BLOCKMODELS WITH MAXIMUM CONCENTRATION

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Abstract

There are many circumstances in which binary relations are defined between pairs of objects: in sociology there are social relations between people; in business there are trading relations between firms; in design there are functional dependencies between components. In all of these the clustering of objects into densely interconnected blocks reveals something of the structure of the system. In this paper a criterion is presented which permits the construction of blocks to be formulated as a quadratic programme. The method is applied to two illustrative cases: the pattern of elective choices by MBA students and the performance assessment of British universities. The method is shown to give results which are readily interpreted and, for the purpose of performance ranking, leads to a more realistic description of achievement.

Keywords quadratic programming
blockmodel
multiple criteria analysis

BLOCKMODELS WITH MAXIMUM CONCENTRATION

1. Introduction

A blockmodel is a description of the structural relations between a number of objects. For example, the objects may be people and the relations “like” or “talk to”. The problem of describing the relations may be seen as part of structural modelling (Harary, Norman and Cartwright, 1965; Lendaris, 1980; Hage and Harary, 1983) or social network analysis (Wasserman and Faust, 1997; Scott, 2000). The purpose of the model is to define groups of objects that exhibit a high degree of interconnectedness, and in this sense is a form of cluster analysis. Two model forms may be distinguished: that in which an object may belong to only one of a number of disjoint groups, and that in which there is no such restriction and so the groups may overlap. Although nomenclature varies a little groups of the first type are generally called blocks and those of the second type are called cliques. Although what follows is concerned with blockmodels it may be noted that cliques have for some time been of interest in sociometry (Luce, 1950; Harary and Ross, 1957; Arabie, 1977) and in the analysis of design problems (Alexander, 1964; Chermayeff and Alexander, 1966; Elms, 1983) in which application a relation exists if the solution chosen for one object influences the choice of solution for another. The sociometric concept of a clique requires, quite naturally, that an individual will in general be a member of a number of cliques and that position and power are in part a result of this multiple membership. Again, in the dissagregation of a design problem into smaller, and so more easily resolved, sub-problems it is thought reasonable that an object – a door, say – will form part of more than one sub-problem and that this should be recognised in the disaggregation. Whether this overlap has much practical use remains an open question. Clique detection methods rely heavily on network theoretic results, as do structural modelling methods in general.

2. Blockmodels

Relations between objects are encoded as either present or not. This may be inherent in the nature of the relation, "father of", for instance, or it may be arrived at by applying a cutoff level to a continuous measure such as a correlation coefficient. The relations need not be symmetric. A block may be visualised as a group of nodes in a network which are highly interconnected or, equivalently, as a high density region of the incidence matrix obtained by rearrangement of the rows and columns. Most of the blockmodel methods refer primarily to the matrix idea though some are based on network models (e.g. Everett, 1982). The rationale of the blockmodel relies on the idea of structural equivalence. Two objects are structurally equivalent if they have the same pattern of interaction with all objects in the set. Ideally all such equivalent objects are grouped into a block and in doing so no information is lost. In practice this ideal is unachievable and so some acceptably good approximate blocking must be achieved.

The blockmodel approach was introduced by White, Boorman and Breiger (1976). Overviews are given by Arabie, Boorman and Levitt (1978), Light and Mullins (1979) and in Wasserman and Faust (1997) and Scott (2000). These authors also provide numerous examples of application, primarily sociological in nature. In general, there may be more than one relation to be considered ("likes" and "helps", say) and there are two ways of constructing a model in this situation. First, each relation may be modelled separately and the results compared to identify if two or more patterns of interrelation are practically the same (multiplexity). Second, the criterion underlying the blockmodel may be extended to calculate just one partition for all relations (e.g. as by Boorman and Levitt, 1983). The method described in this paper will be for a single relation but the application to either of these cases will be obvious.

To state the problem formally, consider a binary network X of n nodes in which $x_{ij} = 1$ if the relation being studied exists between the objects represented as nodes i and j and 0 if it does not. It is not usual that diagonal elements have any meaningful interpretation and so, solely for convenience in what follows, let $x_{ii} = 1$.

