Ribbon curling

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The procedure of curling a ribbon by running it over a sharp blade is commonly used when wrapping presents. Despite its ubiquity, a quantitative explanation of this everyday phenomenon is still lacking. We address this using experiment and theory, examining the dependence of ribbon curvature on blade curvature, the longitudinal load imposed on the ribbon and the speed of pulling. Experiments in which a ribbon is drawn steadily over a blade under a fixed load show that the ribbon curvature is generated over a restricted range of loads, the curvature/load relationship can be non-monotonic, and faster pulling (under a constant imposed load) results in less tightly curled ribbons. We develop a theoretical model that captures these features, building on the concept that the ribbon under the imposed deformation undergoes differential plastic stretching across its thickness, resulting in a permanently curved shape. The model identifies factors that optimize curling and clarifies the physical mechanisms underlying the ribbon’s non-linear response to an apparently simple deformation.

elasticity | plasticity | mechanics | stress relaxation

Abbreviations: None

Classification: Physical sciences; applied physical sciences

Significance statement: The forming of thin film structures through intrinsic stress relaxation is an important design tool to create flexible shapes such as rolls, spirals and origamis from the macro- to the nano-scale. We exploit the everyday process inspired from gift wrapping, of curling an initially straight ribbon by pulling it over a blade, to probe the mechanical shear response of thin polymer sheets. Experiments show that curling occurs over a limited range of loads applied to the ribbon, with the curl radius reaching a maximum at intermediate loads. A theoretical model reveals several patterns of irreversible yielding across the ribbon, and the dependence of curl radius on pulling speed shows that the stress relaxes dynamically as the ribbon passes over the blade.

Results

Experimental results. A schematic side-view diagram of the experimental apparatus is shown in Fig. 1A. A polymer ribbon [10] of thickness $H^* = 100 \pm 3$ µm and width $W^* = 10.0 \pm 0.3$ mm, made from PVC transparency film, is pulled steadily over a blade at a prescribed rate (by attaching the ribbon to a rotating drum) and under a prescribed load (provided by a weight attached to one end). We used machined blades

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with radius of curvature $R^*$ = 1, 1.5 and 2 mm, and a fourth ‘sharp’ blade with a much higher curvature ($R^* < 0.1$ mm). Pulling imprints a permanent curvature on the ribbon, which is measured after the ribbon is removed from the blade (Fig. 1C) (see SI Appendix). Images of the experimental ribbon configurations over each blade are shown in Fig. 1B for loads in the range $50 \leq m^* \leq 1530$ g. In each case, the ribbon’s resistance to bending led to slightly different ribbon geometries over the blade for increasing loads. This was most significant for the sharp blade where the ribbon configuration varied significantly over the entire range of loads, and could not conform to the radius of curvature of the blade. Measurable changes in configuration were also found for the $R^* = 1$ mm blade over this range of loads; for $R^* = 1.5$ mm and $R^* = 2$ mm, the ribbon approximately adopted the curvature of the blade at $m^* = 1030$ g and $m^* = 510$ g, respectively, so that the geometry of the ribbon remained unchanged for higher loads.

Dimensional permanent curvature measurements are shown using symbols in Fig. 2A, as a function of axial load applied to the ribbon for each of the four blades. The data corresponding to experiments with the sharp and $R^* = 1$ mm blades exhibit a characteristic triangular shape, where curvature increases approximately linearly with increasing load to a maximum in curvature that is larger the sharper the blade. The curvature then decreases monotonically upon further increase of the load. In both cases, the maximum load applied was determined by the smallest value of curvature that could be reliably measured at high loads. In contrast, for $R^* = 1.5$ mm and 2 mm, the curvature increased monotonically up to a maximum load, beyond which the ribbon ruptured. For $R^* = 1.5$ mm, the curvature appears to have reached a maximum, whereas for $R^* = 2$ mm, only a small increase in curvature could be observed before the load exceeded its threshold value for rupture. The threshold load was found to increase as the curvature of the blade was reduced (Fig. 2A). The four sets of experimental data also suggest that a critical load needs to be exceeded in order for the ribbon to curl, and this critical load increases significantly with reduction in blade curvature. Hence, the modest loads required to bend the ribbon over the sharp and 1 mm blades meant that the ribbon geometry varied with load over the entire range investigated (Fig. 1B), whereas the larger loads required to bend the ribbon over the $R^* = 1.5$ mm and 2 mm blades exceeded the values at which the ribbon geometry reached a constant configuration. The effect on ribbon curling of the pulling speed was investigated for $R^* = 1$ mm and two applied loads ($m^* = 960$ and 689 g). Experimental data shown with symbols in Fig. 2B indicate that the curvature of the ribbon curl decreases monotonically with linear pulling speed.

In order to inform comparison with the theoretical model, the material properties of the PVC ribbon were measured with uniaxial tensile tests performed using an Instron 3345 (L2957) universal testing system. The Young’s modulus $E^*$ was determined by linear least-square fit of average stress-strain curves measured in the elastic regime to take a value of $E^* = 2.5 \pm 0.4$ GPa. The viscoplastic behaviour of the material was investigated with creep experiments, where stress was applied to six different ribbon samples in successive step changes of variable magnitude. The average rate of plastic strain creep was determined with a linear fit to the time variation of strain data following each step change in applied stress. The average strain rate is shown in Fig. 2C as a function of applied stress, where each symbol indicates experiments performed on an individual ribbon sample. The strain rate is approximately zero below a critical yield stress, and increases approximately
Comparison between experimental measurements and theoretical predictions. (A) Symbols show experimental measurements of the permanent curvature of the curled ribbon as a function of stress applied to the end of the ribbon (see Fig. 1A), for the four blades: sharp blade (green diamond), $R^* = 1$ mm (blue circle), $R^* = 1.5$ mm (red square) and $R^* = 2$ mm (black triangle). In all cases, curling is observed above a threshold value of the applied load which depends of the radius of curvature of the blade. For the sharp blade, $R^* = 1$ mm and $R^* = 1.5$ mm, the curvature increases approximately linearly above the threshold load to a maximum value that decreases as $R^*$ is increased. The curvature then decreases approximately linearly with applied load, and for $R^* = 1$ mm, the minimum values of curvature measurable experimentally are recovered. The vertical dotted lines correspond to the typical loads at which the ribbon ruptured on the blade. For $R^* = 1.5$ mm, the rupture load is very close to the maximum of curvature, so the decrease of curvature with load cannot be observed. For $R^* = 2$ mm, the curvature does not show evidence of a maximum below the rupture load. The pulling speed is $V^* = 4.9$ mm/s. Lines show model predictions using parameters $V^* = 4.9$ mm/s and $E^* = 2.5$ GPa. Values of the yield stress and plastic relaxation time were adjusted to $Y^* = 28$, $30$, $31$, $40$ MPa and $t_p^* = 0.15$, $0.18$, $1.0$, $0.9$ s for the sharp, 1mm, 1.5mm and 2mm blades respectively. The angles at which the ribbon is pulled are similar to those shown in Fig. 1B. (B) Symbols show experimentally measured curvature as a function of pulling speed $V^*$ for two different weights: $m^* = 600$ g (blue circles) and $m^* = 960$ g (cyan diamonds), using the blade with radius $R^* = 1$ mm. The curvature decreases approximately by a factor of three over the range of speeds investigated. Lines show model predictions using parameters for the 1mm blade. (C) Average strain rate (symbols) measured as a function of applied stress during uniaxial tensile creep tests. A linear fit to the growing part of the curve (solid line) is extrapolated to zero strain rate to determine the yield stress $Y^* = 39$ MPa. Each type of symbol denotes a series of measurements performed on an individual ribbon sample. In each of these experiments, the imposed stress was incremented from zero in steps, and the strain was allowed to creep upwards at each step until it reached an approximately constant value. An average strain rate was estimated at each step by a linear fit to the data.

