual η_{j, J_K} measures the jumps on the faces of K of the approximate solution $u_{j, hp}$:

$$\eta_{j, J_K}^2 = \frac{1}{2} \sum_{F \in \mathcal{F}_I(K)} \frac{\gamma^2 p_F^3}{h_F} \|\llbracket u_{j, hp} \rrbracket\|_{0, F}^2 + \sum_{F \in \mathcal{F}_B(K)} \frac{\gamma^2 p_F^3}{h_F} \|\llbracket u_{j, hp} \rrbracket\|_{0, F}^2.$$

Summing (4.30) on all elements we obtain the global error estimator η_j :

$$\eta_j^2 = \sum_{K \in \mathcal{T}} \eta_{j, K}^2. \quad (4.31)$$

Remark 4.2 As already remarked in [9, 11] for the two-dimensional case and in [8] for the three-dimensional case, the weight in the jump residual η_{j,J_K} is of the different order $p_F^3 h_F^{-1}$, which is suboptimal with respect to the powers of p_F when compared to the jump terms in the interior penalty form $A_{hp}(u, v)$, which is of order $p_F^2 h_F^{-1}$ on each face. This suboptimality is due to the possible presence of hanging nodes in the underlying mesh \mathcal{T} . On meshes without irregular nodes, Theorem 4.5 holds true with the following jump residual:

$$\widehat{\eta}_{j,J_K}^2 = \frac{1}{2} \sum_{F \in \mathcal{F}_I(K)} \frac{\gamma^2 p_F^2}{h_F} \|\llbracket u_{j, hp} \rrbracket\|_{0,F}^2 + \sum_{F \in \mathcal{F}_B(K)} \frac{\gamma^2 p_F^2}{h_F} \|\llbracket u_{j, hp} \rrbracket\|_{0,F}^2,$$

with associated estimator $\widehat{\eta}$:

$$\widehat{\eta}_j^2 = \sum_{K \in \mathcal{T}} \widehat{\eta}_{j,K}^2 \quad \text{with} \quad \widehat{\eta}_{j,K}^2 = \eta_{j,R_K}^2 + \eta_{j,F_K}^2 + \widehat{\eta}_{j,J_K}^2. \quad (4.32)$$

In Section 5 we show that both η_j and $\widehat{\eta}_j$ lead to exponential convergence to the true solution on the sequence of adaptively refined meshes.

4.2 Reliability

In order to prove the reliability, we decompose a computed eigenfunction $u_{j, hp}$ into a conforming part and a remainder:

$$u_{j, hp} = u_{j, hp}^c + u_{j, hp}^r,$$

where $u_{j, hp}^c = I_{hp} u_{j, hp} \in S_{\underline{p}}^c(\widetilde{\mathcal{T}}) \subset H_0^1(\Omega)$ is defined using the averaging operator I_{hp} in Theorem 4.1 and the remainder $u_{j, hp}^r$ is given by $u_{j, hp}^r = u_{j, hp} - u_{j, hp}^c \in S_{\underline{p}}^r(\widetilde{\mathcal{T}})$. It is straightforward to show that $\|u_j - u_{j, hp}\|_{E, \mathcal{T}} \leq \|u_j - u_{j, hp}\|_{E, \widetilde{\mathcal{T}}}$, therefore, since $u_j - u_{j, hp}^c \in H_0^1(\Omega)$,

$$\begin{aligned} \|u_j - u_{j, hp}\|_{E, \mathcal{T}} &\leq \|u_j - u_{j, hp}\|_{E, \widetilde{\mathcal{T}}} \leq \|u_j - u_{j, hp}^c\|_{E, \widetilde{\mathcal{T}}} + \|u_{j, hp}^r\|_{E, \widetilde{\mathcal{T}}} \\ &= \|u_j - u_{j, hp}^c\|_{E, \mathcal{T}} + \|u_{j, hp}^r\|_{E, \widetilde{\mathcal{T}}} \end{aligned} \quad (4.33)$$

Then to prove reliability for eigenfunctions it is just necessary to bound both terms in the right hand side of (4.33) using η_j . The proof that

$$\|u_{j, hp}^r\|_{E, \widetilde{\mathcal{T}}} \lesssim \eta_j, \quad (4.34)$$

is equivalent to [8, Lemma 4.1] and we omit it for brevity.

On the other hand, to bound $\|u_j - u_{j,hp}^c\|_{\mathbb{E},\mathcal{T}}$ in (4.33), we split $A_{hp}(\cdot, \cdot) = D_{hp}(\cdot, \cdot) + K_{hp}(\cdot, \cdot)$ where

$$\begin{aligned} D_{hp}(u, v) &= \sum_{K \in \mathcal{T}} \int_K \nabla u \cdot \nabla v \, dx + \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \int_F \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds, \\ K_{hp}(u, v) &= - \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \{\!\!\{ \nabla u \}\!\!\} \cdot \llbracket v \rrbracket \, ds - \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \{\!\!\{ \nabla v \}\!\!\} \cdot \llbracket u \rrbracket \, ds. \end{aligned}$$

The form $D_{hp}(u, v)$ is well-defined for $u, v \in \mathcal{S}(h)$, whereas $K_{hp}(u, v)$ is only well-defined for discrete functions $u, v \in S_{\underline{p}}(\mathcal{T})$. Furthermore, we have

$$A(u, v) = D_{hp}(u, v) \quad \forall u, v \in H_0^1(\Omega), \quad (4.35)$$

as well as

$$A_{hp}(u, v) = D_{hp}(u, v) + K_{hp}(u, v) \quad \forall u, v \in S_{\underline{p}}(\mathcal{T}). \quad (4.36)$$

We also recall the standard hp -approximation results from [9, Lemma 3.7]: for any $v \in H_0^1(\Omega)$, there exists a function $v_{hp} \in S_{\underline{p}}(\mathcal{T})$ such that

$$\begin{aligned} p_K^2 h_K^{-2} \|v - v_{hp}\|_{0,K}^2 &\lesssim \|\nabla v\|_{0,K}^2, \\ \|\nabla(v - v_{hp})\|_{0,K}^2 &\lesssim \|\nabla v\|_{0,K}^2, \\ p_K h_K^{-1} \|v - v_{hp}\|_{0,\partial K}^2 &\lesssim \|\nabla v\|_{0,K}^2, \end{aligned} \quad (4.37)$$

for any element $K \in \mathcal{T}$.