Nodes are partitioned into m sets or blocks ($m \le n$) via the membership matrix Λ in which $\lambda_{ik} = 1$ if node i is in block k and 0 if it is not. Each node must belong to just one block and so

$$\sum_k \lambda_{ik} = 1 \quad ; \quad \forall\, i \tag{1}$$

The number of nodes in block k is

$$s_k = \sum_i \lambda_{ik} \tag{2}$$

The density matrix, D, describes interactions between the m blocks as the proportion of possible inter-block links realised in the network. Typically , for blocks k and l, the maximum number of inter-block connections is $s_k s_l$ and so the inter-block density is

$$d_{kl} = \sum_i \sum_j x_{ij} \lambda_{ik} \lambda_{jl} \; / \; s_k s_l \tag{3}$$

A special case is the intra-block density, or just block density,

$$d_{kk} = \sum_i \sum_j x_{ij} \lambda_{ik} \lambda_{jk} \; / \, s_k^2 \qquad (4)$$

From the density matrix, D, may be derived a binary matrix called an image matrix, Y, via a cutoff value α:

$$\begin{aligned} y_{kl} &= 1 \text{ if } d_{kl} \geq \alpha \\ &= 0 \text{ otherwise} \end{aligned} \qquad (5)$$

If α=0 the result is a "zeroblock" or lean fit image, since only for zero densities will the density and image be the same. Similarly, if α=1 the result is a "oneblock" or fat fit. Other values are called α-fit images. A convenient value for α is the density of the whole matrix X so that the image matrix shows those inter-block densities above and below the mean. This last stage is not always required, as will be the case in the illustrations below.

Construction of a blockmodel requires the determination of the partition Λ.

Measured by the number of published applications the two most popular methods of blockmodel construction are BLOCKER and, particularly, CONCOR described by Light and Mullins (1979) as being respectively deductive and inductive. BLOCKER (Heil and White, 1976) requires that an hypothesised structure (image) is provided as input and then seeks permutations of the network to give best fit solutions. The hypothesis is justified *a priori* by reference to some body of theory outwith BLOCKER. However, it is not often that such a hypothesis is available, rather it is required that some structure inherent in the data is revealed by the analysis. CONCOR (Breiger, Boorman and Arabie, 1975) does just this by a process of repeated correlation. The calculations are observed to lead to a useable result in that blocks are produced but by a process with no theoretical justification. Schwartz (1977) criticised the method as being obscure and a poor substitute for a principle component analysis. Despite these reservations CONCOR continues to be used, for example by Gerlach (1992) in his study of corporate relations in Japan.

An alternative, and in principle a more straightforward, strategy is to find some criterion for model performance and then a blockmodel which is optimal. Such criteria may be of two types: those which measure the goodness of fit of model to data and those which describe some characteristic of the blockmodel structure.

Alternative measures of goodness of fit are described by, among others, Arabie, Boorman and Levitt (1978), Carrington and Heil (1979) and Wasserman and Faust (1997: Ch. 16) as ways of describing the adequacy of the description provided by the model of the data after the blocks have been constructed. As a criterion for block construction the COBLOC algorithm proposed by Carrington and Heil (1981) uses a chi-squared measure to compare the density and image matrices, D and Y, as the basis for a hierarchical clustering procedure giving partitions of varying coarseness wherein the clusters are determined by the measure. Panning (1982) takes the values of the image matrix as predictors of the interactions in the data matrix, X, and uses the correlation between elements in these two $n \times n$ matrices as a measure of goodness of fit to be optimised, and shows that in this case blockmodelling is equivalent to regression.

While choosing a model to maximise goodness of fit is a common enough approach to model building generally, it will always be more satisfactory if the model is derived from some other considerations and the goodness of fit calculated only after the model is formed.

In one sense an ideal of a block structure is a rearrangement of matrix rows and columns to create a high density diagonal band. Katz (1947) takes this idea and uses the distance (number of cells) that an interaction lies above or below the diagonal. The sum of squares of these distances, $\Sigma\Sigma x_{ij}(i\text{-}j)^2$, provides a function to be minimised in constructing the partition. Beum and Brundage (1950) give an alternative algorithm for the same objective.

