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Minimal Disconnected Cuts in Planar Graphs^{*}

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Abstract. The problem of finding a disconnected cut in a graph is NP-hard in general but polynomial-time solvable on planar graphs. The problem of finding a minimal disconnected cut is also NP-hard but its computational complexity is not known for planar graphs. We show that it is polynomial-time solvable on 3-connected planar graphs but NP-hard for 2-connected planar graphs. Our technique for the first result is based on a structural characterization of minimal disconnected cuts in 3-connected $K_{3,3}$ -free-minor graphs and on solving a topological minor problem in the dual. We show that the latter problem can be solved in polynomial-time even on general graphs. In addition we show that the problem of finding a minimal connected cut of size at least 3 is NP-hard for 2-connected apex graphs.

1 Introduction

A *cutset* or *cut* in a connected graph is a subset of its vertices whose removal disconnects the graph. The problem STABLE CUT is that of testing whether a connected graph has a cut that is an independent set. Le, Mosca, and Müller [12] proved that this problem is NP-complete even for K_4 -free planar graphs with maximum degree 5. A connected graph $G = (V, E)$ is *k-connected* for some integer k if $|V| \geq k + 1$ and every cut of G has size at least k . It is not hard to see that if one can solve STABLE CUT for 3-connected planar graphs in polynomial-time then one can do so for all planar graphs (in particular the problem is trivial

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if the graph has a cut-vertex or a cut set of two vertices that are non-adjacent). Hence, the problem is NP-complete for 3-connected planar graphs.

Due to the above it is a natural question whether one can relax the condition on the cut to be an independent set. This leads to the following notion. For a connected graph $G = (V, E)$, a subset $U \subseteq V$ is called a *disconnected cut* if U disconnects the graph and the subgraph induced by U is disconnected as well, that is, has at least two (connected) components. This problem is NP-complete in general [13] but polynomial-time solvable on planar graphs [8]. However, the property of the cut being disconnected can be viewed to be somewhat artificial if one considers the 4-vertex path $P_4 = p_1p_2p_3p_4$, which has two disconnected cuts, namely $\{p_1, p_3\}$ and $\{p_2, p_4\}$. Both these cuts contain a vertex, namely p_1 and p_4 , respectively, such that putting this vertex out of the cut and back into the graph keeps the graph disconnected. Therefore, Ito et al. [7] defined the notion of a *minimal disconnected cut* of a connected graph $G = (V, E)$, that is, a disconnected cut U so that $G[(V \setminus U) \cup \{u\}]$ is connected for every $u \in U$ (more generally, we call a cut that satisfies the later condition a *minimal cut*). Here, the graph $G[S]$ denotes the subgraph of G induced by $S \subseteq V(G)$. We note that every vertex of a minimal cut U of a connected graph $G = (V, E)$ is adjacent to every component of $G[V \setminus U]$. See Figure 1 for an example of a planar graph with a minimal disconnected cut.

The corresponding decision problem is defined as follows.

MINIMAL DISCONNECTED CUT

Instance: a connected graph G .

Question: does G have a minimal disconnected cut?

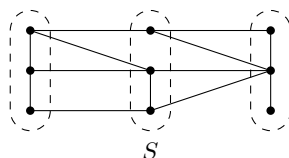


Fig. 1. An example of a planar graph with a minimal disconnected cut, namely the set S .

Ito et al. [7] showed that MINIMAL DISCONNECTED CUT is NP-complete. However its computational complexity remains open for planar graphs. It can be seen, via a straightforward reduction, that the problem of deciding whether a graph has a minimal stable cut is NP-complete for any graph class (and thus for the class of planar graphs) for which STABLE CUT is NP-complete. Moreover, the problem of deciding whether a graph has a minimal cut (that may be connected or disconnected) is polynomial-time solvable: given a vertex cut U we can remove vertices from U one by one until the remaining vertices in U form a minimal cut.