Physical interpretation. Figure 3 illustrates the mechanism that we propose to explain ribbon curling. The ribbon’s complex constitutive properties are idealised by treating it as an isotropic elastic-viscoplastic material with a yield stress $Y^*$ and a stress-relaxation timescale $t_p^*$, such that the material behaves elastically over timescales much less than $t_p^*$, but stresses in excess of $Y^*$ relax over a timescale $t_p^*$ via irreversible deformation of the material; in line with experimental observations, we allow ourselves some latitude in defining precise values of $Y^*$ and $t_p^*$. We ignore friction between the ribbon and the blade, so that the ribbon bears a uniform load along its length and remains isothermal. We consider the motion of an element of ribbon as it passes onto, over and off the blade. In doing so, the curvature of the element (i) rises smoothly while the ribbon is off the blade, (ii) adopts the curvature of the blade while in contact with it, and then (iii)- (v) falls once off the blade (see Fig. 3, top), giving rise to stress distributions across the ribbon illustrated in Fig. 3 for low and high loads (cases A and B respectively). Although the ribbon adopts a steady shape, material elements experience a time-varying curvature as they pass over the blade. The off-blade curvature distributions (i, iii-v) are regulated by a balance between the ribbon’s bending resistance (which we assume is unaffected by any plastic deformation) and the imposed axial load. We assume that, as it passes over the blade, the ribbon element experiences a transverse strain gradient proportional to its instantaneous curvature, stretching the ribbon on its outer surface and compressing it (relatively) at its inner surface. Thus, upstream of the blade (i), the curvature-induced strain induces a transverse gradient of stress through an initial elastic response, which acts in addition to the axial loading on the ribbon. Where the stress exceeds $Y^*$, however, the ribbon...
starts to yield irreversibly (see (i, ii) of Fig. 3); initially, this takes place close to the ribbon’s outer surface. As the stress in the yielded region relaxes, the yielded region widens in order that the ribbon element can support a constant net axial load. Passing off the blade, the stress in the yielded region relaxes towards \(Y^*\), leaving the ribbon irreversibly elongated at its outer surface ((iii) in Fig. 3).

Passage of the element further off the blade leads to a reduction in curvature and hence in the transverse strain gradient. Thus, via an initial elastic response, there is a corresponding reduction in the transverse stress gradient; this may be visualized as counter-clockwise pivoting of the stress distribution about the ribbon’s mid-line. If the yielded region remains confined to the upper half of the ribbon ((iv, v) in case A, Fig. 3), then no further yielding occurs. However if the yielded region penetrates into the lower half of the ribbon, pivoting of the stress lowers the stress near the outer wall but creates a new zone near the ribbon’s mid-line where the stress exceeds \(Y^*\), ((iv) in case B, Figure 3). This, in turn, induces a second phase of stress relaxation, involving widening of the central yielded region ((v) in case B, Fig. 3), and further irreversible elongation of the ribbon; this process is promoted by further curvature reduction as the ribbon element straightens out. Ultimately, the yielded region extends to the inner surface of the ribbon, reducing the gradient of irreversible strain. When unloaded, the curvature of the ribbon element is determined by the overall gradient of the net plastic strain; this gradient grows as the ribbon yields near its outer surface ((v) in case A, Fig. 3) but falls if there is additional yielding near the inner surface ((v) in case B, Fig. 3).

The minimal load required to induce a curl (Fig. 2A) can therefore be associated with the threshold required to induce yield at the ribbon’s outer surface; the increase of curvature with load is associated with thickening of this yielded region ((i)-(iii) in case A, Fig. 3); the reduction of curvature with load at higher load is due to the compensating yield near the inner surface ((iv, v) in case B, Fig. 3). The reduction of curvature with pulling speed (Fig. 2B) arises because the ribbon element has limited time in which to undergo stress relaxation while on the blade. Curling is maximized by driving the on-blade yield surface to the ribbon centreline (but not beyond), and by ensuring the ribbon moves slowly enough for the stress to relax fully before leaving the blade.

We can use experimentally measured parameters to estimate the load required to induce curling. We represent the load as an axial stress \(Y^*\) and define the ratio of the ribbon’s thickness \(H^*\) to the blade radius of curvature as \(\epsilon \equiv H^*/R^*\), where \(\epsilon \ll 1\). The ribbon will conform tightly to the blade if the bending length \(L^*_b \equiv (E^*H^*/Y^*)^{1/2}\) (treating the ribbon as a loaded elastic beam) is small compared to \(R^*\), i.e. \(Y^* \gg \epsilon^2E^*\). Pre-blade, we require \(Y^* < Y^*\) (to avoid large-scale yielding) and the mean axial strain is \(O(Y^*/E^*)\). The additional strain at the outer ribbon surface induced by curling the ribbon over the blade is \(O(\epsilon\epsilon^2E^*)\). For yielding to take place, we therefore require \(Y^*\) to lie in a window of width of order \(\epsilon\epsilon^2E^*\) below \(Y^*\). This explains why the threshold for curling is lower for sharper blades (Fig. 2A), but does not explain why the maximum load leading to curling falls for sharper blades. When the dimensionless Curling number,

\[
\mathcal{C} \equiv \frac{H^*E^*}{R^*Y^*},
\]

is small, curling takes place in a narrow window of loads with the ribbon conforming tightly to the blade; the maximum
equilibrium ribbon curvature can be expected to be comparable in magnitude to that of the blade. For larger \( \mathcal{C} \), the curling window extends to low loads, encompassing (at the lowest loads) the case in which the ribbon bends gently over the blade. For sufficiently large \( \mathcal{C} \), the resulting maximum equilibrium curvature is limited by the bending length of the ribbon to be of magnitude \( 1/(R'\mathcal{C}) = Y^*/(H^*Y^*) \) (i.e. \( 1/L_b^* \) with \( E^*(H^*/L_b^*) = O(Y^*) \)). Curling is reduced further if the on- or off-blade transit time falls beneath the stress-relaxation timescale, i.e. for \( V't_b^* \gtrsim \min(R',L_b^*) \).

We now develop this qualitative explanation with a quantitative model, summarised briefly below and explained more fully in Materials and Methods and SI Appendix.

**Model predictions.** Model predictions are shown using lines in Fig. 2A. The model predicts that curling takes place for loads satisfying

\[
\Sigma^* > Y^* - \frac{1}{2}\kappa_{\max}E^*
\]  

where the dimensionless curvature \( \kappa_{\max} \) is the minimum of the curvature of the blade \( H'/R' \) (which arises at higher loads, when the ribbon is in line contact with the blade, as in Fig. 3A) or the maximum beam curvature \( \sqrt{48\Sigma^*/E^*} \) (this arises at lower loads, when the ribbon is bent through 90° and in point contact with the blade). Thus if \( 0 < \mathcal{C} < 2 \), the curvature of the ribbon at the lower threshold for curling is set by the blade and the cut-off lies at \( (1-\frac{1}{2}\mathcal{C})Y^*/2 \); for larger values of \( \mathcal{C} \) the cut-off is close to zero load (being \( O(Y^*/\sqrt{2}) \)) and the ribbon curvature at the onset of curling is regulated by bending effects.

The predicted ribbon curvature is composed of two curves, one rising with load and the second falling (Fig. 2A). On the rising curve, yield is confined to the upper half of the ribbon cross-section; on the falling curve, yield extends into the lower half of the cross-section. The threshold between the curves depends on geometric and material parameters, and the speed at which the ribbon passes over the blade. Choosing values of \( Y^* \) and \( V't_b^* \) (within the range of experimentally determined values) to match measurements of peak curvature, the model underestimates the minimum load for the onset of curling (although it provides a qualitative explanation for this behaviour). Just as stress relaxation measurements reveal a range of yield stress values (see SI Appendix), it was not possible to identify a single parameter set appropriate for all four blades, reflecting limitations of the constitutive model. The maximum load for curling is closely related to the assumed yield stress (see (2)); experimental data show that the sharp blade induces yielding at a lower effective yield stress, which is reflected by choosing a lower value of \( Y^* \). Curling is promoted by allowing for full stress relaxation while the ribbon is on the blade. As independent uniaxial tensile tests (SI Appendix, Fig. S1) show stress relaxation occurring more rapidly for larger strains, we adopt a smaller relaxation time for the experiments with sharper blades.

Lines on Fig. 2B show how the curvature is predicted to fall with increasing speed for a fixed load, using parameters for the Inland blade. The rate of decay of curvature with speed is captured reasonably well, and the model confirms that greater curvature may generally be achieved at lower speeds (and higher loads) by allowing for complete stress relaxation in the upper half of the ribbon. Although not evident in the experimental data, the model predicts that this effect may be offset at very low speeds (to the left of the kink in predicted curves), when the whole of the curvature is generated in the lower half of the ribbon: in this case the model suggests that slightly greater curvatures can be achieved under lower loads.