The goal of Boorman and Levitt (1983) is to determine that partition which separates as effectively as possible high density from low density regions. To this end they maximise the weighted sum of squares of block densities from the mean density, or densities if more than one relation is being modelled simultaneously.

3. Concentration

Just as Boorman and Levitt had separation as a motivating idea for block construction so we propose a criterion based upon the blocks themselves: that we prefer large dense blocks. Large blocks are those which have a large number of members, typically s_k for block k. The extent to which a size distribution tends to a small number of large blocks has long been studied by industrial economists when looking at the degree of concentration in an economy, in particular the consideration of the distribution of sizes of firms in a sector. A popular measure is the Herfindahl-Hirschman Index (Herfindahl, 1950; Hirschman, 1964), *HHI*, which is just the sum of squares of the size of each firm. In this formulation size is expressed as a proportion of the whole so that concentration indices for sectors of different absolute size may be compared. When considering the distribution of block sizes this total, the total number of objects in the system, is the same for all possible block configurations and so we may use just the sum of squared block sizes:

$$HHI = \sum_k s_k^2 \tag{6}$$

The value of this index increases with increased concentration and so we seek a blockmodel which maximises *HHI*. This criterion may be justified on grounds of both parsimony and clarity in that we implicitly seek the smallest number of (large) blocks as a model. In so doing the most compact description is sought.

An acceptable density is set by requiring blocks to have a density no less than the parameter β, and so, from (4),

$$\sum_i \sum_j x_{ij}\lambda_{ik}\lambda_{jk} \; / \, s_k^2 \;\geq\; \beta \tag{7}$$

Substituting for s_k from (2) gives the programme:

$$\left.\begin{array}{lll} \text{choose } \Lambda \text{ to maximise} & \sum_k \left(\sum_i \lambda_{ik} \right)^2 & \\ \text{such that} & \sum_i \sum_j x_{ij}\lambda_{ik}\lambda_{jk} - \beta\left(\sum_i \lambda_{ik} \right)^2 \;\geq\; 0 & ; \; \forall\, k \\ \text{and} & \sum_k \lambda_{ik} \;=\; 1 & ; \; \forall\, i \end{array}\right\} \quad (8)$$

4. Illustration 1: Elective choice

The most frequent application of blockmodel construction is the formation of socially interacting groups. As an example thirty MBA students were studied. On their programme each student must choose five elective courses from sixteen offered. To the extent that students choose the same electives they may be said to constitute a block and knowledge of these blocks will help in the understanding of common interests and so the structure of the programme as experienced by the students. The number of electives common to each pair of students was used to form the binary relations by coding

x_{ij} = 1 if students i and j have 3 or more electives in common
= 0 otherwise

The result of making maximum density blocks ($\beta = 1$) using (8) is shown in Figure 1. Rows and columns represent students and each shaded cell represents an interaction ($x_{ij} = 1$). The ten blocks are labelled A to J and their sizes given in the last column of Table 1. The concentration for this model is $HHI = 136$. The main blocks may be described as:

(A) *Corporate*	mainly interested in finance and strategy
(B) *Marketeers*	also an interest in finance and strategy but with a stronger common interest in marketing
(C) *Changers*	focus on change management and negotiating
(D) *Entrepreneurs*	concerned with small business management and entrepreneurship

Block densities may be appreciated from Figure 1 and are also given in Table 1 from which it would be easy to make an image matrix by choosing a cutoff value, but this is not the focus of this paper. The blocks and their interaction as measured by the density matrix provide a structural description of the interests of this group of students. That finance and strategy are important to MBA students is hardly a surprise and the analysis reflects this. The small group of *changers* is perhaps less expected and may indicate a possible syllabus development.