Our Results. As a start we observe that MINIMAL DISCONNECTED CUT is polynomial-time solvable for outerplanar graphs (as these graphs do not contain

$K_{2,3}$ as a minor, any minimal cut has size at most 2). In Section 2 we prove that MINIMAL DISCONNECTED CUT is also polynomial-time solvable on 3-connected planar graphs. The technique used by Ito et al. [8] for solving DISCONNECTED CUT in polynomial-time was based on the fact that a planar graph either has its treewidth bounded by some constant or else contains a large grid as a minor. However, grids (which are 3-connected planar graphs) do not have minimal disconnected cuts. Hence, we need to use a different approach, which we describe below.

We first provide a structural characterization of minimal disconnected cuts for the class of 3-connected $K_{3,3}$ -free-minor graphs, which contains the class of planar graphs. In particular we show that any minimal disconnected cut of a 3-connected planar graph G has exactly two components and that these components are paths. In order to find such a cut we prove that it suffices to test whether G contains, for some fixed integer r , the biclique $K_{2,r}$ as a contraction. We show that G has such a contraction if and only if its dual contains for some fixed r the multigraph D_r , which is obtained from the r -vertex cycle by replacing each edge by two edges, as a subdivision (see also Figure 2). We then present a characterization of any graph that contains such a subdivision. Next we use this characterization to prove that the corresponding decision problem is polynomial-time solvable even on general graphs.

In Section 3 we give our second result, namely that MINIMAL DISCONNECTED CUT stays NP-complete for the class of 2-connected planar graphs. This proof is based on a reduction from STABLE CUT and as such different from the NP-hardness proof for general graphs [7], the gadget of which contains large cliques. In the same section we show that the problem of finding a minimal *connected* cut of size at least 3 is NP-complete for 2-connected *apex graphs* (graphs that can be made planar by removing one vertex); to the best of our knowledge the computational complexity of this problem has not yet been determined even for general graphs. We note that the problem of finding whether a graph contains a (not necessarily minimal) connected cut of size at most k that separates two given vertices s and t is linear-time FPT when parameterized by k [14].

We finish our paper with some further observations on related problems in Section 4.

Related Work. Vertex cuts play an important role in graph connectivity. In the literature various kinds of vertex cuts, besides stable cuts, have been studied extensively and we briefly survey a number of results below.

A cut U of a graph $G = (V, E)$ is a clique cut if $G[U]$ is a clique, a k -clique cut if $G[U]$ has a spanning subgraph consisting of k complete graphs; a strict k -clique cut if $G[U]$ consists of k components that are complete graphs; and a matching cut if $E_{G[U]}$ is a matching. It follows from a classical result of Tarjan [17] that determining whether a graph has a clique cut is polynomial-time solvable. Whitesides [18] and Cameron et al. [3] proved that the problem of testing whether a graph has a k -clique cut is solvable in polynomial time for $k = 1$ and $k = 2$, respectively. Cameron et al. [3] also proved that testing whether a graph has a strict 2-clique cut can be solved in polynomial time. As mentioned the problem

of testing whether a graph has a stable cut is NP-complete. This was first shown for general graphs by Chvátal [4]. Also the problem of testing whether a graph has a matching cut is NP-complete. This was shown by Brandstädt et al. [2]. Bonsma [1] proved that this problem is NP-complete even for planar graphs with girth 5 and for planar graphs with maximum degree 4.

The SKEW PARTITION problem is that of testing whether a graph $G = (V, E)$ has a disconnected cut U so that $V \setminus U$ induced a disconnected graph in the complement of G . De Figueiredo, Klein and Reed [5] proved that even the list version of this problem, where each vertex has been assigned a list of blocks in which it must be placed, is polynomial-time solvable. Afterwards, Kennedy and Reed [11] gave a faster polynomial-time algorithm for the non-list version.