The model predicts net axial elongation (in addition to curling) that undergoes a transition from modest to steep increase with load at approximately the load required for maximum curvature (see Fig. 2A). Hence, net axial elongation is most significant along the falling part of the curvature-load curve. The experimental data confirm this prediction (SI Appendix, Fig. S2).

**Discussion**

Perhaps the most surprising feature of the experimental data reported here is the non-monotonic dependence of curvature on load, showing that the applied load need not be carefully tuned in order to maximise permanent ribbon curvature when using a blade of given radius. The load applied to the ribbon serves multiple purposes: it wraps the ribbon over the blade, forcing it to curve; it elevates the axial stress in the ribbon towards the yield stress; and it regulates the pattern of plastic deformation across the cross-section of the ribbon. When the ribbon is curved, stretching of the ribbon at its outer surface may be sufficient to induce plastic deformation locally. This deformation is applied to a length of ribbon by running the ribbon over the blade, at a speed that is sufficiently slow for part of the ribbon’s cross-section to stretch irreversibly. If the stress relaxes while the ribbon is in a curved configuration, then straightening of the ribbon as it leaves the blade elevates the stress on the inner surface of the ribbon. If the load is sufficiently great, this can induce further plastic deformation, reducing the transverse strain gradients that lead to permanent curvature.

Experiments characterising the material properties of the ribbon under elongation demonstrate surprisingly complex constitutive properties, that we have not attempted to represent in full detail, choosing instead to work within the framework of a relatively simple (quasi-linear viscoelastic) constitutive model. Our semi-quantitative predictions are sufficient to provide the physical insight needed to rationalise ribbon curling, during which extensional, shear and viscous effects interact. Our model discountsfrictional effects that may induce heating or surface deformations; these may contribute to curling in other circumstances.

The experimental protocol described here offers novel insights into material properties under shear of thin materials that are stiff but which yield at relatively low loads. The yield stress and relaxation time can be hard to define unambiguously for the polymer materials that often constitute ribbons, even in simple extensional tests. The curling number (1) and the dimensionless transit speed \( V't_b^*/R' \) are useful in characterising the range of loads over which curling arises and the tightness of the resulting curls.

**Materials and Methods**

**Model description.** A full description of the mathematical derivation and solution of the model can be found in SI Appendix. The following highlights the model’s key aspects.

To allow physical insight, our model seeks to capture the essential features of the experiment using a minimal number of parameters. We impose a strain field on a ribbon element experiencing transverse strain gradients, induced by the imposed load at approximately the load required for maximum curvature (see Fig. 2A). Hence, net axial elongation is significant along the falling part of the curvature-load curve. The experimental data confirm this prediction (SI Appendix, Fig. S2).

\[
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To allow physical insight, our model seeks to capture the essential features of the experiment using a minimal number of parameters. We impose a strain field on a ribbon element as it moves from state to state and compute the resulting stress field. We calculate strain profiles by modeling the ribbon as an Euler–Bernoulli beam which is subject to an applied load and the constraint that it wraps around the blade for a portion of its length. This yields a curvature profile which is uniform on the blade and decays over a distance \( L_b^* = (E'H'^*/\Sigma_{\min})^{1/2} \) off it (Fig. 3, top). The ribbon element experiences a transverse strain gradient, induced by the imposed curvature, superimposed on a transversely uniform axial strain. As shown in SI Appendix, the ribbon is found to be in point contact with the blade at low loads (when \( L_b^* \geq \sqrt{\beta R'} \), where the maximum curvature is \( \sqrt{\beta L_b^*} \)); it is in point contact with maximum curvature \( 1/R' \) for slightly higher loads and in line contact with
The axial stress resultant balances the imposed load, giving

\[ \sigma = \frac{1}{2} \max_{h,t} \left\{ \sum \right\} - \frac{1}{2} \min_{h,t} \left\{ \sum \right\}, \]

for \( \sigma > 0 \) appears as a parameter in (3); there is plastic deformation wherever \( \sigma \) exceeds the yield stress \( \sigma^* \). For \( \sigma > \sigma^* \) we use the material parameter \( \eta \) to pass over the blade. The dimensionless compliance parameter \( \eta = Y^*/E^* \) is assumed small. In terms of the transverse coordinate \( t \), the axial strain distribution is ascribed to the outer wall of an element of ribbon, the axial strain distribution is

\[ e(h,t) = \eta \int_0^t \kappa(h,t) \, \mathrm{d}t + e_p(h,t), \]

where \( \kappa \) is the ribbon’s centreline having a constant equilibrium curvature \( \kappa_c \) and average strain \( e_c = \frac{E_c}{E^*} \). The blade-induced residual strain \( e_p(h) \) means the stress in this state is \( \sigma(h) = (e_c(h) - e_p(h))/\eta \). We enforce force and moment balance under zero applied load and couple,

\[ \int_{-1/2}^{1/2} h \sigma \, \mathrm{d}h = 0, \quad \text{and} \quad \int_{-1/2}^{1/2} \sigma \, \mathrm{d}h = 0, \]

which was solved numerically up until a time at which the stress had fully relaxed.

Once the load is removed from the ribbon, each element relaxes to form a coil with the ribbon’s centreline having a constant equilibrium curvature \( \kappa_c \) and average strain \( e_c \). The blade-induced residual strain \( e_p(h) \) means the stress in this state is \( \sigma(h) = (e_c(h) - e_p(h))/\eta \). We enforce force and moment balance under zero applied load and couple,

\[ \int_{-1/2}^{1/2} h \sigma \, \mathrm{d}h = 0, \quad \text{and} \quad \int_{-1/2}^{1/2} \sigma \, \mathrm{d}h = 0, \]

and \( \kappa_c \).

Solutions of the numerical model are shown in Figs 3 and 5.7-510.

In addition, we show in SI Appendix how (5) can be simplified in the limit of rapid stress relaxation to deduce an analytic approximation for the rising part of the curve relating equilibrium curvature \( \kappa_c \) and imposed load \( \Sigma \), namely

\[ \kappa_c = 1 + 2 \left( \frac{2(1 - \Sigma)}{\kappa_{\max}} \right)^{3/2} \left( \frac{2(1 - \Sigma)}{\kappa_{\max}} \right)^3 \]

where \( \kappa_{\max} = \min(e, \sqrt{2E^*/Y^*}) \) for \( C < \kappa_{\max} < \frac{1}{2} \). Values of \( \kappa_{\max} \) (see SI Appendix, Fig. S11). In this limit the ribbon is predicted to be entirely in point contact with the blade for \( C > 8 \), with maximum \( \kappa^* = 4/CR^* \), whereas for \( C < 8 \) the maximum predicted curvature is \( \kappa^* = 2/CR^* \) with the ribbon in line contact with the blade. In practice, finite relaxation times reduce measured curvatures below the upper bound predicted by this approximation.

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Ribbon curling - Supplement

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In this document we follow the convention of the main paper that all dimensional quantities will be labelled with a superscript and non-dimensionalised variables will be unmarked.

Experimental Materials and Methods

Experiments were performed with polyvinyl chloride (PVC) transparency films of a thickness of $H^* = 100 \pm 3 \, \mu m$, which were cut into ribbons of length of $260 \pm 0.3 \, mm$ or $292 \pm 0.3 \, mm$, and width $W^* = 10 \pm 0.3 \, mm$ (West Films, Write-On FLM510200 and FLM510210). One end of the ribbon was bonded with double-sided tape to a rotating cylinder (of radius $12.60 \pm 0.05 \, mm$ diameter), which was rotated uniformly about its axis. The rotation of the cylinder was driven by a d.c. motor (Maxtor) connected to a 100:1 gearbox, resulting in the winding of the ribbon around the cylinder at constant linear pulling speeds of $0.9 \, mm/\text{s} \, V^* < 25 \, \text{mm/s}$. The other end of the ribbon was attached to a clamp, which could be loaded with weights, so that the ribbon was pulled vertically downward due to gravitational acceleration by masses in the range of $0 \leq m^* \leq 500 \, g$, which was machined out of aluminium and wider than the ribbon, was positioned at a distance of $45 \, mm$ away from the axis of the cylinder, so that the weight-loaded ribbon was forced to bend sharply over the surface of the blade from an approximately horizontal to a vertical configuration. The blades had radii of curvature $R^* = 1.04 \pm 0.07 \, mm$, $1.51 \pm 0.10 \, mm$, $2.03 \pm 0.06 \, mm$ except for the sharp blade which had $R^* < 0.1 \, mm$.