5. Illustration 2: Performance ranking

Ranking according to aggregated performance measures is increasingly popular, despite the practical difficulties frequently encountered: it is not uncommon that the constituent measures are chosen as much for their availability as for their desirability. In addition, the relative importance given to each constituent is, though sensible, usually somewhat arbitrary. This uncertainty about weights must necessarily result in some doubt as to whether, in all cases, those organisations being assessed really do exhibit performances significantly different from each other. Despite these problems such rankings will continue to be published. We examine here the second difficulty; uncertainty about weights. The difficulties surrounding the selection and measurement of appropriate characteristics, while real, do not undermine what follows as an illustration of blockmodel construction.

The Times annually publishes a ranking of the 97 British Universities. The ranking published on 14th. April 2000 was based on nine attributes:

1. Teaching quality assessment score

2. Research assessment exercise score
3. A level score for entrants
4. Student / staff ratio
5. Library and computer spending per student
6. Spending on facilities per student
7. Percentage graduating with a First or 2(i)
8. Percentage finding employment within six months of graduating
9. Completion rate

In each case a total or mean was taken across all departments for the whole university. Values were scaled as proportions of the maximum score achieved for each attribute. The base measures all logically have a lower bound of zero which is never observed since even the weakest institution can register some level of activity. Consequently the more common scaling to a [0,1] scale via a value function using maxima and minima found in the data is preferred and the results of both calculations are shown in Table 2 for the top twenty universities of *The Times* listing. Even this change has nontrivial effects: Warwick is elevated from ninth to fourth and King's falls from fifteenth to twentieth. However, the main purpose here is to examine the effects of uncertainty about weights. *The Times* gave teaching quality a relative weight of 2.5, research 1.5 and the rest 1.0. For calculation weights were found by scaling these relative values to sum to 1. Illustrative levels of uncertainty were modelled for each by a rectangular distribution with limits ± 25% of the weight. The requirement that weights sum to 1 means that they cannot be treated as independent random variables and so simulation was used to find the standardised difference, z_{ij} , for each pair, i and j, of universities:

$$z_{ij} = (q_i - q_j) / \sigma_{ij} \tag{9}$$

where q_i is the weighted aggregate score for university i and σ_{ij} is the standard deviation of the difference $(q_i - q_j)$. The matrix Z is recoded to give X according to whether the difference is statistically significant:

$$x_{ij} = 1 \text{ if } |z_{ij}| < z^*$$
$$= 0 \text{ otherwise}$$

where z^* is chosen to correspond to a given significance level. In this example, conservatively, $z^* = 3$. The resulting blockmodel is shown in Figure 2 and Table 3 and in the last column of Table 2.

Rather than the strict ranking of the twenty universities only three seem to be clearly distinct: Cambridge, Oxford and King's, with Imperial nearly so. These universities have performance levels significantly different from all others. UCL and Lancaster are also distinctive. The three main blocks are, in performance order,

Warwick, LSE, Bristol
Nottingham, Durham, Bath, York
Manchester, Sheffield, Birmingham, Newcastle, SOAS

These three blocks account for twelve of the twenty universities. Two aspects of this structure are notable. First, the complete lack of interaction between the three blocks, suggesting that the differences between them are substantial. Second, that the size of the block increases, albeit slightly, as one moves down the list reflecting, perhaps, the distinctiveness of superior performers.

Made in this way performance assessments are given in blocks of universities, the performance of the members of each being sensibly indistinguishable, together with some universities with performance levels distinctly different from others. It is not uncommon for these singletons to be found at the head of a ranking. Such a mix of blocks and singletons provides a more natural articulation of performance differences than an uneasily enforced strict ranking dependent, in part, upon spuriously precise weights.

6. Illustration 3: Recovering a known pattern

Clustering methods uncover structure but the structure uncovered depends in part on the method: it is not *the* structure which is found, rather *a* structure which is suggested. It is therefore not possible to prove a method in the normal sense because that would require that a true structure was known in advance, which could only occur with problems of such simplicity that they provide no real test at all. Nonetheless, an illustration is offered in Figure 3. The data are artificial. The underlying structure of three blocks and three singletons was decided and then the noise provided by off-diagonal interactions added in a haphazard way. The model successfully detected the initial pattern. This small example may permit some confidence that the method has value.

