A monoidal interval of isotone clones on a finite chain

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Abstract

Let $\mathbf{k}$ denote a $k$-element chain, $k \geq 3$. Let $M$ denote the clone generated by all unary isotone operations on $\mathbf{k}$ and let $Pol \subseteq$ denote the clone of all isotone operations on $\mathbf{k}$. We investigate the interval of clones $[M, Pol \subseteq]$. Among other results, we describe completely those clones which contain only join (or meet) homomorphisms, and describe the interval completely for $k \leq 4$.

1 Introduction

Let $A$ be a finite set. A clone on $A$ is a set of finitary operations on $A$ closed under superposition (composition) and containing all projections. If $X$ is an $m \times n$-matrix with entries from $A$, and $f$ is an $n$-ary operation, then the column $f(X)$ is calculated row-wise. For an $m$-ary relation $\theta$ on $A$, the clone $Pol \theta$ consists of all operations $f$ such that $f(X)$ belongs to $\theta$ whenever all columns of $X$ do. If $f$ belongs to $Pol \theta$ we say that $f$ preserves the relation $\theta$.

Let $\rho$ be a preorder on $A$, i.e. a binary relation on $A$ which is reflexive and transitive. Let $M$ denote the clone generated by all unary operations
preserving $\rho$ and as above, denote the clone of all operations preserving $\rho$ by $Pol\rho$. It is proved in [4] that if the interval of clones $[M, Pol\rho]$ is finite then the preorder must be a chain, and that for $|A| = 3$ it is indeed the case that the interval is finite. Our purpose in this note is to further study the monoidal interval $[M, Pol \leq]$ where $\leq$ is the natural ordering on the set $k = \{1, 2, \ldots, k\}$ for $k \geq 3$. We refer the reader to [3, 4] and Chapter 3 of [11] for a discussion of the general problem of determining monoidal intervals, and to [8, 9, 11] for standard results and notation.

Before we state our results, we need some notation. Let $3 \leq h \leq k$ and let $\mu_h$ denote the $h$-ary relation consisting of all tuples $(a_1, \ldots, a_h)$ such that $a_1 \leq a_2 \leq \cdots \leq a_h$ and such that $|\{a_1, \ldots, a_h\}| < h$. For $1 \leq h \leq k$ let $P_h$ denote the clone of all isotone operations $f$ which are either essentially unary or such that the image of $f$ contains at most $h$ elements. Notice that $P_1 = M$ and that $P_k = Pol \leq$.

Let $\lor^o$ denote the 3-ary relation consisting of all tuples $(a, b, a \lor b)$ where $\lor$ denotes the join operation of the chain, and similarly for the relation $\land^o$ where $\land$ is the meet operation of the chain $k$. Notice that since the order we consider is a chain, we have that $M \subseteq Pol \lor^o$ and $M \subseteq Pol \land^o$.

It is difficult to state our main result in one short theorem. Therefore we shall refer to Figure 1 and describe its main properties and where in the text their proofs can be found. The figure depicts the (partial) Hasse diagram of the interval $[M, Pol \leq]$ for $k \geq 3$.

1. The interval has three maximal elements, $Pol \lor^o$, $Pol \land^o$ and $Pol \mu_k$; this is proved in Lemma 2.4.

2. Each solid line segment indicates, as usual, a covering relation. This follows from Lemmas 2.5 and 2.6 and Theorem 3.15.

3. Let $C$ be a clone in the interval $[M, Pol \leq]$. Suppose that $C$ in not one of $M$, $Pol \leq$, $P_h$, $Pol \lor^o \cap P_h$, $Pol \land^o \cap P_h$, $Pol \mu_h$, for any $h$. Then $C$ is contained in an interval $[P_h, Pol \mu_{h+1}]$ for some $3 \leq h \leq k - 1$. This is Theorem 3.15. These intervals are depicted by curved lines in Figure 1.

Notice that the above is sufficient to describe the interval if $k = 3$ (this was first done in [4]), see Figure 2. In section 4 we describe completely the interval for the case $k = 4$: 

2
Figure 1: The interval $[M, Pol \leq]$. 

3
Figure 2: The interval $[M, Pol \leq]$ for $k = 3$.

Figure 3: The interval $[M, Pol \leq]$ for $k = 4$. 
Theorem 1.1 For $k = 4$, the interval $[M, Pol \leq]$ consists of exactly 11 clones, as shown in Figure 3.