Finally, for an integer $k \geq 1$, a cut U of a connected graph G is a k -cut of G if $G[U]$ contains exactly k components. For $k \geq 1$ and $\ell \geq 2$, a k -cut U is a (k, ℓ) -cut of a graph G if $G[V \setminus U]$ consists of exactly ℓ components. Ito et al. [8] proved that testing if a graph has a k -cut is solvable in polynomial time for $k = 1$ and NP-complete for every fixed $k \geq 2$. In addition they showed that testing if a graph has a (k, ℓ) -cut is polynomial-time solvable if $k = 1$, $\ell \geq 2$ and NP-complete otherwise [8]. The same authors show, by using the approach for solving DISCONNECTED CUT on planar graphs, that both problems are polynomial-time solvable on planar graphs.

Terminology. Let $G = (V, E)$ be a connected simple graph. A maximal connected subgraph of G is called a *component* of G . Recall that, for a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G induced by S , which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. A vertex $u \in V \setminus S$ is *adjacent* to a set $S \subseteq V \setminus \{u\}$ if u is adjacent to a vertex in S . We say that two disjoint sets $S \subset V$ and $T \subset V$ are *adjacent* if S contains a vertex adjacent to T , or equivalently if T contains a vertex adjacent to S .

Let G be a graph. We define the following operations. The *contraction* of an edge uv removes u and v from G , and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to u or v in G . Unless we explicitly say otherwise we remove all self-loops and multiple edges so that the resulting graph stays simple. The *subdivision* of an edge uv replaces uv by a new vertex w with edges uw and vw . Let $u \in V(G)$ be a vertex that has exactly two neighbours v, w , and moreover let v and w be non-adjacent. The *vertex dissolution* of u removes u and adds the edge vw .

A graph G contains a graph H as a *minor* if H can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. We say that G contains H as a *contraction*, denoted by $H \leq_c G$, if H can be obtained from G by a sequence of edge contractions. Finally, G contains H as a *subdivision* if H can be obtained from G by a sequence of vertex deletions, edge deletions and vertex dissolutions, or equivalently, if G contains a subgraph H' that is a *subdivision* of H , that is, H can be obtained from H' after applying zero or more vertex dissolutions. We say that a vertex in H' is a *subdivision vertex* if we need to dissolve it in order to obtain H ; otherwise it is called a *branch vertex* (that is, it corresponds to a vertex of H).

For some of our proofs the following global structure is useful. Let G and H be two graphs. An H -witness structure \mathcal{W} is a vertex partition of a (not necessarily proper) subgraph of G into $|V(H)|$ nonempty sets $W(x)$ called (H -witness) bags, such that

- (i) each $W(x)$ induces a connected subgraph of G ,
- (ii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G if x and y are adjacent in H .

In addition, we may require the following additional conditions:

- (iii) for all $x, y \in V(H)$ with $x \neq y$, bags $W(x)$ and $W(y)$ are adjacent in G *only* if x and y are adjacent in H ,
- (iv) every vertex of G belongs to some bag.

By contracting all bags to singletons we observe that H is a minor or contraction of G if and only if G has an H -witness structure such that conditions (i)-(ii) or (i)-(iv) hold, respectively. We note that G may have more than one H -witness structure with respect to the same containment relation.

We denote the complete graph on k vertices by K_k and the complete bipartite graph with bipartition classes of size k and ℓ , respectively, by $K_{k,\ell}$.

2 The Algorithm

We first present a necessary and sufficient condition for a 3-connected $K_{3,3}$ -minor-free graph to have a minimal disconnected cut.

Theorem 1. *A 3-connected $K_{3,3}$ -minor-free graph G has a minimal disconnected cut if and only if $K_{2,r} \leq_c G$ for some $r \geq 2$.*

Proof. Let $G = (V, E)$ be a 3-connected graph that has no $K_{3,3}$ as a minor. First suppose that G has a minimal disconnected cut U . Let p and q be the number of components of $G[U]$ and $G[V \setminus U]$, respectively. Because U is a disconnected cut, $p \geq 2$ and $q \geq 2$. By definition, every vertex of every component of $G[U]$ is adjacent to all components in $G[V \setminus U]$. Hence, G contains $K_{p,q}$ as a contraction. Because G has no $K_{3,3}$ as a minor, G has no $K_{3,3}$ as a contraction. This means that $p \leq 2$ or $q \leq 2$. Because $p \geq 2$ and $q \geq 2$ holds as well, we find that $K_{2,r} \leq_c G$ for some $r \geq 2$.