Lemma 4.3 *For any $v \in H_0^1(\Omega)$, for all (λ_j, u_j) solving (1.2) and for all $(\lambda_{j,hp}, u_{j,hp})$ solving (2.10), we have*

$$\int_{\Omega} \lambda_j u_j (v - v_{hp}) \, dx - D_{hp}(u_{j,hp}, v - v_{hp}) + K_{hp}(u_{j,hp}, v_{hp}) \lesssim \left(\eta_j + \frac{h}{p} \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0 \right) \|v\|_{\mathbb{E},\mathcal{T}},$$

where, $v_{hp} \in S_{\underline{p}}(\mathcal{T})$ is the hp -approximation of v satisfying (4.37).

Proof. For brevity, let us set

$$T = \int_{\Omega} \lambda_j u_j (v - v_{hp}) \, dx - D_{hp}(u_{j,hp}, v - v_{hp}) + K_{hp}(u_{j,hp}, v_{hp}).$$

Integrating the volume terms by parts we obtain

$$\begin{aligned}
T &= \sum_{K \in \mathcal{T}} \int_K (\lambda_j u_j + \Delta u_{j,hp})(v - v_{hp}) dx - \sum_{F \in \mathcal{F}(\mathcal{T})} \frac{\gamma p_F^2}{h_F} \int_F \llbracket u_{j,hp} \rrbracket \cdot \llbracket v - v_{hp} \rrbracket ds \\
&\quad - \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_F \llbracket \nabla u_{j,hp} \rrbracket \llbracket \{v - v_{hp}\} \rrbracket ds - \sum_{F \in \mathcal{F}(\mathcal{T})} \int_F \llbracket \nabla v_{hp} \rrbracket \cdot \llbracket u_{j,hp} \rrbracket ds \\
&\equiv T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the approximation properties (4.37) we have that

$$\begin{aligned}
T_1 &= \sum_{K \in \mathcal{T}} \int_K (\lambda_{j,hp} u_{j,hp} + \Delta u_{j,hp})(v - v_{hp}) dx + \sum_{K \in \mathcal{T}} \int_K (\lambda_j u_j - \lambda_{j,hp} u_{j,hp})(v - v_{hp}) dx \\
&\lesssim \left(\sum_{K \in \mathcal{T}} \eta_{j,R_K}^2 \right)^{\frac{1}{2}} \|v\|_{\mathbf{E},\mathcal{T}} + \frac{h}{p} \|\lambda u - \lambda_{hp} u_{j,hp}\|_0 \|v\|_{\mathbf{E},\mathcal{T}}.
\end{aligned}$$

For term T_2 , we again exploit the Cauchy-Schwarz inequality to conclude that

$$T_2 \leq \left(\sum_{F \in \mathcal{F}(\mathcal{T})} \gamma^2 p_F^3 h_F^{-1} \|\llbracket u_{j,hp} \rrbracket\|_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}(\mathcal{T})} p_F h_F^{-1} \|\llbracket v - v_{hp} \rrbracket\|_{0,F}^2 \right)^{\frac{1}{2}}.$$

Thus, from (2.3), (2.6) and (4.37), we obtain the bound

$$T_2 \lesssim \left(\sum_{K \in \mathcal{T}} \eta_{j,J_K}^2 \right)^{\frac{1}{2}} \|v\|_{\mathbf{E},\mathcal{T}}.$$

Similarly, term T_3 can be bounded by

$$\begin{aligned}
T_3 &\leq \left(\sum_{F \in \mathcal{F}_I(\mathcal{T})} p_F^{-1} h_F \|\llbracket \nabla u_{j,hp} \rrbracket\|_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{F \in \mathcal{F}_I(\mathcal{T})} p_F h_F^{-1} \|\llbracket \{v - v_{hp}\} \rrbracket\|_{0,F}^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{K \in \mathcal{T}} \eta_{j,F_K}^2 \right)^{\frac{1}{2}} \|v\|_{\mathbf{E},\mathcal{T}}.
\end{aligned}$$

In a similar way we use Cauchy-Schwarz inequality, (2.3) and (2.6) for term T_4 :

$$T_4 \lesssim \gamma^{-1} \left(\sum_{F \in \mathcal{F}(\mathcal{T})} \gamma^2 p_F^2 h_F^{-1} \|\llbracket u_{j,hp} \rrbracket\|_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}} p_K^{-2} h_K \|\nabla v_{hp}\|_{0,\partial K}^2 \right)^{\frac{1}{2}}.$$

From the standard hp -version inverse trace inequality, see [29], we conclude that

$$T_4 \lesssim \gamma^{-1} \left(\sum_{K \in \mathcal{T}} \eta_{j,J_K}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}} \|\nabla v_{hp}\|_{0,K}^2 \right)^{\frac{1}{2}},$$

furthermore, using the approximation properties in (4.37),

$$\sum_{K \in \mathcal{T}} \|\nabla v_{hp}\|_{0,K}^2 \lesssim \sum_{K \in \mathcal{T}} \|\nabla(v - v_{hp})\|_{0,K}^2 + \sum_{K \in \mathcal{T}} \|\nabla v\|_{0,K}^2 \lesssim \|v\|_{\mathbf{E},\mathcal{T}}^2.$$

Hence

$$T_4 \lesssim \gamma^{-1} \left(\sum_{K \in \mathcal{T}} \eta_{j,J_K}^2 \right)^{\frac{1}{2}} \|v\|_{\mathbf{E},\mathcal{T}}.$$

The bounds for T_1 , T_2 , T_3 , and T_4 imply the assertion. \blacksquare

We are now ready to bound $\|u_j - u_{j,hp}^c\|_{\mathbf{E},\mathcal{T}}$ in (4.33).