After the ribbon had been pulled over the blade to form at least one curl, it was carefully removed from the cylinder, cut to size and transferred to a back-lit translucent Perspex plate for visualisation. The curled ribbon was positioned on its edge and allowed to relax between one and five minutes before a photograph was taken with a digital camera (Nikon D80), as shown in the left-hand image of Fig. 1C in the main paper. The images were processed in MATLAB by firstly detecting the outer edge of the circular ribbon profile using the “Canny” method, and secondly applying the Hough transform to fit a circle to the measured outline, whose inverse curvature below the maximum value for each blade exhibited a very small decrease in Young’s modulus with a value approximately a factor of 4. Tensile tests were performed after curling on the sharp and 1mm blades. Curled ribbons with curvature below the maximum value for each blade exhibited a very small decrease in Young’s modulus with a value approximately 2% below the value of $E^* = 2.5 \pm 0.4 \, GPa$ measured for straight ribbons. However, beyond the maximum curvature, where the net extension of the ribbon was significant,

for small strain before reaching a plateau, which is associated with visco-plastic yield of the material. Reduction in stress is observed at all set values of strain, indicating that relaxation occurs in time scales longer than the pulling timescale in all cases. The maximum value of the pulling timescale is 9 s for a strain $\varepsilon = 0.045$. The associated variation of stress with time is shown in Fig. S1B over a period of 180 s. The maximum rate of stress relaxation increases monotonically with applied strain. However, the results shown in Fig. S1 also indicate that once the ribbon yields, the stress relaxes to a ranges of values $32 \leq \Sigma^* \leq 39 \, MPa$, where the upper bound is the measured value of yield stress (Fig. 2C in the main paper). These results demonstrate that the material is associated with a range of yield stress values, and that its viscoplastic time scale depends on strain. This complex behaviour is linked to the proximity of the glass transition (which occurs for PVC at a temperature $T_g \approx 80 \, ^\circ C$ that decreases upon addition of plasticisers) to the laboratory temperature of $21 \pm 1 \, ^\circ C$, resulting in load dependent stress-relaxation behaviour shown in Fig. S1 [1, 2].

The extension of the ribbon relative to its initial length, or net strain, was measured after curling the ribbon over the sharp and $R^* = 1 \, mm$ blades at $V^* = 4.9 \,\text{mm/s}$. Lines were drawn across the ribbon with a thin-tipped permanent marker prior to experiment in order to delimit a 120 mm long segment of ribbon. The measurement segment was photographed with high resolution before and after curling, and the curvature below the maximum value for each blade exhibited a very small decrease in Young’s modulus with a value approximately 2% below the value of $E^* = 2.5 \pm 0.4 \, GPa$ measured for straight ribbons. However, beyond the maximum curvature, where the net extension of the ribbon was significant,

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the Young’s modulus was found to decrease by up to 14% for curls formed with the sharp blade. Hence, the softening of the ribbon beyond the load yielding maximum curvature may contribute to the large values of total strain obtained experimentally in Fig. S2A. In addition, the contribution from shear effects neglected in the model (i.e., strain gradients above the gradient given by the curvature of the ribbon centreline), which increases with the curvature of the blade, may also add to the discrepancy in total strain values between experiments and numerical results.

Model derivation

In the following we describe how bending effects are used to determine the strain in the ribbon and then explain how the resulting stress field induces yielding. Numerical solutions of the model are supplemented with an asymptotic approximation in the limit of rapid stress relaxation.

Deriving the strain. We first represent the ribbon as an Euler–Bernoulli beam [3], defined by its unit tangent vector \( \mathbf{d}_1 \) and unit normal vector \( \mathbf{d}_2 \), with

\[
\mathbf{d}_1 = (\cos \theta(s^*), \sin \theta(s^*)), \quad \mathbf{d}_2 = (-\sin \theta(s^*), \cos \theta(s^*))
\]

where \( \theta(s^*) \) is the angle made with the \( x^- \)-axis, \( \theta(0) = 0 \), and arclength \( s^* \) is measured from the centre of contact with the blade and the ribbon (Fig. S3A). The ribbon has an associated cross-sectional force \( \mathbf{n} = n_1 \mathbf{d}_1 + n_2 \mathbf{d}_2 \) and couple \( \mathbf{m} = D^* \theta(s^*) \mathbf{d}_1 \times \mathbf{d}_2 \); the subscript \( s^* \) denotes differentiation with respect to arclength and \( D^* \) is a material parameter representing the coefficient of bending of the ribbon. As the ribbon has a rectangular cross section, \( D^* = E^* H^* W^*/12 \) [4], with \( E^* \) the Young’s modulus, \( H^* \) the ribbon thickness and \( W^* \) its width. The balance equations for the ribbon are [3]

\[
\mathbf{n}^* + \mathbf{f}^* = 0, \quad \mathbf{m}^* + \mathbf{d}_1 \times \mathbf{n}^* = 0, \quad \mathbf{f}^* = f^*_1 \mathbf{d}_1 + f^*_2 \mathbf{d}_2 \tag{2}
\]

with \( \mathbf{f}^* \) a force density acting along the length of the ribbon.

The ribbon is pulled over a circular blade by a net axial force of magnitude \( N^* = H^* W^* \Sigma^* \), directed at an angle \( \theta_0 \) with respect to the \( x^- \)-axis (Fig. S3A). For the sake of simplicity we treat the ribbon as infinitely long, assuming that \( \theta \to \pm \theta_0 \) as \( s^* \to \pm \infty \). We assume that the ribbon wraps...
around the blade adopting its curvature \( \kappa_{2} \) for a sub-domain \( \theta \in [-\theta_{0}, \theta_{0}] \subset [-\theta_{e}, \theta_{e}] \). For low applied loads it will transpire that the domain \([-\theta_{0}, \theta_{0}]\) reduces to a single point of contact.

By treating the ribbon as an Euler–Bernoulli beam we do not allow the possibility of shearing deformations (off-diagonal components of the strain tensor). More advanced formulations, such as the Timoshenko beam [5], allow for such effects. However, unlike the Euler–Bernoulli beam, they are not generally integrable systems [6]. We have chosen to use the simpler shear-free system in order to obtain closed-form expressions for the ribbon’s kinematics; this greatly simplifies the plastic modelling in the proceeding section and is ultimately justified by the quality of the full model predictions. For the same reasons we ignore the effects of friction in what follows.

It is convenient to introduce the following non-dimensional quantities,

\[
\begin{align*}
\mathbf{n}^* &= Y^*H^2\mathbf{n}, \quad \mathbf{f}^* = Y^*H^2\mathbf{f}, \quad s^* = sH^*, \\
\kappa^* &= \kappa/H^*, \quad N^* = Y^*H^2N, \quad D^* = Y^*H^4D,
\end{align*}
\]

with \( Y^* \) is the ribbon’s yield stress. In component form, (2) becomes

\[
\begin{align*}
n_{1s} - \theta_{e}n_{2} + f_{1} &= 0, \\
n_{2s} + \theta_{e}n_{1} + f_{2} &= 0, \\
D\theta_{es} + n_{2} &= 0.
\end{align*}
\]

Neglecting gravity, a body force \( \mathbf{f} \) is only present where there is contact between the ribbon and the blade. We construct solutions in two parts: one for \(|s| < s_{0}\), say, in which the ribbon has uniform curvature \( \theta_{e} = \kappa_{0} \equiv H^*/R^* \) (where \( R^* \) is the blade radius) and a non-zero contact force \( f_{2} \) (on-blade); the other for \(|s| > s_{0}\) in which the free ribbon’s curvature decays from \( \kappa_{0} \) at the point where it leaves the blade to \( 0 \) far from the blade (off-blade). We using matching conditions to create a closed system. The solution is symmetric about \( \theta = 0 \), as suggested by the symmetry of the boundary conditions under the transformation \( s \rightarrow -s \).

On-blade, with \( \theta_{e} = \kappa_{0} \) and \( \theta_{es} = 0 \), (4) implies that \( n_{2} \) vanishes so that the system reduces to

\[
\begin{align*}
n_{1s} + f_{1} &= 0, \quad \kappa_{0}n_{1} + f_{2} &= 0.
\end{align*}
\]

In the absence of any tangential forces (e.g. friction), \( f_{1} = 0, n_{1} \) is constant and \( f_{2} = -n_{1}/\kappa_{0} \). To assign a value to \( n_{1} \) we need to consider the matching conditions, requiring continuity of \( \theta_{e} \) across \( s = s_{0} \) where \( \theta = \theta_{0} \) (Fig. S3B).