Now suppose that $K_{2,r} \leq_c G$ for some $r \geq 2$. Throughout the remainder of the proof we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_\ell\}$. We refer to the bags in a $K_{k,\ell}$ -witness structure of G corresponding to the vertices in X and Y as x -bags and y -bags, respectively. Because $K_{2,r} \leq_c G$, there exists a $K_{2,r}$ -witness structure \mathcal{W} of G that satisfies conditions (i)-(iv). Note that $W(x_1) \cup W(x_2)$ is a disconnected cut. However, it may not be minimal.

Suppose that $W(x_1)$ contains a vertex u that is adjacent to some but not all y -bags, i.e., the number of y -bags to which u is adjacent is h for some $1 \leq h < r$. Then we move u to a y -bag that contains one of its neighbors unless $W(x_1) \cup W(x_2)$

no longer induce a disconnected graph (which will be the case if u is the only vertex in $W(x_1)$). We observe that $G[W(x_1) \setminus \{u\}]$ may be disconnected, namely when u is a cut vertex in $G[W(x_1)]$. We also observe that u together with its adjacent y -bags induces a connected subgraph of G . Hence, the resulting witness structure \mathcal{W}' is a $K_{q,r'}$ -witness structure of G with $q \geq 2$ (as the resulting vertices in $W(x_1) \cup W(x_2)$ still induce a disconnected graph) and $r' = r - (h - 1)$. Because $1 \leq h < r$, we find that $2 \leq r' \leq r$. We repeat this rule as long as possible. During this process, $W(x_2)$ does not change, and afterwards, we do the same for $W(x_2)$. Let \mathcal{W}^* denote the resulting witness structure that is a K_{q^*,r^*} -witness structure satisfying conditions (i)-(iv) for some $q^* \geq 2$ and $2 \leq r^* \leq r$.

We will now prove the following claim. Afterwards, we are done; due to this claim and because there are at least two x -bags and at least two y -bags in \mathcal{W}^* , the x -bags of \mathcal{W}^* form a minimal disconnected cut U of G .

Claim 1. Every vertex of each x -bag of \mathcal{W}^ is adjacent to all y -bags.*

We prove Claim 1 as follows. First suppose that there exists an x -bag of \mathcal{W}^* , say $W^*(x_1)$, that contains a vertex u adjacent to some but not to all y -bags of \mathcal{W}^* , say u is not adjacent to $W^*(y_1)$. By our procedure we would have moved u to an adjacent y -bag unless that makes the disconnected cut connected. Hence we find that there are exactly two witness bags $W^*(x_1)$ and $W^*(x_2)$ and that $W^*(x_1) = \{u\}$. In our procedure we only moved vertices from x -bags to y -bags. This means that u belonged to an x -bag of the original witness structure \mathcal{W} . This x -bag was adjacent to all y -bags of \mathcal{W} (as \mathcal{W} was a $K_{2,r}$ -witness structure). As we only moved vertices from x -bags to y -bags, this means that there must still exist a path from u to a vertex in $W^*(y_1)$ that does not use any vertex of $W^*(x_2)$; a contradiction. Hence every x -bag of \mathcal{W}^* only contains vertices that are either adjacent to all y -bags or to none of them.

Now, in order to obtain a contradiction, suppose that an x -bag, say $W^*(x_1)$, contains a vertex u not adjacent to any y -bag. Because G is 3-connected, G contains three vertex-disjoint paths P_1, P_2, P_3 from u to a vertex in $W^*(y_1)$ (by Menger's Theorem). Each P_i contains a vertex v_i in $W^*(x_1)$ whose successor on P_i is outside $W^*(x_1)$. Hence, by our assumption, v_i has a neighbour in every y -bag (including $W^*(y_1)$). Recall that the number of y -bags is $r^* \geq 2$. Then the subgraph induced by the vertices from $W^*(y_1)$ and $W^*(y_2)$ together with the vertices on the three paths P_1, P_2, P_3 form a $K_{3,3}$ -minor of G . This is not possible. Hence, every vertex of each x -bag of \mathcal{W}^* is adjacent to all y -bags. This completes the proof of Claim 1 and thus the proof of Theorem 1. \square

By Theorem 1 we may restrict ourselves to finding a $K_{2,r}$ -contraction for some $r \geq 2$ in a 3-connected planar graph. Below we state some additional terminology.