Lemma 4.4 *Let $(\lambda_{j,hp}, u_{j,hp})$ be a computed eigenpair of (2.10) and let (λ_j, u_j) be an eigenpair of (1.2). Then we have for $u_{j,hp}^c = I_{hp} u_{j,hp}$ that:*

$$\|u_j - u_{j,hp}^c\|_{\mathbf{E},\mathcal{T}} \lesssim \eta_j + \left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0.$$

Proof. Since $u_j - u_{j,hp}^c \in H_0^1(\Omega)$, we have that

$$\|u_j - u_{j,hp}^c\|_{\mathbf{E},\mathcal{T}}^2 = A(u_j - u_{j,hp}^c, v), \quad (4.38)$$

where $v = u_j - u_{j,hp}^c$. To bound the right-hand side of (4.38), we note that, by (4.35),

$$A(u_j - u_{j,hp}^c, v) = \int_{\Omega} \lambda_j u_j v \, dx - A(u_{j,hp}^c, v) = \int_{\Omega} \lambda_j u_j v \, dx - D_{hp}(u_{j,hp}^c, v).$$

It is straightforward to see that $D_{hp}(u_{j,hp}^c, v) = D_{hp}(u_{j,hp}, v) + R$, with

$$R = - \sum_{\tilde{K} \in \tilde{\mathcal{T}}} \int_{\tilde{K}} \nabla u_{j,hp}^r \cdot \nabla v \, dx.$$

Furthermore, from (2.10) and (4.36), we have

$$\int_{\Omega} \lambda_{j,hp} u_{j,hp} v_{hp} \, dx = D_{hp}(u_{j,hp}, v_{hp}) + K_{hp}(u_{j,hp}, v_{hp}),$$

where $v_{hp} \in S_{\underline{p}}(\mathcal{T})$ is the hp -approximation of v . Combining these results, we thus arrive at

$$\begin{aligned} A(u_j - u_{j, hp}^c, v) &= \int_{\Omega} (\lambda_j u_j - \lambda_{j, hp} u_{j, hp}) v_{hp} \, dx + \int_{\Omega} \lambda_j u_j (v - v_{hp}) \, dx \\ &\quad - D_{hp}(u_{j, hp}, v - v_{hp}) + K_{hp}(u_{j, hp}, v_{hp}) - R. \end{aligned} \quad (4.39)$$

Using Poincaré's inequality and (4.37) we have

$$\|v_{hp}\|_{0, \Omega} \lesssim \frac{h}{p} \|\nabla v\|_{0, \Omega} + \|v\|_{0, \Omega} \leq \left(\frac{h}{p} + C_p\right) \|\nabla v\|_{0, \Omega},$$

then from (4.39) we obtain:

$$\begin{aligned} A(u_j - u_{j, hp}^c, v) &\leq \left(\frac{h}{p} + C_p\right) \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_{0, \Omega} \|v\|_{\mathbb{E}, \mathcal{T}} + \int_{\Omega} \lambda_j u_j (v - v_{hp}) \, dx \\ &\quad - D_{hp}(u_{j, hp}, v - v_{hp}) + K_{hp}(u_{j, hp}, v_{hp}) - R. \end{aligned}$$

The estimate in Lemma 4.3 now yields

$$A(u_j - u_{j, hp}^c, v) \lesssim \left(\eta_j + \left(C_p + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_0\right) \|v\|_{\mathbb{E}, \mathcal{T}} + |R|. \quad (4.40)$$

It remains to bound $|R|$; from the Cauchy-Schwarz inequality and (4.34), we readily obtain

$$|R| \lesssim \|u_{j, hp}^r\|_{\mathbb{E}, \tilde{\mathcal{T}}} \|v\|_{\mathbb{E}, \mathcal{T}} \lesssim \eta_j \|v\|_{\mathbb{E}, \mathcal{T}}. \quad (4.41)$$

The desired result now follows from (4.40) and (4.41). \blacksquare

The proof of Theorem 4.5 readily follows from (4.33), (4.34) and Lemma 4.4.

Theorem 4.5 (Reliability for eigenfunctions) *Let $(\lambda_{j, hp}, u_{j, hp})$ be a computed eigenpair of (2.10) converging to the true eigenvalue λ_j of multiplicity $R \geq 1$. Then we have that:*

$$\text{dist}(u_{j, hp}, E_1(\lambda_j))_{\mathbb{E}, \mathcal{T}} \lesssim \eta_j + \left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_0,$$

where u_j is the minimizer of (3.17).

Proof. From (4.33), (4.34) and Lemma 4.4 we have that:

$$\begin{aligned} \text{dist}(u_{j, hp}, E_1(\lambda_j))_{\mathbb{E}, \mathcal{T}} &\leq \|u_j - u_{j, hp}^c\|_{\mathbb{E}, \mathcal{T}} + \|u_{j, hp}^r\|_{\mathbb{E}, \tilde{\mathcal{T}}} \\ &\lesssim \eta_j + \left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_0. \end{aligned} \quad \blacksquare$$

Theorem 4.6 (Reliability for eigenvalues) *Let $(\lambda_{j,hp}, u_{j,hp})$ be a computed eigenpair of (2.10) and converging to λ_j of multiplicity $R \geq 1$. Then we have that:*

$$|\lambda_j - \lambda_{hp}| \lesssim \eta_j^2 + G ,$$

where

$$G = \left(1 + \frac{h}{p}\right)^2 \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0^2 + 2\eta_j \left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0 + 2|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j,hp})| ,$$

where u_j is the minimizer of (3.17) and \hat{u}_j is the minimizer of (3.18).

