On the expected diameter of planar Brownian motion

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Abstract

Known results show that the diameter $d_1$ of the trace of planar Brownian motion run for unit time satisfies $1.595 \leq \mathbb{E}d_1 \leq 2.507$. This note improves these bounds to $1.601 \leq \mathbb{E}d_1 \leq 2.355$. Simulations suggest that $\mathbb{E}d_1 \approx 1.99$.

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Let $(b_t, t \in [0, 1])$ be standard planar Brownian motion, and consider the set $b[0, 1] = \{b_t : t \in [0, 1]\}$. The Brownian convex hull $\mathcal{H}_1 := \text{hull} b[0, 1]$ has been well-studied from Lévy [5, §52.6, pp. 254–256] onwards; the expectations of the perimeter length $\ell_1$ and area $a_1$ of $\mathcal{H}_1$ are given by the exact formulae $\mathbb{E}\ell_1 = \sqrt{8\pi}$ (due to Letac and Táncs [4,6]) and $\mathbb{E}a_1 = \pi/2$ (due to El Bachir [1]).

Another characteristic is the diameter

$$d_1 := \text{diam} \mathcal{H}_1 = \text{diam} b[0, 1] = \sup_{x,y \in b[0,1]} \|x - y\|,$$

for which, in contrast, no explicit formula is known. The exact formulae for $\mathbb{E}\ell_1$ and $\mathbb{E}a_1$ rest on geometric integral formulae of Cauchy; since no such formula is available for $d_1$, it may not be possible to obtain an explicit formula for $\mathbb{E}d_1$. However, one may get bounds.

By convexity, we have the almost-sure inequalities $2 \leq \ell_1/d_1 \leq \pi$, the extrema being the line segment and shapes of constant width (such as the disc). In other words,

$$\frac{\ell_1}{\pi} \leq d_1 \leq \frac{\ell_1}{2}.$$

The formula of Letac and Táncs [4,6] says that $\mathbb{E}\ell_1 = \sqrt{8\pi}$, so we get:

**Proposition 1.** $\sqrt{8/\pi} \leq \mathbb{E}d_1 \leq \sqrt{2\pi}$. 

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Note that $\sqrt{8/\pi} \approx 1.5958$ and $\sqrt{2\pi} \approx 2.5066$. In this note we improve both of these bounds.

For the lower bound, we note that $b[0, 1]$ is compact and thus, as a corollary of Lemma 6 below, we have the formula

$$d_1 = \sup_{0 \leq \theta \leq \pi} r(\theta), \quad (1)$$

where $r$ is the parametrized range function given by

$$r(\theta) = \sup_{0 \leq s \leq 1} (bs \cdot e_\theta) - \inf_{0 \leq s \leq 1} (bs \cdot e_\theta),$$

with $e_\theta$ being the unit vector $(\cos \theta, \sin \theta)$. Feller [2] established that

$$E r(\theta) = \sqrt{8/\pi} \quad \text{and} \quad E(r(\theta)^2) = 4 \log 2, \quad (2)$$

and the density of $r(\theta)$ is given explicitly as

$$f(r) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp\{-k^2 r^2/2\}, \quad (r \geq 0). \quad (3)$$

Combining (1) with (2) gives immediately $Ed_1 \geq Er(0) = \sqrt{8/\pi}$, which is just the lower bound in Proposition 1. For a better result, a consequence of (1) is that $d_1 \geq \max\{r(0), r(\pi/2)\}$. Observing that $r(0)$ and $r(\pi/2)$ are independent, we get:

**Lemma 2.** $Ed_1 \geq E \max\{X_1, X_2\}$, where $X_1$ and $X_2$ are independent copies of $X := r(0)$.

It seems hard to explicitly compute $E \max\{X_1, X_2\}$ in Lemma 2, because although the density given at (3) is known explicitly, it is not very tractable. Instead we obtain a lower bound. Since

$$\max\{x, y\} = \frac{1}{2} (x + y + |x - y|)$$

we get

$$E \max\{X_1, X_2\} = EX + \frac{1}{2} E|X_1 - X_2|. \quad (4)$$

Thus with Lemma 2, the lower bound in Proposition 1 is improved given any non-trivial lower bound for $E|X_1 - X_2|$. Using the fact that for any $c \in \mathbb{R}$, if $m$ is a median of $X$, $E|X - c| \geq E|X - m|$, we see that

$$E|X_1 - X_2| \geq E|X - m|. \quad$$

Again, the intractability of the density at (3) makes it hard to exploit this. Instead, we provide the following as a crude lower bound on $E|X_1 - X_2|$.

**Lemma 3.** For any $a, h > 0$,

$$E|X_1 - X_2| \geq 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h).$$

**Proof.** We have

$$E|X_1 - X_2| \geq E \left[ |X_1 - X_2| \mathbb{1}\{X_1 \leq a, X_2 \geq a + h\} + |X_1 - X_2| \mathbb{1}\{X_2 \leq a, X_1 \geq a + h\} \right] \geq h \mathbb{P}(X_1 \leq a) \mathbb{P}(X_2 \geq a + h) + h \mathbb{P}(X_2 \leq a) \mathbb{P}(X_1 \geq a + h) = 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h),$$

which proves the statement. \qed
This lower bound yields the following result.