Recall that D_n is the multigraph obtained from the cycle on $n \geq 3$ vertices by doubling its edges. We let D_2 be the multigraph that has two vertices with four edges between them. The *dual* graph G_d of a plane graph G has a vertex for each face of G , and there exist k edges between two vertices u and v in G_d if and only if the two corresponding faces share k edges in G . Note that the dual of a graph may be a multigraph. As 3-connected planar graphs have a unique embedding

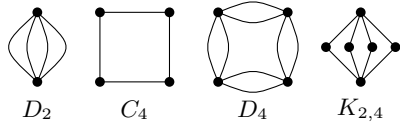


Fig. 2. The graphs D_2 , C_4 , D_2 , $K_{2,4}$. Note that the dual of $C_4 = K_{2,2}$ is D_2 , that D_4 is obtained from C_4 by duplicating each edge and that D_4 is the dual of $K_{2,4}$.

(see e.g. Lemma 2.5.1, p.39 of [16]) we can speak of the dual of a 3-connected planar graph. We need the following lemma. Its proof, which we omit, follows from using a result from [9].

Lemma 1. *Let G be a 3-connected planar graph. Then G contains $K_{2,r}$ as a contraction for some $r \geq 2$ if and only if the dual of G contains D_r as a subdivision.*

By Lemma 1 it suffices to check if the dual of the 3-connected planar input graph contains D_r as a subdivision for some $r \geq 2$. We show how to solve this problem in polynomial time for general graphs. In order to do so we need the next lemma which gives a necessary condition for a graph G to be a yes-instance of this problem. In its proof we use the following notation. For a path $P = v_1v_2 \dots v_p$, we write v_iPv_j to denote the subpath $v_iv_{i+1} \dots v_j$ or v_jPv_i if we want to emphasize that the subpath is to be traversed from v_j to v_i .

Lemma 2. *Let v, w be two distinct vertices of a multigraph G such that there exist four edge-disjoint v - w -paths in G . Then G contains a subdivision of D_r for some $r \geq 2$.*

Proof. We prove the lemma by induction on $|V(G)| + |E(G)|$. Then we can assume that G is the union of the four edge-disjoint v - w -paths. Let us call these paths P_1, P_2, P_3 , and P_4 . If these four paths are vertex-disjoint (apart from v and w) then they form a subdivision of D_2 . Hence, we may assume that there exists at least one vertex of G not equal to v or w that belongs to more than one of the four paths.

First suppose that there exists a vertex u that belongs to all four paths P_1, P_2, P_3 and P_4 . Let G' be the graph consisting of the vertices and edges of the four subpaths vP_1u, vP_2u, vP_3u and vP_4u . As G' does not contain w , it holds that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, G' , and thus G , contains a subdivision of D_r for some $r \geq 2$.

Now suppose that there exists a vertex u that belong to only three of the four paths, say to P_1, P_2 , and P_3 . Let G' be the graph that consists of the vertices and edges of the four paths uP_1w, uP_2w, uP_3w and uP_1vP_4w . As G' does not contain an edge of vP_2u we find that $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. By the induction hypothesis, G' , and thus G , contains a subdivision of D_r for some $r \geq 2$.

From now on assume that every inner vertex of every path P_i ($i = 1, \dots, 4$) belongs to at most one other path P_j ($j \neq i$). We say that two different paths

P_i and P_j cross in a vertex u if u is an inner vertex of both P_i and P_j . Suppose P_i and P_j cross in some other vertex u' as well. Then we say that u is crossed before u' by P_i and P_j if u is an inner vertex of both vP_iu' and vP_ju' .