Proof. Applying (2.15) to Lemma 3.8 and also noticing that $\lambda_j \|\hat{u}_j - u_{j,hp}\|_{0,\Omega}^2 > 0$ we have

$$|\lambda_j - \lambda_{j,hp}| \lesssim \text{dist}(u_{j,hp}, E_1(\lambda_j))_{E,\mathcal{T}}^2 + 2|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j,hp})| .$$

Applying Theorem 4.5

$$|\lambda_j - \lambda_{j,hp}| \lesssim \left(\eta_j + \left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0\right)^2 + 2|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j,hp})| .$$

■

The next two corollaries identify the higher order terms in the results of Theorem 4.5 and Theorem 4.6.

Corollary 4.7 *Under the same assumptions as in Theorem 4.5 and with the extra condition that $E_1(\lambda_j) \subset H^s(\Omega)$, with $s \geq 2$, we have that the term $\left(1 + \frac{h}{p}\right) \|\lambda_j u_j - \lambda_{hp} u_{j,hp}\|_0$ is asymptotically higher in order compared to η_j .*

Proof. By the triangle inequality we have

$$\|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0 \leq \|\lambda_j u_j - \lambda_j u_{j,hp}\|_0 + \|\lambda_j u_{j,hp} - \lambda_{j,hp} u_{j,hp}\|_0 = \lambda_j \|u_j - u_{j,hp}\|_0 + |\lambda_j - \lambda_{j,hp}| .$$

In view of Theorem 3.10 we have that

$$\text{dist}(u_{j,hp}, E_1(\lambda_j))_{E,\mathcal{T}} = \mathcal{O}\left(\frac{h^{\min\{p+1,s\}-1}}{p^{s-3/2}}\right), \quad \|\lambda_j u_j - \lambda_{j,hp} u_{j,hp}\|_0 = \mathcal{O}\left(\frac{h^{\min\{p+1,s\}}}{p^{s-1/2}}\right),$$

leaving the only possibility that η_j is the leading term. ■

Corollary 4.8 *Under the same assumptions as in Theorem 4.6 and with the extra condition that $E_1(\lambda_j) \subset H^s(\Omega)$, with $s \geq 2$, we have that the term G is asymptotically higher in order compared to η_j^2 .*

Proof. From the Corollary 4.7 and from Theorem 3.10(ii) we already know that both η^2 and $|\lambda - \lambda_{hp}|$ are of order $\mathcal{O}\left(\frac{h^{2(\min\{p+1,s\}-1)}}{p^{2s-3}}\right)$. So what remains to be proved is that all the terms in G are higher in order. From simple applications of Theorem 3.10 and Corollary 4.7

$$\|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_0^2 = \mathcal{O}\left(\frac{h^{2(\min\{p+1,s\})}}{p^{2s-1}}\right), \quad \eta_j \|\lambda u - \lambda_{hp} u_{j, hp}\|_0 = \mathcal{O}\left(\frac{h^{2\min\{p+1,s\}-1}}{p^{2s-2}}\right).$$

Then, applying Lemma 3.7 we have

$$|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j, hp})| \lesssim \frac{h^{\min\{p+1, s-1\}}}{p^{s-1}} \|u\|_{s, \Omega} \|\hat{u}_j - u_{j, hp}\|_{E, \mathcal{T}}.$$

Using [3, Theorem 4.1]) we obtain

$$|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j, hp})| \lesssim \frac{h^{\min\{p+1, s-1\} + \min\{p+1, s\} - 1}}{p^{2s-5/2}}.$$

It is possible that for some values of p and s , $\min\{p+1, s-1\} = \min\{p+1, s\} - 1$, making the term $|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j, hp})|$ of the same order in h as η_j^2 . However in general we have that

$$\min\{p+1, s\} - 1 = \min\{p, s-1\} \leq \min\{p+1, s-1\},$$

making $|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j, hp})|$ of order equal or higher. Moreover in p the term $|\mathcal{R}(\hat{u}_j, \hat{u}_j - u_{j, hp})|$ is definitely higher order compared to η_j^2 . \blacksquare

4.3 Efficiency

In this section we prove the efficiency of the error estimator η_j . Unfortunately a proof of efficiency robust in both h and p is not available, so we present a proof robust only in h as in many other works [8, 11, 12].

In the proof we exploit bubble functions, which are in general smooth and positive real valued functions with compact supports and bounded by 1 in the L^∞ norm. Also, these functions have local support, so it is possible to define a bubble function on each element and on each edge in the mesh. Furthermore, it is possible to prove inverse estimates for bubble functions of standard results involving norms, thanks to their regularity. These estimates are collected in the next proposition. We define for any element K a real-valued bubble function ψ_K with support in K which vanishes on the edge of K and for any edge F in the interior of the domain we need a real-valued bubble function ψ_F that vanishes outside the closure of $K_F^+ \cup K_F^-$. In [12, Lemma 3.3], such bubble functions ψ_τ, ψ_f are constructed using polynomials. Moreover, it is proven that ψ_τ, ψ_f satisfy the following lemma:

Lemma 4.9 *We have*

$$\|v\|_{0,K} \lesssim \|\psi_K^{1/2} v\|_{0,K}, \quad (4.42)$$

$$\|\psi_K v\|_{E,K} \lesssim h_K^{-1} \|v\|_{0,K}, \quad (4.43)$$

and on an interior edge F

$$\|w\|_{0,F} \lesssim \|\psi_F^{1/2} w\|_{0,F}, \quad (4.44)$$

$$\|\psi_f w\|_{0,K_F^+ \cup K_F^-} \lesssim h_F^{1/2} \|w\|_{0,F}, \quad (4.45)$$

$$\|\psi_f w\|_{E,K_F^+ \cup K_F^-} \lesssim h_F^{-1/2} \|w\|_{0,F}, \quad (4.46)$$

hold for all $\tau \in \mathcal{T}_h$, all $F \in \mathcal{F}_I(\mathcal{T})$, for all polynomials v and w and where K_F^+ and K_F^- are the two elements sharing F .