7. Computational note

The results discussed above were found as the solutions to the quadratic programme (8). Because this formulation is of a standard form proprietary software may be used, and was here. An alternative is provided by a heuristic based on the construction of blocks, node by node, in decreasing order of block size. Broadly, this is achieved by selecting at each stage the node with the highest connectivity, $C_i = \Sigma_j x_{ij}$. A full description is given by Jessop (2002). Table 4 compares the results given by the two methods. The solutions provided by the quadratic programme are, of course, not inferior. For the electives data the solution is clearly superior to that given by the heuristic, giving larger blocks, a reduced number of singletons, and so a more compact description. For the university ranking data the results are a little different in detail but give the same *HHI* and by the same distribution of block sizes: the optimum is not unique. As can be seen from Figures 1 and 2 the electives matrix is the more dense and so presents the opportunity of a greater number of good, if not optimal, solutions, whereas for the more sparse matrix these opportunities are necessarily fewer.

8. Discussion

The method presented here provides a conceptually simple criterion for the formation of blocks from a binary matrix of interactions without the requirement for any prior specification of desired structure. There are a number of parameters which must be set. First, the cutoff needed to obtain binary relations from some other measure of interaction; here the number of electives, 3, and the value of standardised difference, also 3. Second, the smallest level of density, β, permissible for block formation. Although this may at first appear to present a fine level of control, most of the cases presented for analysis comprise, as do the two illustrations above, twenty or thirty objects and this results in maximum block sizes of about six. In these cases the changes to β required to generate alternative blockmodels are likely to be somewhat coarse. In any case, the interpretation of blocks of less that maximum density may not always be clear. It is likely that leaving $\beta=1$ and varying the cutoff value used to obtain the binary relations from the data will give a more readily interpretable model. As a lower bound, and given the likely presence of two or more singletons, it is necessary to have $\beta>0.5$ to prevent the combination of two unconnected singletons into a spurious block. Third, the level α required to form an image matrix in (5) must be set. This stage was not used in the applications described above. Setting levels for parameters may seem to be somewhat arbitrary but these articulations of judgement are unavoidable just as they are, for instance, when determining confidence levels for statistical inference.

In some applications, notably the sociological, singletons, or even pairs, may not be acceptable for the very idea of a social group would seem to rule out these small blocks. On the other hand, it might be argued that identifying such people (objects) as a first stage is itself useful, for while a group of one may be thought an oxymoron loners do exist. If it is thought desirable to impose a minimum block size then this may be done using a standard linear programming formulation (e.g. Wisniewski and Dacre, 1990: Ch. 10).

The optimum may not be unique. Alternative optima may arise in two ways. First, different distributions of block sizes may have the same value of *HHI*, as in the following mappings of eighteen nodes into four blocks – [8,6,2,2] and [7,7,3,1] – which both have *HHI* = 108. Using a power greater than two in the objective function will resolve this in favour of distributions with larger blocks; the first in this case. However high the power this situation may still arise, though less frequently. Second, it may be that the same distribution of block sizes arises through more than one assignment of objects to blocks, as was the case with the two solutions to the universities model (Table 4). In this case the differences were slight. The practical significance of the existence of more than one optimum blocking will depend on the application. If the purpose of the model is to provide a useful disaggregation of a design problem into smaller sub-problems then it is likely to be unimportant, for what is needed is a disaggregation which is useful rather than in some strict sense optimal. If the purpose is to provide performance rankings then the position of an organisation in those rankings, including the block of which it is a member, may matter. All such analyses, whichever method is used, contain, to some degree, imprecisions and arbitrariness and as a result must be treated with circumspection. The method described here is no different, though the simplicity of formulation, being based on a clearly stated criterion, should assist in the interpretation of results.

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block											
A	B	C	D	E	F	G	H	I	J	block	size
100	38	29	8	25	0	38	19	0	38	A	8
	100	22	11	0	0	0	25	17	0	B	6
		100	22	17	0	17	17	0	0	C	3
			100	0	0	0	0	0	33	D	3
				100	0	0	25	0	0	E	2
					100	0	25	0	0	F	2
						100	25	0	0	G	2
							100	0	0	H	2
								100	0	I	1
									100	J	1

Table 1.
Illustration 1: Electives – density matrix (%) and block sizes.