We now prove the following claim.

Claim 1. If P_i and P_j ($i \neq j$) cross in both u and u' then we may assume without loss of generality that either u is crossed before u' or u' is crossed before u .

We prove Claim 1 as follows. Suppose that u is not crossed before u' by P_i and P_j and similarly that u' is not crossed before u by P_i and P_j . Then we may assume without loss of generality that u is an inner vertex of vP_iu' and that u' is an inner vertex of vP_ju . See Figure 3 for an example of this situation. However, in that case we can replace P_i and P_j by the paths vP_iuP_jw and $vP_ju'P_iw$. These two paths together with the two unused original paths form a subgraph G' of G with fewer edges than G (as for instance no edge on uP_iu' belongs to G'). We apply the induction hypothesis on G' . This completes the proof of Claim 1.

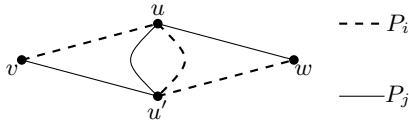


Fig. 3. The paths P_i and P_j where u is not crossed before u' by P_i and P_j and similarly u' is not crossed before u by P_j and P_i . Note that the paths P_i and P_j may have more common vertices, but for clarity this is not been shown.

We need Claim 1 to prove the following claim, which is crucial for our proof.

Claim 2. We may assume without loss of generality that there exists a vertex $u \notin \{v, w\}$ that is on two paths P_i and P_j ($i \neq j$) so that every inner vertex of vP_iu and vP_ju has degree 2 in G .

We prove Claim 2 as follows. By our assumption there exists at least one vertex in G that is on two paths. Let $s \notin \{v, w\}$ be such a vertex, say s belongs to P_1 and P_2 . Assume without loss of generality that every inner vertex of vP_1s has degree 2. Then, by Claim 1, we find that P_1 and P_2 do not cross in an inner vertex of vP_2s .

If every inner vertex of vP_1s and vP_2s has degree 2 in G then the claim has been proven. Suppose otherwise, namely that there exists an inner vertex s' of vP_1s or vP_2s whose degree in G is larger than 2, say s' belongs to vP_2s . As P_1 does not cross vP_2s , we find that s' must belong to P_3 or to P_4 . Choose s' in such a way that every inner vertex of vP_2s' has degree 2 in G . Assume without loss of generality that s' belongs to P_3 .

If every inner vertex of vP_3s' has degree 2 then the claim has been proven (as every inner vertex of vP_2s' has degree 2 as well). Suppose otherwise, namely that there exists an inner vertex s'' of vP_3s' whose degree in G is larger than 2.

Choose s'' in such a way that every inner vertex of vP_3s'' has degree 2 in G . By Claim 1, no inner vertex of vP_3s' belongs to P_2 , so s'' does not lie on P_2 . This means that s'' belongs either to P_1 or to P_4 .

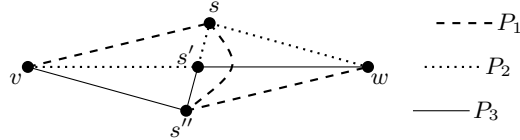


Fig. 4. The paths P_1 , P_2 and P_3 where s belongs to P_1 and P_2 , s' belongs to vP_2s and P_3 and s'' belongs to vP_3s' and P_1 .

Suppose s'' belongs to P_1 . See Figure 4 for an example of this situation. As every inner vertex of vP_1s has degree 2, we find that s is an inner vertex of vP_1s'' . However, we can now replace P_1 , P_2 and P_3 by the three paths vP_1sP_2w , $vP_2s'P_3w$ and $vP_3s''P_1w$. These three paths form, together with P_4 , a subgraph of G with fewer edges than G (for instance, no edge of sP_1s'' belongs to G'). We can apply the induction hypothesis on this subgraph. Hence we may assume that s'' does not belong to P_1 .