Off-blade, where \( \mathbf{f} = 0 \), (2) shows that \( \mathbf{n} \) will be constant. If we consider the section of ribbon in \( s > 0 \), noting that an axial load \( N\mathbf{d}_{1} \) is applied where \( \theta \to \theta_{e} \) for \( s \to \infty \), it follows that \( \mathbf{n} = N(\cos\theta_{e}, \sin\theta_{e}) \). This boundary condition and the first two equations of (4) are satisfied if

\[
\begin{align*}
n_{1} &= N\cos(\theta_{e} - \theta), \\
n_{2} &= N\sin(\theta_{e} - \theta).
\end{align*}
\]

The transformation \( s \rightarrow -s \) and \( \theta \rightarrow -\theta \), for \( n_{1} \rightarrow n_{1} \) and \( n_{2} \rightarrow -n_{2} \) as expected. If we assume both \( \theta_{e} \) and \( n_{1} \) are continuous where the ribbon leaves the blade, i.e. across \( s \in [s_{0}^{*}, s_{0}^{\prime}] \), then we can integrate the first equation of (4) to obtain

\[
[n_{1}]_{s_{0}^{*}}^{s_{0}^{\prime}} = 0 \Rightarrow n_{1}(s_{0}) = N\cos(\theta_{e} - \theta_{0}).
\]

This value will be the same for both branches of the solution. However there must be a discontinuity in the value of \( n_{2} \) across \( s_{0} \), from zero on the blade to a non-zero value off it, necessitating a Dirac \( \delta \)-function contribution to the contact force of the form

\[
f_{2} = -\kappa_{0}n_{1}H_{e}(s_{0} - s) + F\delta(s - s_{0}),
\]

with \( H_{e} \) the Heaviside function. Integrating equation 2 of (4) gives

\[
[n_{2}]_{s_{0}^{*}}^{s_{0}^{\prime}} + F = 0 \Rightarrow F = -N\sin(\theta_{e} - \theta_{0}).
\]

The value of \( F \) reverses for the negative \( s \) branch of the solution. One can integrate \( f \) over the on-blade domain to find the net applied force on the blade from one half of the ribbon to be \( N\cos\theta_{e} \).

To compute the off-blade shape of the ribbon, the third equation of (4) reduces to

\[
D\theta_{es} - N\sin(\theta - \theta_{e}) = 0.
\]

Rescaling \( s \) with \( s = \hat{s}\sqrt{D/N} \), multiplying by \( \theta_{e} \), and integrating gives

\[
\frac{1}{2}(\theta_{e})^{2} + \cos(\theta - \theta_{e}) = C_{1}.
\]

The condition \( \theta_{e} \to 0 \) as \( \theta \to \theta_{e} \) gives \( C_{1} = 1 \), yielding a non-linear pair of ODEs for \( \theta_{e} \):

\[
\theta_{e} = \pm \sqrt{2(1 - \cos(\theta - \theta_{e}))}.
\]

For \( \hat{s} > 0 \), \( \theta \) is increasing so we choose the positive root. We can then write

\[
\hat{s} - s_{0} = \frac{1}{\sqrt{2}} \int_{\theta_{0}}^{\theta} \frac{d\theta}{\sqrt{1 - \cos(\theta - \theta_{e})}}.
\]
It follows that
\[
\hat{s}(\theta) - \hat{s}_0 = \log \left( \cot \left( \frac{\theta - \theta_0}{4} \right) \right) - \log \left( \cot \left( \frac{\theta - \theta_0}{4} \right) \right).
\]
Writing \( L_0 = \log \left( \cot \left( \frac{\theta_0 - \theta_0}{4} \right) \right) \) yields
\[
\theta(\hat{s}, \theta_0) = \theta_0 - 4 \arccot \left( \frac{\theta - \theta_0}{4} \right) \quad \text{[13]}
\]
From this we obtain the curvature distribution \( \kappa = \frac{\theta}{\sqrt{D/N}} \) where
\[
\hat{\kappa}(\hat{s}, \theta_0) = \theta(\hat{s}, \theta_0) = 2 \sech (\hat{s} - \theta_0) + L_0(\theta_0). \quad \text{[14]}
\]
It remains to compute the value of \( \theta_0 \). Enforcing the requirement that \( \theta(\hat{s}_0) = \hat{\kappa}_0 \), where \( \hat{\kappa}_0 = \kappa_0 \sqrt{D/N} \), gives
\[
L_0 = \log \left( \frac{2 \pm \sqrt{1 - \kappa_0^2}}{\kappa_0} \right),
\]
and hence
\[
\theta_0 = \theta_0 - \arccot \left( \frac{2 \pm \sqrt{1 - \kappa_0^2}}{\kappa_0} \right). \quad \text{[15]}
\]
Unscaling the arclength variable \( \hat{s} = s/\sqrt{D/N} \) gives
\[
\theta_0 = \theta_0 - \arccot \left( \frac{2 \pm \sqrt{1 - \kappa_0^2}}{\sqrt{D/N} \kappa_0} \right). \quad \text{[16]}
\]

The location of the contact point is regulated by the dimensionless parameter \( \sqrt{D/N} \kappa_0 \equiv L_0^* / (12R^*) \) (in terms of the bending length \( L_0^* = (E^* H^2 / \Sigma^*)^{1/2} \) defined in the main paper). Thus in the limit \( N \gg \kappa_0^2 D \) the ribbon wraps tightly around the blade \( (\theta_0 \to \theta_0^*) \).

We summarise the predictions of the bending calculation for which the ribbon is in line contact with the blade in terms of the parameters \( \Sigma \) and \( \eta = \frac{Y}{E^*} \) through the relation \( N/D = 12\eta \Sigma \), parameters appearing in the plastic relaxation calculation below:
\[
\theta(s, \theta_0, \kappa_0, \Sigma, \eta) = \theta_0 - 2 \arccot \left( e^{(s - \kappa_0) \sqrt{12 \Sigma \eta} + L_0} \right). \quad \text{[17]}
\]
\[
\kappa(s, \kappa_0, \Sigma, \eta) = \sqrt{48 \Sigma \eta} \sech \left( (s - \kappa_0) \sqrt{12 \Sigma \eta} + L_0 \right). \quad \text{[18]}
\]
\[
\theta_0(\kappa_0, \Sigma, \eta) = \theta_0(\kappa_0, \Sigma, \eta) - \arccot \left( \frac{2 \pm \sqrt{1 - 4 \Sigma \kappa_0^2}}{\sqrt{12 \Sigma \eta} \kappa_0} \right). \quad \text{[19]}
\]
\[
\tau_0(\kappa_0, \Sigma, \eta) = \frac{\theta_0(\kappa_0, \Sigma, \eta)}{\kappa_0}. \quad \text{[20]}
\]
\[
L_0(\kappa_0, \Sigma, \eta) = \sqrt{12 \Sigma \eta} \log \left( \frac{2 \pm \sqrt{1 - 4 \Sigma \kappa_0^2}}{\sqrt{12 \Sigma \eta} \kappa_0} \right). \quad \text{[21]}
\]

For low net loads \( (\Sigma \in [0, \Sigma_0], \text{say}) \), (19) will not yield real solutions for \( \theta_0 \), implying that the ribbon makes point contact with the blade at \( \theta = s = 0 \). Setting \( \theta_0 = 0 \) in (19) yields the load at which the ribbon changes from point to line contact
\[
\Sigma_b = \frac{\kappa_0^2}{\eta (24 - 24 \cos \theta_0)}. \quad \text{[22]}
\]
This load increases with \( \kappa_0 \) and decreases with \( \theta_0 \), reflecting the fact that at lower angles it becomes harder to pull the ribbon into line contact with the blade. For \( \Sigma < \Sigma_b \), the ribbon’s peak curvature can be found by inserting \( \theta_0 = s_0 = s = 0 \) in (18), giving
\[
\kappa_{\text{max}} = \min(\kappa_0, \sqrt{48 \Sigma \eta} \sin(2\theta_0)). \quad \text{[23]}
\]
In particular if \( \theta_0 = \pi/4 \), as in most of our experiments, we have \( \kappa_{\text{max}} = \sqrt{48 \Sigma \eta} \) provided \( \Sigma < \kappa_0^2 / 48 \eta \), so that the ribbon’s peak curvature falls below the blade curvature at very low loads.