The next section presents some basic results and definitions we shall need. In section 3 we prove all results that lead up to our description of the interval $[M, Pol \leq]$ in the general case. Then in section 4 we prove Theorem 1.1. We conclude with a few comments on the structure of the interval for $k \geq 5$.

2 Preliminaries

We begin with a few auxiliary results and definitions. In the following, the symbol $\subseteq$ shall denote strict inclusion. If $F$ is a set of operations on $\mathbb{k}$ then $\langle F \rangle$ shall denote the clone generated by $F$. To simplify notation we shall write $\langle M, f_1, \ldots, f_n \rangle$ instead of $\langle M \cup \{f_1, \ldots, f_n\} \rangle$.

Definition. Let $\theta$ be an $r$-ary relation on $\mathbb{k}$, $r \geq 1$. Let $i$ and $j$ be distinct, $1 \leq i, j \leq r$. Then let $\theta_{ij}$ denote the set of all pairs $(a_i, a_j)$ such that there exists $(b_1, \ldots, b_r) \in \theta$ with $b_i = a_i$ and $b_j = a_j$. The relation $\theta$ is irredundant if $\theta_{ij}$ is not the equality relation for any $i \neq j$.

Lemma 2.1 Let $\theta$ be an irredundant $r$-ary relation on $\mathbb{k}$, $r \geq 2$. If $M \subseteq Pol \theta$ then $\theta_{ij}$ is one of $\leq$, $\geq$ or $\mathbb{k}^2$.

Proof. This is straightforward. 

Lemma 2.2 (Extension Lemma) Let $P$ be any finite poset and $D$ a non-empty subset of $P$. Let $f : D \rightarrow \mathbb{k}$ be an isotone map. Then there exists a map $g : P \rightarrow \mathbb{k}$ such that (i) $g$ is isotone, (ii) the restriction of $g$ to $D$ is $f$ and (iii) $g$ and $f$ have the same image.

Proof. For each $x \in P$ let $D_x = \{y \in D : y \leq x\}$. Let $T$ denote the image of $f$ and let $a_0$ denote the least element in $T$. Now define

$$g(x) = \begin{cases} \max\{f(y) : y \in D_x\} & \text{if } D_x \neq \emptyset, \\ a_0 & \text{otherwise.} \end{cases}$$

It is easy to see that $g$ satisfies all the requirements.
Lemma 2.3 [7] An n-ary operation f is in Pol $\vee^o$ if and only if
\[ f(x_1, \ldots, x_n) = f_1(x_1) \lor \cdots \lor f_n(x_n) \]
for some $f_i \in M$. (Mutatis mutandis for the clone Pol $\wedge^o$.)

Lemma 2.4 The maximal subclones of Pol $\leq$ containing M are Pol $\vee^o$, Pol $\wedge^o$ and Pol $\mu_k$.

Proof. We refer the reader to [6] for terminology, notation, and auxiliary results used in this proof. The three clones in question are maximal subclones by Theorem 3.4 of [6]. Now we prove that there are no others. If $\theta$ is a binary relation and $M \subseteq Pol \theta$ then by Lemma 2.1 Pol $\theta$ is equal to Pol $\leq$ or the clone of all operations on $\underline{k}$. Then by Lemma 3.1 of [6], if $C$ is a maximal subclone of Pol $\leq$ containing M then it is of type $(C, h)$, $(R, h)$ or $(M, h)$ for $h \geq 3$. Suppose that $C$ is equal neither to Pol $\vee^o$ nor to Pol $\wedge^o$. By Lemmas 3.2 and 3.3 of [6] we may assume that $C = Pol \theta$ where $\theta$ is a chain-like, essential relation of arity $h \geq 3$. By Lemma 2.5 of [6], $\theta$ must contain $\mu_h$. On the other hand, if $\theta$ contains some $h$-tuple not in $\mu_h$, say $(a_1, \ldots, a_h)$ such that $a_1 < a_2 < \ldots < a_h$, let $(b_1, \ldots, b_h)$ be any tuple such that $b_1 < \ldots < b_h$. Then it is easy to find, using the extension lemma above, an $f \in M$ that will map $(a_1, \ldots, a_h)$ to $(b_1, \ldots, b_h)$. Hence $\theta$ is full (i.e. Pol $\theta = Pol \leq$), a contradiction. Thus $\theta = \mu_h$.