University	*The Times* ranking	Revised value function ranking	Block
Cambridge	1	1	E
Oxford	3	2	G
London, Imperial	2	3	F
Warwick	9	4	C
LSE	8	5	C
Bristol	4	6	C
Edinburgh	6	7	D
St. Andrews	7	8	D
Nottingham	12	9	B
Durham	16	10	B
Bath	10	11	B
York	11	12	B
London, UCL	5	13	H
Lancaster	19	14	J
Manchester	18	15	A
Sheffield	20	16	A
Birmingham	13	17	A
Newcastle	17	18	A
London, SOAS	14	19	A
London, King's	15	20	I

Table 2. *The Times* top twenty British Universities.

block											
A	B	C	D	E	F	G	H	I	J	block	size
100	0	0	0	0	0	0	0	0	40	A	5
	100	0	63	0	0	0	75	0	0	B	4
		100	50	0	33	0	0	0	0	C	3
			100	0	0	0	0	0	0	D	2
				100	0	0	0	0	0	E	1
					100	0	0	0	0	F	1
						100	0	0	0	G	1
							100	0	0	H	1
								100	0	I	1
									100	J	1

Table 3.
Illustration 2: Universities – density matrix (%) and block sizes.

	Illustration 1: Electives		Illustration 2: Universities	
block	Quadratic Programme *HHI* = 136	Heuristic *HHI* = 108	Quadratic Programme *HHI* = 60	Heuristic *HHI* = 60
A	5,9,14,16,18,25,26,29	1,3,4,5,16,18,25	13,14,17,18,20	13,14,17,18,20
B	3,4,10,21,23,30	9,14,26,29	10,11,12,16	6,11,12,16
C	1,13,2719,22,24	10,21,23,30	4,8,9	4,8,9
D	2,7	22,24,27	6,7	5,10
E	17,28	2,7	1	1
F	6,20	6,20	2	2
G	8,11	8,28	3	3
H	12	11	5	7
I	15	12	15	15
J		13	19	19
K		15		
L		19		
M		17		

Table 4. Comparison of blocks found by quadratic programme
and by heuristic.

The numbers arbitrarily label students; universities are identified
by the ranks given by *The Times* (Table 2, second column).

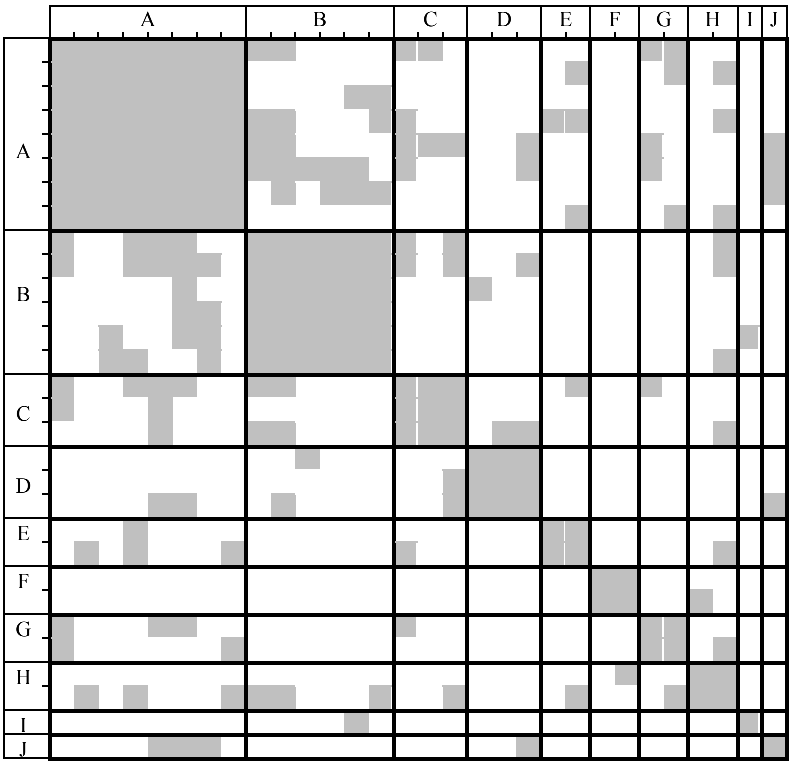

Figure 1.