In the following we bound each single term forming η_j with the energy norm of the error plus, where necessary, high order terms.

Lemma 4.10 *Let $(\lambda_{j,hp}, u_{j,hp})$ be a computed eigenpair of (2.10) converging to λ_j of multiplicity $R \geq 1$. Then we have that:*

$$\left(\sum_{K \in \mathcal{T}_K} \eta_{j,J_K}^2 \right)^{1/2} \lesssim \text{dist}(u_{j,hp}, E_1(\lambda_j))_{E,\mathcal{T}}.$$

Proof. Let u_j be the minimizer of (3.18). Since $\llbracket u_j \rrbracket = 0$ then for any K

$$\begin{aligned} \eta_{j,J_K}^2 &= \frac{1}{2} \sum_{F \in \mathcal{F}_I(K)} \frac{\gamma^2 p_F^3}{h_F} \|\llbracket u_{j,hp} - u_j \rrbracket\|_{0,F}^2 + \sum_{F \in \mathcal{F}_B(K)} \frac{\gamma^2 p_F^3}{h_F} \|\llbracket u_{j,hp} - u_j \rrbracket\|_{0,F}^2 \\ &\lesssim \sum_{F \in \mathcal{F}(K)} \frac{\gamma p_F^2}{h_F} \|\llbracket u_{j,hp} - u_j \rrbracket\|_{0,F}^2 \leq \|u_j - u_{j,hp}\|_{E,\omega_K}^2, \end{aligned}$$

where the set ω_K contains K and its neighbours. The result follows by summing the contribution from all elements. \blacksquare

Lemma 4.11 *Let $(\lambda_{j,hp}, u_{j,hp})$ be a computed eigenpair of (2.10) converging to eigenvalue λ_j of multiplicity $R \geq 1$. Then we have that:*

$$\left(\sum_{K \in \mathcal{T}_K} \eta_{j,R_K}^2 \right)^{1/2} \lesssim \text{dist}(u_{j,hp}, E_1(\lambda_j))_{E,\mathcal{T}} + \frac{h}{p} \|\lambda_{j,hp} u_{j,hp} - \lambda_j u_j\|_{0,\mathcal{T}},$$

where u_j be the minimizer of (3.18).

Proof. For each element K let $W|_K = h_K^2 p_K^{-2} (\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}) \psi_K$, then using (4.42)

$$\eta_{j, R_K}^2 = \frac{h_K^2}{p_K^2} \|\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}\|_{0, K}^2 \lesssim \int_K (\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}) W \, dx .$$

Since $\lambda_j u + \Delta u_j = 0$ is satisfied at least weakly, we then have:

$$\eta_{j, R_K}^2 \lesssim \int_K (\lambda_{j, hp} u_{j, hp} - \lambda_j u_j + \Delta(u_{j, hp} - u)) W \, dx .$$

Then using the fact that $W|_{\partial K} = 0$, we have by integration by parts and using (4.43):

$$\begin{aligned} \eta_{j, R_K}^2 &\lesssim \int_K (\lambda_{j, hp} u_{j, hp} - \lambda_j u_j) W - \nabla(u_{j, hp} - u) \cdot \nabla W \, dx \\ &\leq \|u - u_{j, hp}\|_{E, K} \|W\|_{E, K} + \|\lambda_{j, hp} u_{j, hp} - \lambda_j u_j\|_{0, K} \|W\|_{0, K} \\ &\lesssim \left(h_K^{-1} \|u_j - u_{j, hp}\|_{E, K} + \|\lambda_{j, hp} u_{j, hp} - \lambda_j u_j\|_{0, K} \right) \frac{h_K^2}{p_K^2} \|\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}\|_{0, K} . \end{aligned}$$

Dividing both sides by $h_K p_K^{-1} \|\lambda_{j, hp} u_{j, hp} + \Delta u_{j, hp}\|_{0, K}$ we end up with

$$\eta_{j, R_K} \lesssim p_k^{-1} \|u_j - u_{j, hp}\|_{E, K} + \frac{h_K}{p_K} \|\lambda_{j, hp} u_{j, hp} - \lambda_j u_j\|_{0, K} ,$$

which leads to the result by summing the contributions from all elements and noticing that $p_k^{-1} \leq 1$. \blacksquare

Lemma 4.12 *Let $(\lambda_{j, hp}, u_{j, hp})$ be a computed eigenpair of (2.10) converging to λ_j of multiplicity $R \geq 1$. Then we have that:*

$$\left(\sum_{K \in \mathcal{T}_K} \eta_{j, F_K}^2 \right)^{1/2} \lesssim \text{dist}(u_{j, hp}, E_1(\lambda_j))_{E, \mathcal{T}} + \frac{h^{1/2}}{p^{1/2}} \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_0 ,$$

where u_j is the minimizer of (3.18).

Proof. For each element K let $W = \sum_{F \in \mathcal{F}_I(\mathcal{T})} h_F p_F^{-1} \llbracket \nabla u_{j, hp} \rrbracket \psi_F$, then

using (4.44), $[\![\nabla u_j]\!] = 0$ on interior edges and integration by parts

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} \eta_{j, F_K}^2 &\lesssim \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_F [\![\nabla u_{j, hp}]\!] W \, ds = \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_F [\![\nabla u_{j, hp} - \nabla u_j]\!] W \, ds \\
&= \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\Delta u_{j, hp} - \Delta u_j) W + (\nabla u_{j, hp} - \nabla u_j) \cdot \nabla W \, dx \\
&= \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\Delta u_{j, hp} + \lambda_{j, hp} u_{j, hp}) W + (\nabla u_{j, hp} - \nabla u_j) \cdot \nabla W \, dx \\
&\quad + \sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\lambda_j u_j - \lambda_{j, hp} u_{j, hp}) W \, dx .
\end{aligned}$$