Lemma 2.5 1. $P_{h-1} \subseteq P_h$ for all $2 \leq h \leq k - 1$.
2. $P_{h-1} \subseteq Pol \mu_h$ for every $3 \leq h \leq k$.
3. $P_h \not\subseteq Pol \mu_h$ for all $3 \leq h \leq k$.
4. $Pol \mu_h \cap P_h \not\subseteq Pol \mu_{h-1}$ for every $4 \leq h \leq k$.
5. $Pol \mu_h \subseteq Pol \mu_{h+1}$ for every $3 \leq h \leq k - 1$. 


6. $\text{Pol } \mu_4 \not\subseteq P_h$, for every $3 \leq h \leq k - 1$.

7. $\text{Pol } \mu_3 = P_2$.

8. $\text{Pol } \mu_h \cap P_h \not\subseteq P_{h-1}$ for every $4 \leq h \leq k$.

**Proof.**

1) This is trivial.

2) This inclusion is easy.

3) This is simple, define a binary operation $f$ as follows:

$$f(x, y) = \begin{cases} 
  x + 1 & \text{if } y = k \text{ and } 1 \leq x \leq h - 1, \\
  h & \text{if } y = k \text{ and } x \geq h, \\
  1 & \text{otherwise.}
\end{cases}$$

It is clear that $f \in P_h$ and easy to see that $f \not\in \text{Pol } \mu_h$.

4) This follows from 1), 2) and 3).

5) Note that $\text{Pol } \mu_h \not\subseteq \text{Pol } \mu_{h-1}$ follows from 4). We prove the inclusion as follows: consider the $(h + 1)$-ary relation $\theta$ consisting of all tuples $(a_1, \ldots, a_{h+1})$ such that $a_1 \leq a_2 \leq \ldots \leq a_{h+1}$ and such that there exists $x \in k$ with $(x, a_3, \ldots, a_{h+1}) \in \mu_h$ and $(a_1, a_2, x, a_4, \ldots, a_{h-1}) \in \mu_h$. Since this relation is constructed using only $\mu_h$ and $\leq$ we have that $\text{Pol } \mu_h \subseteq \text{Pol } \theta$. It remains to show that $\theta = \mu_{h+1}$. Let $(a_1, \ldots, a_{h+1}) \in \theta$ and suppose that the $a_i$ are pairwise distinct. Then for some $x \in k$ we have $(x, a_3, \ldots, a_{h+1}) \in \mu_h$ and $(a_1, a_2, x, a_4, \ldots, a_{h-1}) \in \mu_h$. From the first we have that $x = a_3$ and from the second we have that $x = a_2$ or $x = a_4$, a contradiction. Hence $\theta$ is contained in $\mu_{h+1}$. The other inclusion is easy.

6) It suffices by 1) to show that $\text{Pol } \mu_4 \not\subseteq P_{k-1}$. Define a binary operation $f$ on $k$ as follows: let $S$ be the set of pairs $(x, y)$ such that $x + y = k + 1$ and $2 \leq x \leq k - 1$. Let

$$f(x, y) = \begin{cases} 
  x & \text{if } (x, y) \in S, \\
  1 & \text{if } (x, y) < (a, b) \text{ for some } (a, b) \in S, \\
  k & \text{otherwise.}
\end{cases}$$

It is easy to see that $f$ is isotone and that $f \not\in P_{k-1}$. However $f$ is in $\text{Pol } \mu_4$: indeed, suppose it is not; then there exist tuples $(a_1, \ldots, a_4)$ and $(b_1, \ldots, b_1)$ in $\mu_4$ such that $f$ maps $((a_1, b_1), \ldots, (a_4, b_4))$ to some tuple $(c_1, \ldots, c_4)$ not in $\mu_h$. Since $f$ is isotone, this means that $c_1 < c_2 < c_3 < c_4$. 

