Robust Uncertainty Quantification for Measurement Problems with Limited Information.*

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Metrology has an important role in modern science and relies on the accuracy and repeatability of a measurement. However, these measurements are the outcomes of different expensive experiments and noisy due to the epistemic uncertainty associated with these experiments. We express our model by \( y = f(\mu_1, \mu_2, \cdots, \mu_m) \), where \( \mu := (\mu_1, \cdots, \mu_m) \) are \( m \) different inputs. Our main goal is to obtain a confidence interval for \( \hat{\mu} \), which is based on some estimates for \( \mu \).

We use the delta method \([3]\) for uncertainty quantification, which is based on the multivariate normal approximation.

\[
\begin{align*}
\hat{\mu} &\sim N(\mu, \Sigma) \\
\hat{\Sigma} &\approx \Sigma + \nabla f(\mu)^T \Sigma \nabla f(\mu)
\end{align*}
\]

Now, if \( f \) is approximately linear around \( \mu \) for the distributional range of \( \hat{\mu} \), then we approximately have that \( f(\hat{\mu}) \sim N(f(\mu), \nabla f(\mu)^T \Sigma \nabla f(\mu)) \), by Eq. (1) and by the usual linear transformation rule for the covariance matrix. We can use this approximate distribution of \( f(\hat{\mu}) \) to construct a 95% confidence interval for \( f(\mu) \). However, \( f(\hat{\mu}) \) may not be necessarily Gaussian especially if \( f \) is highly non-linear. Additionally, for the variance term \( \nabla f(\mu)^T \Sigma \nabla f(\mu) \), we may need to use the sample standard deviation as we do not know \( \Sigma \), and we may need to use \( \nabla f(\mu) \) as we do not know \( \nabla f(\mu) \).

To avoid these issues, we propose using imprecise probability for uncertainty quantification in metrology, which is a new contribution to the field. Specifically, we propose using p-boxes \([1]\). This helps us to relax distributional assumptions and thereby leads to more robust estimates. Additionally, uncertainty expressed as a p-box can be easily propagated through a range of standard non-linear operators.

We illustrate our results by analysing the uncertainty associated with end gauge calibration \([2]\). Here, we try to determine the length (\( \ell_M \)) of an end gauge (\( M \)) by comparing it with length (\( \ell_5 \)) of a known standard (\( S \)) using the relation

\[
\ell_M = \ell_5 \left( 1 + \frac{\alpha_M + \theta_M}{\alpha_S + \theta_S} \right) + d
\]

where, \( \alpha_M \) and \( \theta_M \) (\( \alpha_S \) and \( \theta_S \)) are thermal expansion coefficient and temperature deviation of \( M \) \( (S) \) and \( d \) is the difference between \( \ell_M \) and \( \ell_5 \). In practice, \( \alpha_M \) and \( \theta_M \) (\( \alpha_S \) and \( \theta_S \)) often have weak correlation between them. Therefore, we use p-boxes to characterise these variables. We inspect their dependence structure for uncertainty propagation and obtain a robust estimate. Finally, we compare our results with the delta method.

References


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