Using Lemma 4.11 and (4.45) we have

$$\begin{aligned}
&\sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\Delta u_{j, hp} + \lambda_{j, hp} u_{j, hp}) W \, dx \\
&\lesssim \left(\|u_j - u_{j, hp}\|_{E, \mathcal{T}} + \frac{h}{p} \|\lambda_{j, hp} u_{j, hp} - \lambda_j u_j\|_{0, \mathcal{T}} \right) \left(\sum_{F \in \mathcal{F}_I(\mathcal{T})} h_F^{-2} \|W\|_{0, K_F^+ \cup K_F^-}^2 \right)^{1/2} \\
&\lesssim \left(\|u_j - u_{j, hp}\|_{E, \mathcal{T}} + \frac{h}{p} \|\lambda_{j, hp} u_{j, hp} - \lambda_j u_j\|_{0, \mathcal{T}} \right) \left(\sum_{K \in \mathcal{T}_h} \eta_{j, F_K}^2 \right)^{1/2} .
\end{aligned}$$

Then, using the continuity and (4.46),

$$\begin{aligned}
\sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\nabla u_{j, hp} - \nabla u_j) \cdot \nabla W \, dx &\lesssim \|u_j - u_{j, hp}\|_{E, \mathcal{T}} \left(\sum_{F \in \mathcal{F}_I(\mathcal{T})} \|W\|_{E, K_F^+ \cup K_F^-}^2 \right)^{1/2} \\
&\lesssim \|u_j - u_{j, hp}\|_{E, \mathcal{T}} \left(\sum_{K \in \mathcal{T}_h} \eta_{j, F_K}^2 \right)^{1/2} .
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_{F \in \mathcal{F}_I(\mathcal{T})} \int_{K_F^+ \cup K_F^-} (\lambda_j u_j - \lambda_{j, hp} u_{j, hp}) W \, dx &\lesssim \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_{0, \Omega} \left(\sum_{F \in \mathcal{F}_I(\mathcal{T})} \|W\|_{0, K_F^+ \cup K_F^-}^2 \right)^{1/2} \\
&\lesssim \frac{h^{1/2}}{p^{1/2}} \|\lambda_j u_j - \lambda_{j, hp} u_{j, hp}\|_{0, \Omega} \left(\sum_{K \in \mathcal{T}_h} \eta_{j, F_K}^2 \right)^{1/2} .
\end{aligned}$$

The proof of the efficiency result Theorem 4.13 follows in a straightforward manner from Lemmas 4.10-4.12. ■

Theorem 4.13 *Let $(\lambda_{j,hp}, u_{j,hp})$ be a computed eigenpair of (2.10) converging to the true eigenvalue λ_j of multiplicity $R \geq 1$. Let also η_j be the error estimator for $(\lambda_{j,hp}, u_{j,hp})$, then we have the bound*

$$\eta_j \lesssim \text{dist}(u_{j,hp}, E_1(\lambda_j))_{E,\mathcal{T}} + \frac{h^{1/2}}{p^{1/2}} \|\lambda_{j,hp} u_{j,hp} - \lambda_j u_j\|_{0,\Omega} .$$

5 Numerical results

In this section we have collected numerical results regarding our *a posteriori* error estimator with the clear aim to show the reliability of the error estimator and the exponential converge of the error on the sequence of adapted meshes.

All the numerics in this section have been carried out using the AptoFEM package (www.aptofem.com) on a single processor desktop machine. In particular we used ARPACK [26] to compute the eigenvalues and MUMPS [27] to solve the linear systems.

The adaptive algorithm that we use is very simple: initially we choose the index j of the eigenvalue that we want to follow, then starting from a conforming coarse mesh we compute the eigenpair $(\lambda_{j,hp}, u_{j,hp})$ and the error estimator η_j . After this we mark elements for refinement using a simple fixed-fraction strategy based on values $\eta_{j,K}$; the choice between refining the marked elements in h or p is made by testing the local analyticity of the computed eigenfunction on the marked elements using the technique developed in [28]. Finally, a refined mesh is generated and the process restarted from the computation of $(\lambda_{j,hp}, u_{j,hp})$ on this refined mesh. The process is halted only when the value of η_j is smaller than a prescribed tolerance or when a maximum number of iterations have been carried out.

5.1 Unit square

The first example that we present, is problem (1.1) on the unit square $[0, 1]^2$. The initial mesh is a conforming structured mesh of 16 elements and the initial order of polynomials is 2. In Figure 1 we plot the true error for the first four eigenvalues against the number of degrees of freedom (DOFs). The solid lines represent the simulations using the error estimator η_j and the dotted lines represent the same simulations, but using the error estimator $\hat{\eta}_j$.

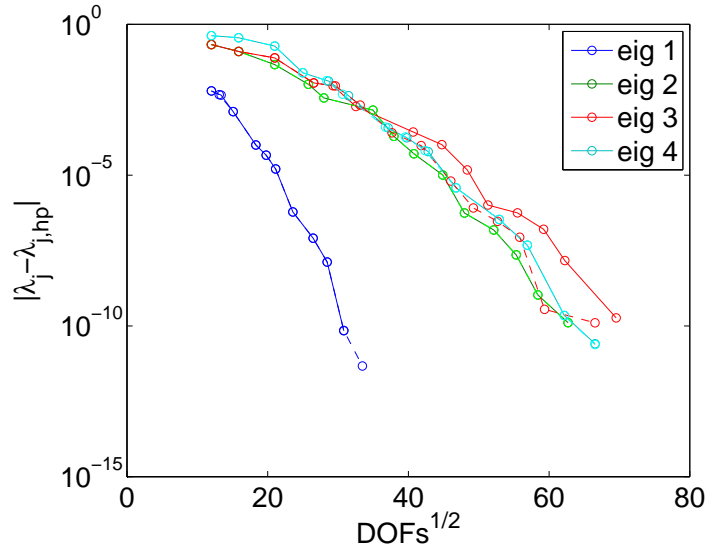


Figure 1: Convergence for the first eigenvalues on the unit square.