7
This means that $1 < c_2 < k$ and $1 < c_3 < k$ and so $(a_2, b_2)$ and $(a_3, b_3)$ are in $S$; but $S$ is an antichain in $\mathbf{k}^2$ so this is impossible by definition of $\mu_4$.

7) By 2) it suffices to prove that $\text{Pol} \mu_3 \subseteq P_2$. By a well-known result of Burle [1] it will suffice to show that $\text{Pol} \mu_3 \subseteq \text{Pol} \theta$ where $\theta$ is the 3-ary relation consisting of all $(a, b, c)$ with $|\{a, b, c\}| \leq 2$. Construct the following 3-ary relation: let $\alpha$ be the set of all $(x_{13}, x_{22}, x_{33})$ such that there exist $x_{ij}$, $1 \leq i, j \leq 3$ satisfying the following:

$$x_{ij} \leq x_{kl} \text{ if } i \leq j \text{ and } k \leq l$$  \hspace{1cm} (1)

$$\begin{align*}
(x_{11}, x_{13}, x_{33}) &\in \mu_3, \\
(x_{11}, x_{31}, x_{33}) &\in \mu_3, \\
(x_{12}, x_{22}, x_{32}) &\in \mu_3, \\
(x_{21}, x_{22}, x_{23}) &\in \mu_3
\end{align*}$$  \hspace{1cm} (2) \hspace{1cm} (3) \hspace{1cm} (4) \hspace{1cm} (5)

Clearly $\text{Pol} \mu_3 \subseteq \text{Pol} \alpha$. We show that $\alpha \subseteq \theta$, the other inclusion is easy. Suppose that there exists $(a, b, c) \in \alpha$ with $a$, $b$ and $c$ distinct. Suppose first that $a$ or $c$ is neither the largest nor the smallest of $a$, $b$ and $c$. Without loss of generality, we may assume that $a > \min\{a, b, c\}$ and $a < \max\{a, b, c\}$. Then by condition (1) we have that $x_{11} < a < x_{33}$ and thus condition (2) is not satisfied. Hence we may assume without loss of generality that $a < b < c$. But then

$$x_{12} \leq a < b < c \leq x_{32}$$

by condition (1) so condition (4) fails.

8) If $h = 4$ consider the binary operation

$$f(x, y) = \begin{cases} 
2 & \text{if } (x, y) = (k, 1), \\
3 & \text{if } (x, y) = (k - 1, 2), \\
1 & \text{if } (x, y) < (k, 1) \text{ or } (x, y) < (k - 1, 2), \\
4 & \text{otherwise.}
\end{cases}$$

It is easy to see that $f$ is in $\text{Pol} \mu_4 \cap P_4$ but not in $P_3$.

Now assume that $h \geq 5$. Define a binary operation as follows: let $S$ be the set of all pairs $(x,y)$ such that $x = k - 1$ and $2 \leq y \leq h - 3$, and let $T$
be the set of pairs \((x, y)\) such that \(x = k\) and \(2 \leq y \leq h - 3\). Let
\[
g(x, y) = \begin{cases} 
2 & \text{if } (x, y) = (k, 1), \\
3 & \text{if } (x, y) \in S, \\
y + 2 & \text{if } (x, y) \in T, \\
h & \text{if } y \geq h - 2, \\
1 & \text{otherwise.}
\end{cases}
\]

It is obvious that \(g \in P_h\) and \(g \notin P_{h-1}\). tuples \(\pi = (1, 2, 4, 5, \ldots, h)\) and The argument that shows that \(g \in Pol \mu_h\) is very similar to the one used in 6).

\[\blacksquare\]

**Lemma 2.6** The clone \(M\) is the intersection of the clones \(Pol \lor^o\) and \(Pol \land^o\).

In fact, \(M = Pol \rho\) where \(\rho\) consists of all \(4\)-tuples of the form \((a, a, b, b)\) with \(a \leq b\) or of the form \((a, b, a, b)\) with \(a \leq b\).