**Predicted ribbon shapes.** Fig. S4A shows representative plots of \( \theta(s^*) \) using the experimentally determined values for \( H^*, W^* \) and \( E^* \) detailed in Experimental Materials and Methods, and a load \( \Sigma = 5 \text{ Mpa} \). The plots for \( R^* = 0.5 \text{ mm} \) depict a case for which the ribbon has only point contact whilst for \( R^* = 3 \text{ mm} \) there is line contact. Fig. S5 shows \( \kappa^2(s^*) \) for the experimental blade radii and a representative set of loads. As the curvature of the blade is increased the width of ribbon-blade contact region decreases. For the sharp blade there is point contact for all loads.

Fig. S6 shows shapes of the ribbon centreline \( r \), obtained by solving \( dr/ds = (\cos \theta(s), \sin \theta(s)) \) subject to \( r(0) = (0,0) \) and
Fig. S5. Plots of curvature as a function of arclength, using the experimental parameters. The plots (A)-(D) are for the $R^* = 2$ mm, 1.5 mm, 1 mm and sharp blades. On each plot the curves are calculated for the loads used in Fig. 1B in the main paper. The ribbon is pulled at an angle of $\theta = \pi/4$ except for the $R^* = 1$ mm blade, where the angle is $\theta = 2/3(\pi/4)$ consistently with the experiments shown in Fig. 1B. The rate of decay off-blade increases with load. For a given curvature, an increase in load leads to an increase in the width of line contact between the ribbon and the blade.

Fig. S6. Plots of the ribbon shape using the experimental parameters. Plots (A)-(D) are for the $R^* = 2$ mm, 1.5 mm, 1 mm and sharp blades. On each plot the curves are calculated for the loads used in Fig. 1B in the main paper. The ribbon is pulled at an angle of $\theta = \pi/4$ except for the $R^* = 1$ mm blade, where the angle is $\theta = 2/3(\pi/4)$ consistently with the experiments shown in Fig. 1B. The circles represent the blades and are drawn to scale. For the larger blades, all the configurations except the 50g case bunch together, similar to the observed configurations of Fig. 1B. The figures are a good match to the experimentally observed configurations shown in Fig. 1B in the main paper. As before, the ribbon is unable to adopt the shape of the sharp blade, even at the highest load.
Modelling plastic deformation. We now consider the passage of an element of the ribbon over the blade, parameterising its position with respect to the blade by a time-like parameter \( t' \), taking \( t' = 0 \) at \( x' = 0 \). We assume the element is subject to a strain distribution

\[
e(h', t') = \tau(t') + \kappa'(s'(t')) h'.
\]  

[24]

The curvature \( \kappa' \) of the element’s centerline (determined by the results of the previous section) creates a strain gradient across the ribbon’s cross section; \( h' \in [-H', 2H', H''] \) measures distance across the cross section and \( \tau \) is the uniform stretch. This prescription of \( \kappa'(t') \) restricts the permissible change in geometry of the ribbon’s centreline to stretching deformations. We neglect feed-back between the plastic relaxation of the system and the curvature distribution in order to keep the problem relatively tractable; this assumption is supported by the observation in Experimental Materials and Methods that the Young’s modulus does not change appreciably when the ribbon curls. Once again we distinguish cases in which the ribbon is in point or line contact with the blade.

To include plastic effects we assume a linear strain decomposition \( e = (\sigma' / E') + e_p \), with \( \sigma'(h', t') \) the axial stress and \( e_p(h', t') \) the plastic strain. We assume a viscoplastic constitutive law

\[
e_p = \frac{\sigma'^*}{E'} + e_p \equiv \frac{\sigma'^*}{E'} + \phi'(\sigma' - Y') H_e(\sigma' - Y').
\]  

[25]

Here \( \phi' \) is the inverse viscosity, so that the material’s relaxation time is \( \tau_e = 1/(E' \phi') \). Plastic deformation takes place for \( \sigma' > Y' \). This corresponds to the Bingham-Maxwell elastic/perfectly-plastic model [7]. An axial stress balance implies

\[
\int_{-H'/2}^{H'/2} \sigma(h', t') \, dh' = H' \Sigma'.
\]  

[26]

Thus in the absence of any plastic deformation, the strain distribution is

\[
e = (\Sigma' / E') + \kappa' h'.
\]  

[27]

The total moment of the system

\[
\int_{-H'/2}^{H'/2} h' \sigma(h', t') \, dh',
\]  

[28]

will change with time from the value \( m' \) obtained from the solutions to (4) detailed in the previous section, dependent on solutions to the system (26), which is complete when the specific forms for \( \kappa'(t') \) from the beam calculation are assumed. Our assumption that \( \kappa'(t') \) does not change under relaxation precludes us from imposing any constraints on the applied couple.

We define non-dimensionalised quantities

\[
\sigma^* = Y' \sigma, \quad t^* = T^* t, \quad h^* = H' h, \quad \eta = Y' / E', \quad \phi^* = \phi / (T^* Y'), \quad \Sigma^* = Y' \Sigma, \quad V^* = (H'/T^*) V,
\]  

[29]

with \( T^* = \pi R' / 2V^* \) the time taken to pass over a quarter of the blade at a velocity \( V^* \). With this rescaling the critical stress is unity, \( h \) spans the domain \([-1/2, 2/1/2]\) and the maximum possible time spent in contact by the element of the ribbon and the blade is 1. The model now has the form

\[
e(h, t) = \eta \tau(t) + \kappa(t) h, \quad e_t = \eta (1 - \phi(\sigma - 1) H(\sigma - 1), \quad \int_{-1/2}^{1/2} \sigma(h, t) \, dh = \Sigma. \]

[30a, 30b, 30c]

At low loads with point contact between ribbon and blade and \( \theta_e = \pi / 4 \), the curvature distribution takes the form

\[
\kappa(t) = \begin{cases} \frac{\sqrt{48 \Sigma \eta \sech(-Vt \sqrt{12 \Sigma \eta} + L_0)} & \text{if } t \leq 0, \\ \frac{\sqrt{48 \Sigma \eta \sech(Vt \sqrt{12 \Sigma \eta} + L_0)} & \text{if } t > 0. \end{cases}
\]

[31]

In the case of line contact, the time spent passing over half the contact region is \( t_0 = s_0 / V \), with \( 2s_0 \) the total arclength of the strip in contact with the blade, and

\[
\kappa(t) = \begin{cases} \frac{\sqrt{48 \Sigma \eta \sech((V - t) - s_0) \sqrt{12 \Sigma \eta} + L_0)} & \text{if } t < -t_0, \\ \frac{\kappa_b}{\sqrt{48 \Sigma \eta \sech((V - t - s_0) \sqrt{12 \Sigma \eta} + L_0)} & \text{if } -t_0 \leq t \leq t_0 \\ \frac{\sqrt{48 \Sigma \eta \sech((V - t - s_0) \sqrt{12 \Sigma \eta} + L_0)} & \text{if } t > t_0. \end{cases}
\]

[32]

For later reference, we note that (30) can be reformulated in terms of the evolving stress as

\[
\eta \sigma = \kappa h + \phi \int_{-1/2}^{1/2} (\sigma - 1) H(\sigma - 1) \, dh - (\sigma - 1) H(\sigma - 1)
\]  

[33]

with \( \int_{-1/2}^{1/2} \sigma \, dh = \Sigma \) where \( \kappa = 0 \). Alternatively, in terms of the evolving plastic strain \( e_p = e - \eta \sigma \), (30) becomes

\[
e_p = \frac{\phi}{\eta} f(e_p) H(f(e_p)), \quad f(e_p) \equiv \int_{-1/2}^{1/2} e_p \, dh + \kappa h + \eta (\Sigma - 1) - e_p,
\]  

[34]

from which the uniform strain is determined via

\[
\bar{e} = \int_{-1/2}^{1/2} e_p \, dh + \Sigma \eta.
\]  

[35]