From the above we conclude that s'' belongs to P_4 . See Figure 5 for an example of this situation. We consider the paths vP_3s'' and vP_4s'' . If every inner vertex of vP_4s'' has degree 2 in G then we have proven Claim 2 (recall that every inner vertex of vP_3s'' has degree 2 in G as well). Suppose otherwise, namely that there exists an inner vertex t of vP_4s'' whose degree in G is larger than 2. Choose t in such a way that every inner vertex of vP_4t has degree 2 in G . By Claim 1 we find that t is not on P_3 . If t is on P_2 we can use a similar replacement of three paths by three new paths as before that enables us to apply the induction hypothesis. Hence, we find that t belongs to P_1 .

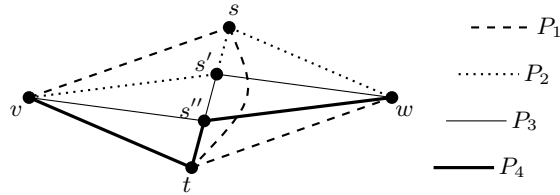


Fig. 5. The paths P_1 , P_2 , P_3 and P_4 where s belongs to P_1 and P_2 , s' belongs to vP_2s and P_3 , s'' belongs to vP_3s' and P_4 and t belongs to vP_4s'' and P_1 .

As every inner vertex of vP_1s has degree 2 in G we find that s is an inner vertex of vP_1t . Then we take the four paths vP_1sP_2w , $vP_2s'P_3w$, $vP_3s''P_4w$ and vP_4tP_1w . These four paths form a subgraph G' of G with fewer edges than G

(as for instance G' contains no edge from sP_1t). We can apply the induction hypothesis on G' . Hence we may assume that such a vertex t cannot exist. Thus we have found the desired vertex and subpaths, namely s'' with subpaths vP_3s'' and vP_4s'' . This completes the proof of Claim 2.

By Claim 2 we may assume without loss of generality that there exists a vertex u that belongs to P_1 and P_2 such that every inner vertex of vP_1u and vP_2u has degree 2. Let G^* be the graph obtained from G by contracting all edges of vP_1u and vP_2u (recall that we remove loops and multiple edges). Let u^* be the new vertex to which all the edges were contracted. Notice that there are four edge-disjoint u^* - w -paths in G^* . Then, by the induction hypothesis, G^* contains a subdivision H of D_r for some $r \geq 2$. If u^* does not belong to H , then G contains H as well and we would have proven the lemma. Assume that u^* belongs to H .

First suppose that u^* is a subdivision vertex of H in G^* . Let u^* have neighbours s_1 and s_2 in H . Take a shortest path Q from s_1 to s_2 in the subgraph of G induced by s_1 , s_2 and the vertices of vP_1u and vP_2u . This results in a graph H' , which is a subgraph of G and which is a subdivision of D_r as well.

Now suppose that u^* is a branch vertex of H in G^* , say u^* corresponds to $z \in V(D_r)$. Note that any vertex in D_r has one neighbour if $r = 2$ and two neighbours if $r \geq 3$. We let s and t be the two branch vertices of H that correspond to the neighbours of z in D_r (note that $s = t$ if $r = 2$). Let s_1 and s_2 be the neighbours of u^* on the two paths from u^* to s , respectively, in H . Similarly, let t_1 and t_2 be the neighbours of u^* on the two paths from u^* to t , respectively, in H . Note that, as G is a multigraph, it is possible that $s_1 = s_2 = s$ and $t_1 = t_2 = t$.

Recall that every internal vertex on vP_1u and on vP_2u has degree 2 in G . As u is an inner vertex of P_1 and P_2 but not of P_3 and P_4 , it has degree 4 in G . As G is the union of P_1 , P_2 , P_3 and P_4 , we find that v has degree 4 as well. Then, after uncontracting u^* , we have without loss of generality one of the following two situations in G . First, u is adjacent to s_1 and s_2 and v is adjacent to t_1 and t_2 . In that case u and v become branch vertices of a subdivision of D_{r+1} in G (to which the internal vertices on the paths uP_1v and uP_2v belong as well, namely as subdivision vertices). Second, u is adjacent to s_1 and t_1 , whereas v is adjacent to s_2 and t_2 . Then u and v become subdivision vertices of a subdivision of D_r in G (and we do not use the internal vertices on the paths uP_1v and uP_2v). This completes the proof of the lemma. \square

Lemma 2 gives us the following result.