**Proof.** Notice that an \(n\)-ary operation \(f\) is in \(Pol \lor^o \cap Pol \land^o\) if and only if it is a lattice homomorphism \(f : k^n \to k\). In particular, the kernel \(\theta\) of \(f\) is a congruence of \(k^n\). But then \(\theta\) must be of the form \(\theta = \theta_1 \times \theta_2 \times \cdots \times \theta_n\) where each \(\theta_i\) is a congruence of the lattice \(k\) (see for example [8], Theorem 2.70).

Suppose that \(f\) is not constant, i.e. that some \(\theta_i\) is not equal to \(k^n\).

Without loss of generality, we may assume that there are \(a_1 < b_1\) such that \(a_1\) and \(b_1\) are not congruent modulo \(\theta_1\). Now suppose that there are \(a_2 < b_2\) with \(a_2\) and \(b_2\) not congruent modulo \(\theta_2\). Then
\[
(a_1, b_2, 0, \ldots, 0) \lor (b_1, a_2, 0, \ldots, 0) = (b_1, b_2, 0, \ldots, 0).
\]

But \(k^n/\theta\) is isomorphic to a chain, hence the join operation is the ‘maximum’, so \((b_1, b_2, 0, \ldots, 0)/\theta = (a_1, b_2, 0, \ldots, 0)/\theta\) or \((b_1, b_2, 0, \ldots, 0)/\theta = (b_1, a_2, 0, \ldots, 0)/\theta\). But by choice of the \(a_i, b_i\) this is not the case. Hence \(\theta_2 = k^n\) and by the same argument the same holds for all \(\theta_i\) with \(i \geq 2\). This means that \(f\) depends only on its first variable, so \(f \in M\) and we are done.

For the second statement: We have that \(Pol \rho \subseteq Pol \rho_{231}\) and \(Pol \rho \subseteq Pol \rho_{231}\) where
\[
\rho_{231} = \{(u, v, w) : (x, u, v, w) \in \rho \text{ for some } x\}
\]
and 
\[ \rho_{231} = \{ (u,v,w) : (w,u,v,x) \in \rho \text{ for some } x \} \]

But \( (x,u,v,w) \in \rho \) iff either \( u = w \geq v \) or \( v = w \geq u \) iff \( w = u \lor v \). In other words, \( \rho_{231} = \lor^2 \). In the same manner one sees that \( \rho_{231} = \land^2 \).

Hence by the result above we have that \( Pol \rho \subseteq M \). On the other hand it is clear that \( M \subseteq Pol \rho \) so we are done.

\[ \blacksquare \]

\section{The interval \([M, Pol \leq], k \geq 3\)}

The next few lemmas will be used to prove the following result:

\textbf{Theorem 3.1} Let \( f \in Pol \lor^2 \) be essentially at least binary, and suppose the image of \( f \) has \( h \) elements, \( 2 \leq h \leq k \). Then \( (M, f) = Pol \lor^2 \cap P_h \). (Mutatis mutandis for \( Pol \land^2 \)).

\textbf{Lemma 3.2} Let \( f \) be an \( n \)-ary operation in \( Pol \lor^2 \), say \( f(x_1, \ldots, x_n) = f_1(x_1) \lor \cdots \lor f_n(x_n) \) with \( f_i \in M \) for all \( 1 \leq i \leq n \). Then \( f \) depends on \( x_i \) if and only if there exist \( u < v \) in the image of \( f_i \) and \( t_j \) in the image of \( f_j \) for all \( j \neq i \) such that \( t_j < v \) for all \( j \).

\textbf{Proof.} Suppose that \( f \) depends on \( x_i \), i.e. there exist \( x_i < x'_i \) and \( x_j \) (\( j \neq i \)) such that \( f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) < f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n) \). Let \( u = f_i(x_i) \) and \( v = f_i(x'_i) \) and \( t_j = f_j(x_j) \) for all \( j \neq i \). Then

\[ t_1 \lor t_2 \lor \cdots t_{i-1} \lor u \lor t_{i+1} \lor \cdots \lor t_n < t_1 \lor t_2 \lor \cdots t_{i-1} \lor v \lor t_{i+1} \lor \cdots \lor t_n \]

implies that \( u < v \) and that no \( t_j \) is greater or equal to \( v \).