Determining the onset of yielding. In the absence of any plastic deformation, with \( \sigma = \eta / \pi \) and \( \pi = \eta \Sigma \), the maximum stress occurs at \( h = 1/2 \) whilst the element is on the blade, where the curvature takes its maximum value

\[
\kappa_0 = \min\left(\kappa_0, \sqrt{48 \Sigma \eta \sin(2\theta_e)}\right).
\]

[36]

There can be plastic deformation only if

\[
\eta \Sigma + \kappa_0 / 2 \geq 1
\]  

[37]

which defines the minimal load \( \Sigma_0 = 1 - \kappa_0 / (2\eta) \) for the onset of curling. If this condition is met then there will be a time \( t_{min} (t_{min} \geq 0) \) at which (37) first becomes an equality. For point contact this value can be obtained by solving

\[
\eta \Sigma + \kappa (t_{min}) / 2 = \eta, \quad \kappa (t_{min}) = \sqrt{48 \Sigma \eta \sech(V t_{min} \sqrt{12 \Sigma \eta} + L_0)}.
\]

[38]

For line contact (37) is satisfied with \( \kappa = \kappa_b \) and \( t_{min} \) satisfies

\[
\eta \Sigma + \kappa (t_{min}) / 2 = \eta, \quad \kappa (t_{min}) = \sqrt{48 \Sigma \eta \sech((V t_{min} - s_0) \sqrt{12 \Sigma \eta} + L_0)}.
\]

[39]

\( s_0 = \theta_0 / \kappa_b \).
Fig. S7. (A) Typical solution for $\sigma_+ = \sigma(1/2, t)$ in the case of point contact, along with the elastic stress in the absence of any plastic deformation. The applied load is $\Sigma^* = 2$ MPa. (B) Typical solution for $\sigma_+$ in the case of line contact and symmetric elastic stress distribution, in the absence of plastic deformation. The applied load is $\Sigma^* = 30$ MPa. The other parameters are $t^*_p = 0.35$ s, $Y^* = 35$ Mpa, $E^* = 2.5$ GPa, $H^* = 100$ $\mu$m and $R^* = 1$ mm.

Fig. S8. Plots of the critical line $h_c(t)$ (blue line) (for which $\sigma(h_c) = 1$) for solutions to (34) corresponding to cases (A) and (B) in Fig. 3 in the main paper. The second critical line shown in (B) (orange line) is $h_{2c}$, which emerges when there is relaxation below the midline. The parameters used in the calculations are $t^*_p = 0.35$ s, $Y^* = 35$ Mpa, $E^* = 2.5$ GPa, $H^* = 100$ $\mu$m and $R^* = 1$ mm, with applied loads $\Sigma^* = 10$ MPa in (A), and $\Sigma^* = 30$ MPa in (B).

Fig. S9. Time-dependence of upper-surface stress $\sigma_+$ when the ribbon in line contact with the blade, for different values of the relaxation time $t^*_p$ shown in the legend. Increasing the value of $t^*_p$ used in Fig. 3 in the main paper by an order of magnitude causes the stress to decay more rapidly to unity, while lowering $t^*_p$ reduces the plastic deformation towards the purely elastic limit. The parameters used in the calculations were $\Sigma^* = 10$ MPa, $Y^* = 35$ Mpa, $E^* = 2.5$ GPa, $H^* = 100$ $\mu$m and $R^* = 1$ mm.

Relaxation to a coiled ribbon. After integration of (34) or (33) to a time $t_f$ at which the ribbon element is straight and stress relaxation is complete, yielding a residual plastic strain distribution $\varepsilon_p(h)$, the load is removed from the ribbon and we assume it relaxes elastically (ignoring gravity) to form a coil with the ribbon’s centreline having a constant curvature $\kappa_c$, average strain $\varepsilon_c$ and hence stress $\sigma(h) = (\varepsilon_c + \kappa_c h - \varepsilon_p(h))/\eta$. Enforcing a force and moment balance under zero applied load
and couple, 
\[ \int_{-1/2}^{1/2} \sigma \, dh = 0, \quad \text{and} \quad \int_{-1/2}^{1/2} \frac{1}{h} \, dh \, dh = 0, \]
gives \( \tau_e \) and \( \kappa_e \), as functions of \( \Sigma \) and the material parameters.

**Numerical results.** We solve (34) numerically as an initial value problem, keeping track of the internal free boundary \( h = h_c \) at which \( \sigma = (e - e_p)/\eta = 1 \). The initial condition of no plastic yielding is \( e_p(h, -t_{\text{min}}) = 0 \) for all \( h \), with \( t_{\text{min}} \) obtained by solving either (38) or (39).

Typical time-dependent relaxation of material is illustrated by tracking the upper surface stress \( \sigma_e \equiv \sigma(1/2, t) \). This is shown in Fig. S8A for \( p_e \) and in Fig. S8B for \( B \) (line-intact). For \( B \), typical cases from \( T_{\text{m}} \) family are shown in Fig. 3 in the main paper, along with the associated curvature distributions. In both cases the shaded region indicates the yield surface of the material. The relaxation of \( \sigma \) from its value in the absence of yielding, \( \sigma_e \equiv e/\eta \), drives \( h_c \) from \( h = 1/2 \) towards the lower boundary in order to maintain stress balance (as depicted in Fig. S8A). The rate at which \( h_c \) migrates is driven by the time varying curvature distribution. The rise of \( \kappa \) (Fig. S8B) toward the mid-line (driven by the time varying curvature) alters sign at some critical load \( \Sigma_c \).

In both cases three distinct regimes can be identified. For low loads the gradient of increase is relatively sharp, coinciding with the ribbon having point contact with the blade. For point contact the maximum curvature \( c_0 \) increases as the square root of the load (see equation (23)), and this rapid change in curvature seems to drive the relatively steep initial gradient in each variable. The gradient then falls as the load increases and the ribbon comes into line contact with the blade, limiting the maximum permissible curvature to \( \kappa_e \). For medium to high loads (dependent on the blade curvature), yielding of the lower part of the ribbon accelerates the growth of curvature and the gradient of the curve significantly. This is also the domain in which \( c_0 \) falls with increasing load, due to yielding of the lower layer of the ribbon (see Fig. S8B) and \( t_{\text{min}} \) is fixed but \( e_p(1/2, t) \) increases, see Fig. 3B (iv) and (v) in the main paper). The dependence of net axial strain on load mirrors experimental observations (Fig. S2).

### Asymptotic solution with rapid stress relaxation

Further insight into the model emerges in the limit of rapid stress relaxation, when a boundary-layer structure emerges in the stress field. We demonstrate how, when the yield surface \( h_c \) is advancing downwards into the ribbon (as in examples (i)-(iii) of Fig. 3 in the main paper), the curvature-load relation can be approximated analytically.

Returning to (33), we write \( \sigma = 1 + F(h) \) with \( F < 0 \) in the range \( c_0 < h < h_c \); \( F > 0 \) above the yield surface (Fig. S8B) and \( F = F^\infty(\xi, t) \) in each region so that (33) becomes

\[ \eta F^\infty_t - h_c, t \eta F^\infty = \kappa_0(h_c + \xi) + \phi[I - F^\infty]^2, \]

\[ \eta F^\infty_t - h_c, t \eta F^\infty = \kappa_0(h_c + \xi) + \phi I_t. \]

Then setting \( F^\infty(\xi, t) = \alpha(t)\xi + \beta(t)\xi^2 + \ldots \) and expanding in \( \xi \) we obtain conditions on the coefficients as follows. At \( O(1) \),

\[ -\gamma a^2 h_c, t = \kappa_0 h_c + \phi I \]

so that if \( h_c, t \neq 0 \) then \( \alpha^2 = \alpha(t) \). The purely elastic response of the unyielded region is represented by \( F^- = \kappa_0\xi/\eta_t \) which requires

\[ 0 = (\kappa_0 h_c, t + \phi I) \]

and hence \( \alpha = \kappa_0 \), \( \beta = 0 \), etc. At \( O(1) \), we find that

\[ \beta^+ = \frac{\phi\kappa_0}{\eta^2 h_c, t} \]

and so on. The ratio \( \alpha/\beta \) identifies a lengthscale \( L \equiv \eta [h_c, t]/\phi \).

When stress relaxation is rapid, so that \( \gamma h_c, t \ll T^* \) and \( \eta \ll \phi \), \( L \) is small, indicating the presence of a boundary layer at the base of the yielded region near \( h = h_c \). An outer solution in the yielded region is obtained by retaining \( \gamma h_c, t \equiv \kappa_0 h_c + \phi I \) and neglecting the time derivative, so that \( F = (\kappa_0 h_c + \phi I) \).