Electives: interaction diagram (density = 29%).
Letters are blocks.

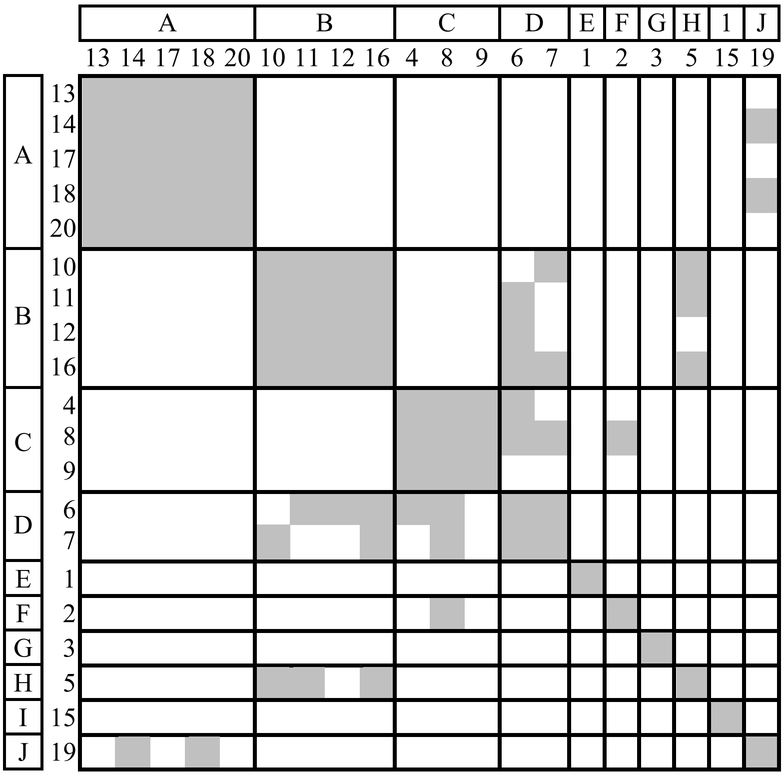

Figure 2.

Universities: interaction diagram (density = 22%).
Numbers show Universities in *The Times* original ranking
(Table 2).

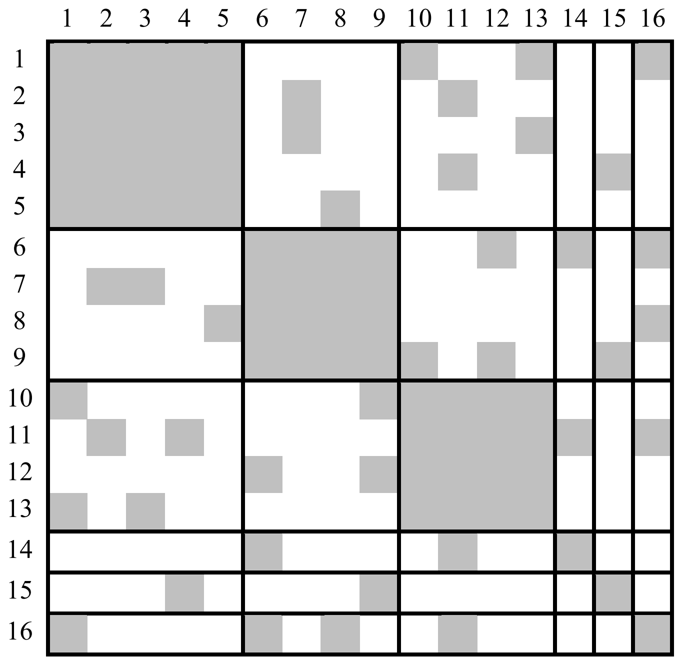

Figure 3.

Test data: model replicates (density = 38%).