Conversely, suppose that there exist \( u, v, t_j \) as in the statement of the lemma. Let \( f_i(x_i) = u \) and \( f_i(x'_i) = v \) and \( f(x_j) = t_j \) for all \( j \neq i \). Then

\[ f(x_1, \ldots, x_n) = t_1 \lor t_2 \lor \cdots t_{i-1} \lor u \lor t_{i+1} \lor \cdots \lor t_n < \]

\[ < v = t_1 \lor t_2 \lor \cdots t_{i-1} \lor v \lor t_{i+1} \lor \cdots \lor t_n = \]

\[ = f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n). \]
Lemma 3.3 Let $\phi$ be an $n$-ary operation in $\text{Pol} \vee^o$. Then there exist $g_i \in M$ such that $\phi$ and $r(t) = \phi(g_1(t), \ldots, g_n(t))$ have the same image.

Proof. Let $T = \{a_1 < a_2 < \ldots < a_h\}$ be the image of $\phi$. We find elements $b_1, \ldots, b_h$ of $\mathbb{K}^n$ such that (1) $\phi(b_i) = a_i$ for all $1 \leq i \leq h$ and (2) $b_i \leq b_{i+1}$ for all $1 \leq i \leq h - 1$. Indeed, choose $c_1, \ldots, c_h$ such that $\phi(c_i) = a_i$ for all $1 \leq i \leq h$. Let $b_i = c_1 \vee c_2 \vee \cdots \vee c_i$ for all $1 \leq i \leq h$. Certainly the $b_i$'s satisfy the second condition, and to see that they satisfy the first, just notice that

$$\phi(b_i) = \phi(c_1 \vee \cdots \vee c_i) = \phi(c_1) \vee \cdots \vee \phi(c_i) = a_1 \vee \cdots \vee a_i = a_i.$$ 

Now it suffices to define the maps $g_i$ $(i \in \mathbb{N})$ as follows: consider the set of first coordinates of the tuples $b_1, \ldots, b_h$, say $B_1 = \{b_1(1) \leq b_2(1) \leq \cdots \leq b_h(1)\}$. We may find an isotone map $g_1$ from $\mathbb{K}$ onto $B_1$ such that $g_1(i) = b_i(1)$ for all $1 \leq i \leq h$ (easy). Do the same for each coordinate. Then of course $b_i = (g_1(i), g_2(i), \ldots, g_n(i))$ for all $i$, so $\phi(g_1(i), \ldots, g_n(i)) = \phi(b_i) = a_i$ for all $i$ and we are done.

Lemma 3.4 Let $\phi \in \text{Pol} \vee^o$ be an essentially at least binary operation. Then there exists $\psi \in \langle M, \phi \rangle$ such that (1) $\psi$ is essentially binary and (2) $\psi$ and $\phi$ have the same image.

Proof. Let $\phi$ be an $n$-ary operation in $\text{Pol} \vee^o$, and suppose without loss of generality that $n \geq 3$ and that $\phi$ depends on its first two variables. Let $T$ be the image of $\phi$. By Lemma 2.3 we have that $\phi(x_1, \ldots, x_n) = f_1(x_1) \vee \cdots \vee f_n(x_n)$ for some $f_i \in M$. Consider the operation $F(x_2, \ldots, x_n) = f_2(x_2) \vee \cdots \vee f_n(x_n)$, and let $B$ denote its image. By Lemma 3.3, we may find $g_2, \ldots, g_n \in M$ such that the map $h(t) = F(g_2(t), \ldots, g_n(t))$ has image equal to $B$.

We claim that the map $\psi(x, y) = f_1(x) \vee h(y)$ is the one we’re looking for. Indeed, $\psi$ is in the clone $\langle M, \phi \rangle$ since $\psi(x, y) = \phi(x, g_2(y), \ldots, g_n(y))$.

The image of $\phi$ is clearly the set of all $z_1 \vee \cdots \vee z_n$ such that $z_i$ is in the image of $f_i$. Similarly, the image of $F$ is the set of all $z_2 \vee \cdots \vee z_n$ such that $z_i$ is in the image of $f_i$ for $2 \leq i \leq n$. Hence the image of $\phi$ is equal to the
set of all $z_1 \lor b$ such that $z_1$ is in the image of $f_1$ and $b \in B$, which is also the image of $\psi$. Hence $\phi$ and $\psi$ have the same image.