(See the main text for the original source and details.)
the unyielded region, are both linear in the transverse coordinate, consistent with numerical predictions of the stress in cases (i)-(iii) of Fig. 3 in the main paper. We can then approximate the integral by neglecting any contribution from the boundary layer, so that

\[
I \approx \int_{h_0}^{h_1} \left( \frac{k \phi}{\phi} \right) dh \tag{45}
\]

from which it follows that \( I = (k_t/2\phi)(1 - h_t) \). Combining this with (43) implies \( 0 = k_t(h_t + 1/2) + 2\kappa_{h,\Sigma} \) and hence \( \kappa(h_t + 1/2) = C^2 \) for some constant \( C \). Thus the location of the yield point is controlled directly by the magnitude of the curvature, as long as the curvature is evolving. The onset of yield is given by the condition (37), when the stress first reaches unity at \( h_c = 1/2 \), i.e. \( \sigma = \Sigma + \kappa h / \eta = 1 \) at \( h = 1/2 \). Thus \( \kappa = 2\eta(1 - \Sigma) \) when \( h_c = 1/2 \) and so

\[
\kappa(h_c + 1/2)^2 = 2\eta(1 - \Sigma). \tag{46}
\]

Thus the yield point reaches the ribbon midline if \( \kappa \geq 2\eta(1 - \Sigma) \), but never reaches the opposite side of the ribbon while \( \kappa \) is increasing.

The outer solution in the yielded region therefore approximates the limit \( F^+ = \kappa(h_t/\phi) + I = (k_t/2\phi)(h_t + 1/2) \) as \( \xi \to 0 \). The boundary layer is described by returning to (41a) and discarding the time derivative \( \dot{F}^+ \). Writing \( \xi = L \xi \) and \( F^+ = (k_t/\phi)\hat{F}(\hat{\xi}) \), and noting that \( h_{\Sigma}/|h_{\Sigma}|h_{\Sigma} = -1 \) as the yield point advances into the ribbon, (41a) reduces to

\[
\hat{F}_t = \frac{1}{2} (h_t + \frac{1}{2}) - \hat{F} \tag{47}
\]

for \( L \ll 1 \). This has the solution \( \hat{F} = \frac{1}{2}(h_t + \frac{1}{2})(1 - e^{-\xi}) \). Its outer limit matches the inner limit of the outer solution. The inner limit \( \hat{F} = \frac{1}{2}(h_t + \frac{1}{2})(\xi - \frac{1}{2} \xi^2 + \ldots) \) recovers \( F^+ = \alpha \xi + \frac{1}{2} \beta^2 \xi^2 + \ldots \) as required.

To interpret (46), we substitute the full solution for \( F^+ \) (ignoring the short boundary layer) into the force balance (30c). This yields

\[
\Sigma = \kappa \frac{2\eta}{2\eta} (\frac{1}{2} + h_t)^2 + \frac{\kappa_t}{2\phi} (\frac{1}{2} - h_c) \tag{48}
\]

representing elastic and viscous contributions from the unyielded and yielded portions of the ribbon respectively. For rapid relaxation, with \( \kappa_t/\phi \ll \kappa/\eta \) (off-blade, this this condition becomes \( V \sqrt{\Sigma} \ll \phi/\eta \) using (31, 32)), the elastic contribution dominates and recovers (46). Because of this, once the curvature reaches its on-blade plateau, \( h_t \) becomes stationary and the stress in the yielded region relaxes to unity. This will cause an adjustment to the position of \( h_c \). However this can be neglected as the force balance does not depend on the load carried in the yielded region to leading order.

We can now use the outer solution to calculate the residual curvature \( \kappa_{e,t} \), for cases in which the ribbon yields only in the upper half of its cross-section. Letting \( \epsilon_p \) be the irreversible strain, the moment balance (40) gives

\[
\int_{-1/2}^{1/2} \kappa \phi^2 dh = \int_{-1/2}^{1/2} \epsilon_p h dh, \tag{49}
\]

where, we recall \( \epsilon_{p,t} \equiv \phi(\sigma - 1)H(\sigma - 1) \). Thus

\[
\kappa_{e,t} = \frac{12}{\pi} \int_{h_0}^{1/2} \phi F dh = \frac{10}{\pi} \int_{h_0}^{1/2} (h^2 + \frac{1}{2} - \frac{h}{h_c}) dh \tag{50}
\]

which reduces to \( \kappa_{e,t} = \kappa_t(\xi - h_c)(\xi + h_c) \). But (46) implies

\[
h_c = -\frac{1}{2} + \sqrt{\frac{2\eta(1 - \Sigma)}{\kappa}} \tag{51}
\]

so we may write \( \kappa = 2\eta(1 - \Sigma)K \), \( \kappa_c = 2\eta(1 - \Sigma)K_e \) and consider \( K \) rising from 1 at the onset of yield to a value \( K_{\max} \) representing the plateau \( \kappa_{\max} \) (23) in curvature associated with line contact over the blade. Then integrating over the time interval over which the ribbon curvature rises, and using \( h_c \) to parametrize time, we obtain

\[
K_e = \int_{-1/2 + K_{\max}^{-1/2}}^{1/2} \frac{(1 - 2y)(y^2 + 2y + y^2)}{(y + \frac{1}{2})^3} dy \tag{52}
\]

\[
= -2y - \frac{1}{(y + \frac{1}{2})^2} \bigg|_{-1/2 + K_{\max}^{-1/2}}^{1/2} \tag{53}
\]

\[
= K_{\max} + \frac{2}{K_{\max}^{1/2}} - 3 \tag{54}
\]

This can be re-expressed as

\[
\kappa_{e,t} = K_{\max} = 1 + 2 \left( \frac{2\eta(1 - \Sigma)}{\kappa_{\max}} \right)^{3/2} - 3 \left( \frac{2\eta(1 - \Sigma)}{\kappa_{\max}} \right) \tag{55}
\]

giving a curve that rises monotonically from \( (\Sigma, \kappa_e) = (1 - \kappa_{\max}/(2\eta), 0) \) (where \( h_c = 0 \) to \( (1 - \kappa_{\max}/(8\eta), \kappa_{\max}/2) \) (where \( h_c = 0 \). For a ribbon in line contact with the blade, \( \kappa_{\max} = \kappa_b \equiv \epsilon \), whereas in point contact \( \kappa_f = \sqrt{8\pi\Sigma}/C \) when \( \theta_c = \pi/4 \). This analysis takes no account of the reduction in \( \kappa_e \) that occurs at high loads if the yield surface enters the lower half of the ribbon. The estimate of maximum curvature \( \kappa_{\max}/2 \) overestimates the measured maxima in Fig. 2A, suggesting that stress relaxation effects are significant in experiments.

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The curvature-load relation in analytic form. Nevertheless, it is revealing to examine (55) and its dependence on the model's governing parameters, particularly the curling number $C = \epsilon/\eta$. For line contact, with $0 < C < 2$, (55) becomes

$$\kappa^* R^* = 1 + 2 \left( \frac{2}{C} (1 - \Sigma) \right)^{3/2} - \frac{6}{C} (1 - \Sigma), \quad 1 - \frac{1}{2} C < \Sigma < 1 - \frac{1}{4} C.$$  \[56\]

Thus curves differ for different curling numbers $C$: thinner ribbons with smaller $C$ curl over narrower ranges of loads. For $C > 2$, point contact takes place at low loads. Writing $\Sigma = \epsilon \Sigma$ and neglecting terms of $O(\epsilon)$, the line contact relation (56) terminates close to zero load at $\Sigma = 10^{-3}$. Further increases in $C$ shrink the range of loads over which curvature is generated and reduce the maximum curvature.

Fig. S11 shows how the predicted rising branch of the curvature-load relationship depends on $\epsilon$ and $C$, varying each while holding $C / \epsilon = E^*/Y^*$ constant. Thicker ribbons can be curled at lower loads but curl less when very thick. The graph shows how the curvature-load curve (55) transitions between self-similar forms (56) and (57) when the ribbon is respectively in pure line ($C < 2$) or pure point ($C > 8$) contact with the blade.

3. Antman, S. S. (2005), Nonlinear Problems of Elasticity (Chapter 8), Springer.