**Proposition 4.** For \( a, h > 0 \) define

\[
    g(a, h) := h \left( \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{2a^2} \right\} \right) \left( 1 - \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8(a + h)^2} \right\} \right).
\]

Then \( E \mathbb{d}_1 \geq \sqrt{8/\pi} + g(1.492, 0.337) \approx 1.6014. \)

*Proof.* Consider \( Z := \sup_{0 \leq s \leq 1} |b_s \cdot e_0|. \)

Then it is known (see [3]) that for \( x > 0, \)

\[
    \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8x^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{8x^2} \right\} \leq \mathbb{P}(Z < x) \leq \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8x^2} \right\}.
\]

Moreover, we have

\[
    Z \leq X \leq 2Z.
\]

Since \( X \leq 2Z, \) we have

\[
    \mathbb{P}(X \leq a) \geq \mathbb{P}(Z \leq a/2) \geq \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{2a^2} \right\},
\]

by the lower bound in (5). On the other hand,

\[
    \mathbb{P}(X \geq a + h) \geq \mathbb{P}(Z \geq a + h) \geq 1 - \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8(a + h)^2} \right\},
\]

by the upper bound in (5). Combining these two bounds and applying Lemma 3 we get \( \mathbb{E}|X_1 - X_2| \geq 2g(a, h). \) So from (4) and the fact that \( \mathbb{E}X = \sqrt{8/\pi} \) by (2) we get \( \mathbb{E}d_1 \geq \sqrt{8/\pi} + g(a, h). \) Numerical evaluation using MAPLE suggests that \( (a, h) = (1.492, 0.337) \) is close to optimal, and this choice gives the statement in the proposition.

We also improve the upper bound in Proposition 1.

**Proposition 5.** \( \mathbb{E}d_1 \leq \sqrt{8 \log 2} \approx 2.3548. \)

*Proof.* First, we claim that

\[
    d_1^2 \leq r(0)^2 + r(\pi/2)^2.
\]

It follows from (6) and (2) that

\[
    \mathbb{E}(d_1^2) \leq \mathbb{E}(X_1^2 + X_2^2) = 2\mathbb{E}(X^2) = 8 \log 2.
\]

The result now follows by Jensen’s inequality.

It remains to prove the claim (6). Note that the diameter is an increasing function, that is, if \( A \subseteq B \) then \( \text{diam } A \leq \text{diam } B. \) Note also, that by the definition of \( r(\theta), \)

\[ b[0, 1] \subseteq z + [0, r(0)] \times [0, r(\pi/2)] =: R_z \text{ for some } z \in \mathbb{R}^2. \]

Since the diameter of the set \( R_z \) is attained at the diagonal,

\[
    \text{diam } R_z = \sqrt{r(0)^2 + r(\pi/2)^2},
\]

for all \( z \in \mathbb{R}^2, \) and we have \( \text{diam } b[0, 1] \leq \text{diam } R_z, \) the result follows. \( \square \)
We make one further remark about second moments. In the proof of Proposition 5, we saw that \( \mathbb{E}(d_1^2) \leq 8 \log 2 \approx 5.5452 \). A bound in the other direction can be obtained from the fact that \( d_1^2 \geq \ell_1^2/\pi^2 \), and we have (see [7, §4.1]) that
\[
\mathbb{E}(\ell_1^2) = 4\pi \int_{-\pi/2}^{\pi/2} d\theta \int_0^\infty du \cos \theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh \left( \frac{(2\theta + \pi)u}{4} \right) \approx 26.1677,
\]
which gives \( \mathbb{E}(d_1^2) \geq 2.651 \).

Finally, for completeness, we state and prove the lemma which was used to obtain equation (1).

**Lemma 6.** Let \( A \subset \mathbb{R}^d \) be a nonempty compact set, and let \( r_A(\theta) = \sup_{x \in A} (x \cdot e_\theta) - \inf_{x \in A} (x \cdot e_\theta) \). Then
\[
\text{diam } A = \sup_{0 \leq \theta \leq \pi} r_A(\theta).
\]

**Proof.** Since \( A \) is compact, for each \( \theta \) there exist \( x, y \in A \) such that
\[
r_A(\theta) = x \cdot e_\theta - y \cdot e_\theta = (x - y) \cdot e_\theta \leq \|x - y\|.
\]
So \( \sup_{0 \leq \theta \leq \pi} r_A(\theta) \leq \sup_{x,y \in A} \|x - y\| = \text{diam } A \).

It remains to show that \( \sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq \text{diam } A \). This is clearly true if \( A \) consists of a single point, so suppose that \( A \) contains at least two points. Suppose that the diameter of \( A \) is achieved by \( x, y \in A \) and let \( z = y - x \) be such that \( \hat{z} := z/\|z\| = e_{\theta_0} \) for \( \theta_0 \in [0, \pi] \). Then
\[
\sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq r_A(\theta_0) \geq y \cdot e_{\theta_0} - x \cdot e_{\theta_0} = z \cdot \hat{z} = \|z\| = \text{diam } A,
\]
as required. \( \square \)

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**References**