We now show that $\psi$ depends on both variables.

We prove that $\psi$ depends on $x$: $\phi$ depends on $x_1$, so by Lemma 3.2, there exist $u < v$ in the image of $f_1$ and $t_j$ in the image of $f_j$ $(2 \leq j \leq n)$ such that $t_j < v$ for all $j \geq 2$. Let $b = t_2 \lor \cdots \lor t_n$. Then $b \in B$, and $b < v$ so by Lemma 3.2 $\psi$ depends on its first variable.

We prove that $\psi$ depends on $y$: $\phi$ depends on $x_2$, so by Lemma 3.2, there exist $z < z'$ in the image of $f_2$ and $t_j$ in the image of $f_j$ $(j \neq 2)$ such that $t_j < z'$ for all $j \neq 2$. Let $u = z \lor t_3 \lor \cdots \lor t_n$ and $v = z' \lor t_3 \lor \cdots \lor t_n$. Notice that both $u$ and $v$ are in $B$. Now $v = z'$ and $u < z'$ so $u < v$, and $t_1$ is an element of the image of $f_1$ which is less than $v$, By Lemma 3.2, $\psi$ depends on its second variable.

\[\square\]

**Definition.** Let $2 \leq h \leq k$ and let $T = \{a_1 < a_2 < \ldots < a_h\}$ be a subset of $\mathbb{k}$. Define an element $\alpha_T$ of $M$ by

$$\alpha_T(x) = \begin{cases} a_1 & \text{if } x \leq a_1, \\ a_i & \text{if } a_{i-1} < x \leq a_i \text{ for some } 1 < i \leq h - 1, \\ a_h & \text{otherwise.} \end{cases}$$

Notice that $\alpha_T$ is a retraction onto $T$, i.e. $\alpha_T^2 = \alpha_T$.

For each $n \geq 2$ we define $n$-ary operations $J_T^{(n)} = J_T$ and $M_T^{(n)} = M_T$ as follows: $J_T(x_1, \ldots, x_n) = \alpha_T(x_1 \lor x_2 \lor \cdots \lor x_n)$ and $M_T(x_1, \ldots, x_n) = \alpha_T(x_1 \land x_2 \land \cdots \land x_n)$ for all $x_i \in \mathbb{k}$. Notice that we have $J_T(x_1, \ldots, x_n) = \alpha_T(x_1) \lor \cdots \lor \alpha_T(x_n)$ and similarly for $M_T$. ($J$ and $M$ stand for ‘join’ and ‘meet’). Notice also that we have nice ‘identities’ such as $J_T(x, J_T(y, z)) = J_T(x, y, z)$, etc. (hence the convenient abuse of notation).

**Lemma 3.5** Let $\phi \in \text{Pol} \lor^0$ be an essentially binary operation, say $\phi(x, y) = f(x) \lor g(y)$ where $f, g \in M$. Let $T$ denote the image of $g$. Then the operation $\Gamma(x, y) = f(x) \lor \alpha_T(y)$ is in $(M, \phi)$, it has the same image as $\phi$ and depends on both variables.
Proof. Let $T = \{c_1 < \ldots < c_r\}$. Choose $b_i \in \mathbb{R}$ such that $g(b_i) = c_i$ for all $1 \leq i \leq r$. Of course we have that $b_1 < \ldots < b_r$. Define

$$h(t) = \begin{cases} 
  b_1 & \text{if } t \leq c_1, \\
  b_i & \text{if } c_{i-1} < t \leq c_i \text{ for some } 1 < i \leq r - 1, \\
  b_r & \text{otherwise.}
\end{cases}$$

Then $gh(y) = \alpha_T(y)$, hence $\phi(x, h(y)) = f(x) \lor \alpha_T(y)$. In particular, this operation is in $\langle M, \phi \rangle$. Since the image of $\phi$ consists of all $u \lor v$ with $u$ in the image of $f$ and $v$ in the image of $g$, it is clear that $\Gamma$ has the same image. By Lemma 3.2 it is clear that $\Gamma$ depends on both variables since $\phi$ does.

\[\blacksquare\]

Lemma 3.6 Let $\phi \in \mathrm{Pol} \lor^o$ be an essentially at least binary operation and let $T$ denote its image. Let $a_1$ denote the least element of $T$. Then there is an operation $F(x, y) = f(x) \lor g(y)$ in the clone $\langle M, \phi \rangle$ such that (1) the image of $F$ is $T$, (2) the images of $f$ and $g$ are contained in $T$, (3) $a_1$ is in the image of $f$ and $g$ and (4) $F$ is essentially binary.