Theorem 2. *It is possible to find in $O(mn^2)$ time whether a graph G with n vertices and m edges contains D_r as a subdivision for some $r \geq 2$.*

Proof. Let G be a graph with m edges. We check for every pair of vertices s and t whether G contains four edge-disjoint paths between them. We can do this via a standard reduction to the maximum flow problem. Replace each edge uv by the arcs (u, v) and (v, u) . Give each arc capacity 1. Introduce a new vertex s' and an arc (s', s) with capacity 4. Also introduce a new vertex t' and an arc

(t, t') with capacity 4. Check if there exists an (s', t') -flow of value 4 by using the Ford-Fulkerson algorithm. As the maximum value of an (s', t') -flow is at most 4, this costs $O(m)$ time per pair, so $O(mn^2)$ time in total.

If there exists a pair s, t in G with four edge-disjoint paths between them then G has a subdivision of D_r , for some $r \geq 2$, by Lemma 2. If not then we find that G has no subdivision of any D_r ($r \geq 2$) as any subdivision of D_r immediately yields four edge-disjoint paths between two vertices and our algorithm would have detected this. \square

We are now ready to state our main result.

Theorem 3. *MINIMAL DISCONNECTED CUT can be solved in $O(n^3)$ time on 3-connected planar graphs with n vertices.*

Proof. Let G be a 3-connected planar graph with n vertices. By Theorem 1 it suffices to check whether $K_{2,r} \leq_c G$ for some $r \geq 2$. By Lemma 1, the latter is equivalent to checking whether the dual of G , which we denote by G^* , contains D_r as a subdivision for some $r \geq 2$. To find G^* we first embed G in the plane using the linear-time algorithm from Mohar [15]. As the number of edges in a planar graph is linear in the number of vertices, G^* has $O(n)$ vertices and $O(n)$ edges and can be constructed in $O(n)$ time. We are left to apply Theorem 2. \square

3 Hardness

We show the following result, which complements Theorem 3. We omit its proof.

Theorem 4. *MINIMAL DISCONNECTED CUT is NP-complete for the class of 2-connected planar graphs.*

A cut S in a graph G is a *minimal connected* cut if $G[S]$ is connected and for all $u \in S$ we have that $G[(V \setminus S) \cup \{u\}]$ is connected. We call the problem of testing whether a graph has a minimal connected cut of size at least k the *MINIMAL CONNECTED CUT(k)* problem. By modifying the proof of Theorem 4 we obtain the following result (proof omitted).

Theorem 5. *MINIMAL CONNECTED CUT(3) is NP-complete even for the class of 2-connected apex graphs.*

We cannot use the reduction in the proof of Theorem 5 to get NP-hardness for *MINIMAL CONNECTED CUT(1)*, the reason being that the gadget graph constructed in our omitted proof contains minimal disconnected cuts of size 2.

4 Conclusions

We proved that *MINIMAL DISCONNECTED CUT* is NP-complete for 2-connected planar graphs and polynomial-time solvable for planar graphs that are 3-connected. Our proof technique for the latter result was based on translating the problem to

a dual problem, namely the existence of a subdivision of D_r for *some* r , for which we obtained a polynomial-time algorithm even for general graphs. One can also solve the problem of determining whether a graph contains D_r as a subdivision for some *fixed* integer r by using the algorithm of Grohe, Kawarabayashi, Marx, and Wollan [6] which tests in cubic time, for any fixed graph H , whether a graph contains H as a subdivision. However, when r is part of the input we can show the following result via a reduction from HAMILTON CYCLE (proof omitted).

Theorem 6. *The problem of deciding whether a graph contains the graph D_r as a subdivision is NP-complete if r is part of the input.*

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