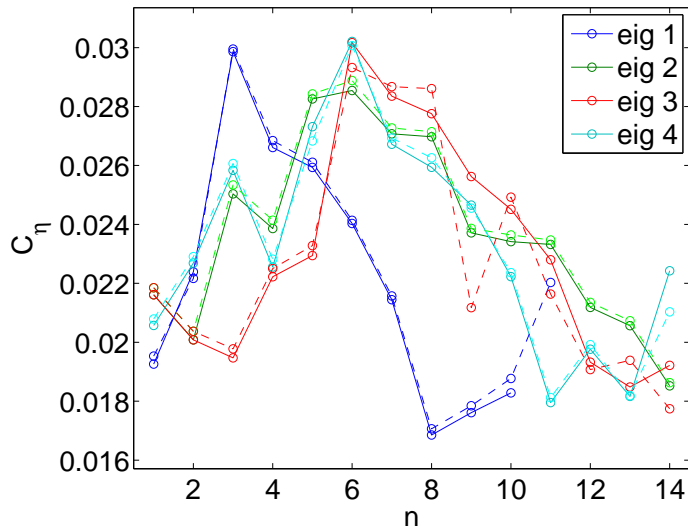


Figure 2: Values of the constant C_η for the first four eigenvalues.

As can be seen, both error estimators give very similar results and in both cases the plots are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this smooth problem.

Moreover, in Figure 2 we plot the computed values of the hidden constant C_η in Theorem 4.6, *i.e.*, $C_\eta = |\lambda_j - \lambda_{hp}|/\eta_j^2$. The solid lines represent the values of C_η for η_j and the dotted lines represent the values of C_η for $\hat{\eta}_j$. The fact that all the values of C_η are in a very small range, support the fact that both the error estimators η_j and $\hat{\eta}_j$ are reliable and efficient and that all the extra terms in the bound in Theorem 4.6 really are higher order terms. Also, the range of values for C_η seems independent of both the error estimator used and the index of the eigenvalue that has been considered. Just for comparison we plot in Figure 3 the convergence lines for the first eigenvalue using η_j either with h -adaptivity or hp -adaptivity. In Figure 4 we show the mesh generated by the hp -adaptivity using the error estimator η_j for the first eigenvalue and after 11 iterations of mesh refinements.

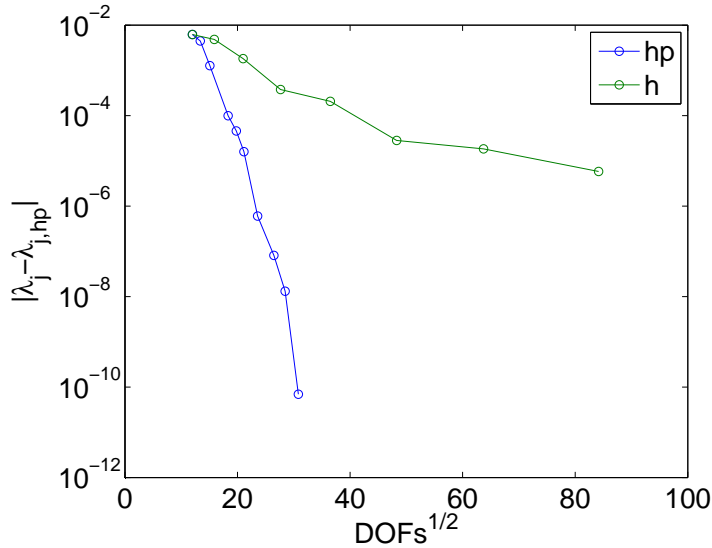


Figure 3: Comparison between h and hp adaptivity for the first eigenvalue on the unit square.

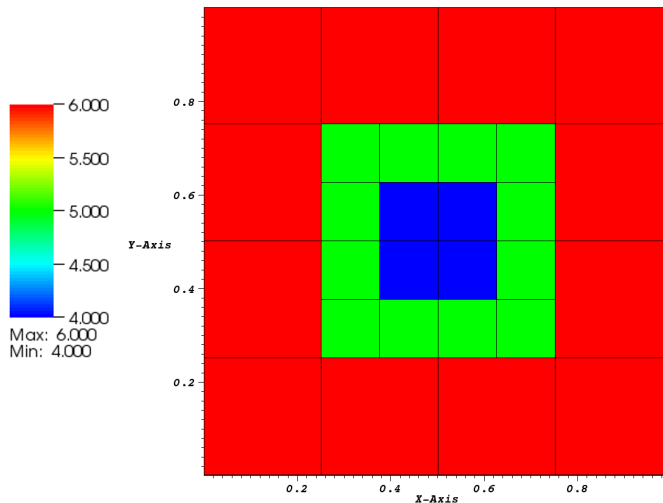


Figure 4: hp -adapted mesh obtained for the first eigenvalue on the unit square.

5.2 L-shaped domain

The second example is problem (1.1) on the L-shaped domain $\Omega = [0, 1]^2 / ([0.5, 1] \times [0, 0.5])$. This problem is of particular interest because it is not smooth due to the reentrant corner. The initial mesh is a conforming structured mesh of 12 elements and the initial order of polynomials is 2. In Figure 5 we plot the true error for the first four eigenvalues against the number of degrees of freedom. As before, the solid lines represent the simulations using the error estimator η_j and the dotted lines represent the same simulations, but using the error estimator $\hat{\eta}_j$. As can be seen, both error estimators give very similar results and in both cases the plots are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this non-smooth problem.

Moreover, in Figure 6 we plot the computed values of the hidden constant C_η in Theorem 4.6, in the same way as in the previous example. From the plots it is clear that we can also draw the same conclusions as previously. Just for comparison we plot in Figure 7 the convergence lines for the first

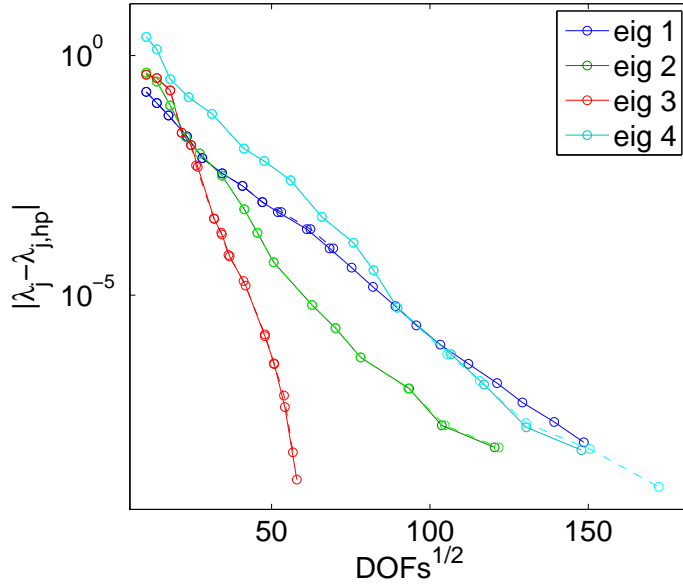


Figure 5: Convergence for the first eigenvalues on the L-shaped domain.

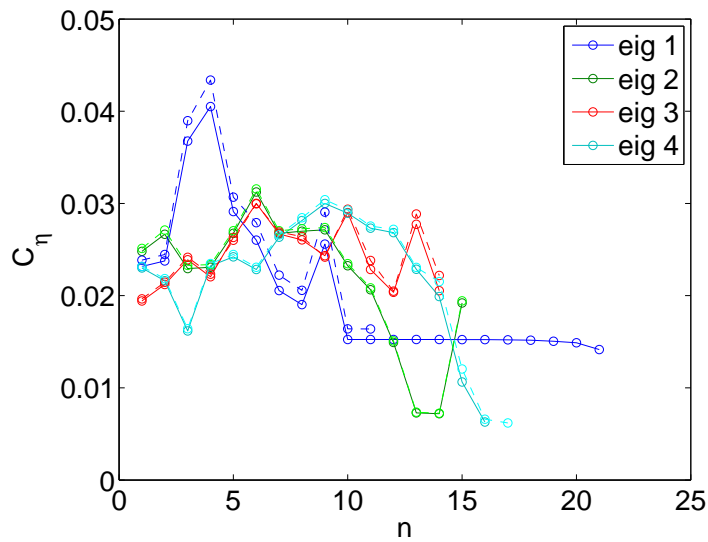


Figure 6: Values of the constant C_n for the first four eigenvalues.

eigenvalue using η_j either with h -adaptivity or hp -adaptivity. In Figure 8 we show the mesh generated by the hp -adaptivity using the error estimator η_j for the first eigenvalue and after 21 iterations of mesh refinements. Unsurprisingly the elements are very small around the reentrant corner, where the singularity sits and the orders of polynomials increase moving away from the singularity.

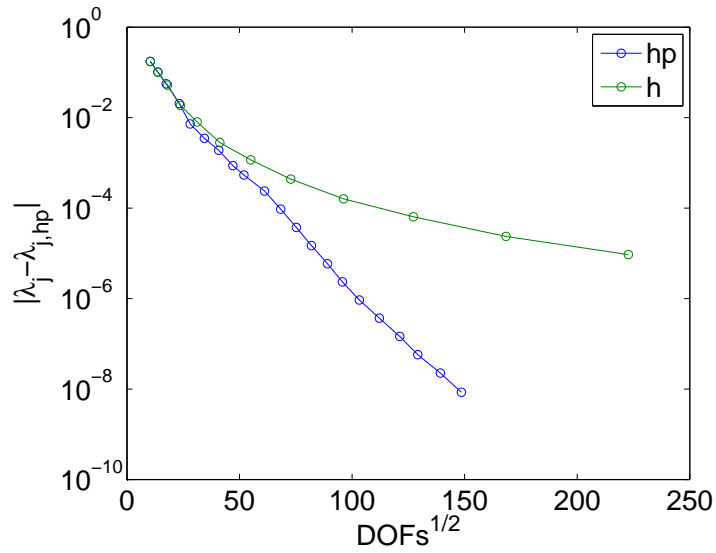


Figure 7: Comparison between h and hp adaptivity for the first eigenvalue on the L-shaped domain.

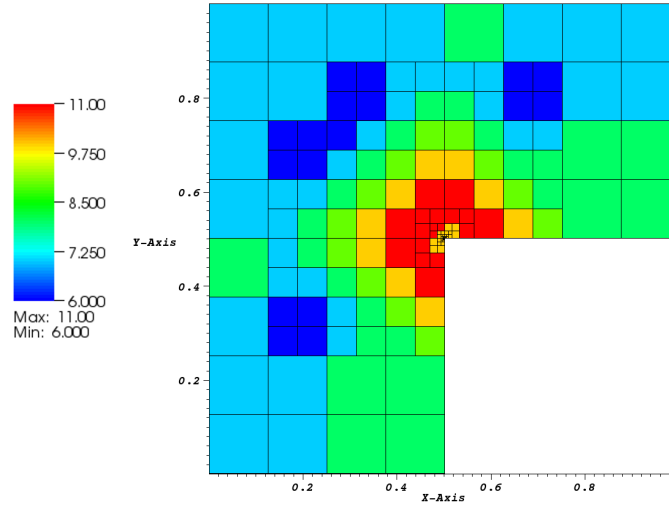


Figure 8: hp -adapted mesh obtained for the first eigenvalue on the L-shaped domain.

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