Proof. By Lemma 3.4 we may assume without loss of generality that $\phi$ is essentially binary, say $\phi(x, y) = p(x) \lor q(y)$ for some $p, q \in M$. Since the map $\alpha_T$ is a retraction onto $T$ we have that

$$\phi(x, y) = \alpha_T(\phi(x, y)) = \alpha_T p(x) \lor \alpha_T q(y)$$

Let $f(x) = \alpha_T p(x)$ and $g(y) = \alpha_T q(y)$. Then $F = \phi$ satisfies the conclusion of the lemma. Indeed, it is clear that the image of $f$ and of $g$ is contained in $T$. This implies that $f(x) \lor g(y) \geq a_1$ for all $x$ and $y$, and since $a_1$ is in the image of $\phi$, $a_1$ must be in the image of $f$ and of $g$. Since $\phi$ satisfies (1) and (4) we are done.

\[\blacksquare\]

Lemma 3.7 Let $\phi \in \mathrm{Pol} \lor^o$ be an essentially at least binary operation and let $T$ denote its image. Let $a_1$ denote the least element of $T$. Then there exists a subset $D$ of $T$ with $|D| \geq 2$ and containing $a_1$ such that the operation $G(x, y) = \alpha_T(x) \lor \alpha_D(y)$ is in $\langle M, \phi \rangle$. Furthermore, $G$ depends on both variables and has image equal to $T$.
Proof. By Lemma 3.6, there exists an operation \( F \in \langle M, \phi \rangle \) such that \( F(x, y) = f(x) \lor g(y) \) and such that \( T \) contains the image of \( f \) and \( g \), \( a_1 \) is contained in the image of \( f \) and \( g \), \( F \) is essentially binary and has image equal to \( T \). Let \( U \) and \( V \) denote the image of \( f \) and \( g \) respectively. By Lemma 3.5, we have that the operation \( F'(x, y) = f(x) \lor \alpha_V(y) \) is in \( \langle M, \phi \rangle \), is essentially binary and has image equal to \( T \). Applying Lemma 3.5 again, we get that the operation \( F''(x, y) = \alpha_U(x) \lor \alpha_V(y) \) is in \( \langle M, \phi \rangle \), is essentially binary and has image equal to \( T \).

For convenience, let us put \( f = \alpha_U \) and \( g = \alpha_V \) and \( \phi = F'' \).

We may assume without loss of generality that \( a_h \) is in the image of \( g \). Consider the operation

\[
G(x, y, z) = \phi(\phi(x, y), z).
\]

Clearly \( G \) is in the clone \( \langle M, \phi \rangle \). Now we have

\[
G(x, y, z) = f(f(x) \lor g(y)) \lor g(z) \\
= f(x) \lor fg(y) \lor g(z) \\
= f(x) \lor g(z) \lor fg(y) \\
= \phi(x, z) \lor fg(y)
\]

By Lemma 3.3 we may find operations \( h_1 \) and \( h_2 \) in \( M \) such that \( f'(t) = \phi(h_1(t), h_2(t)) \) has the same image as \( \phi \), namely \( T \). So we can construct the operation

\[
H(x, y) = G(h_1(x), y, h_2, (x)) = \phi(h_1(x), h_2(x)) \lor fg(y) = f'(x) \lor fg(y)
\]

where the image of \( f' \) is \( T \). Notice that \( H \) depends on both variables: indeed, we have that \( fg(1) = f(a_1) = f(f(1)) = f(1) = a_1 \). Thus by Lemma 3.2 \( H \) depends on \( y \). To show that \( H \) depends on \( y \) it suffices to find some element in the image of \( fg \) which is greater than \( a_1 \). If this is not the case, then we have that \( fg \) is constant so \( a_1 = fg(1) = fg(k) = f(a_h) \). Hence \( f(a) = a_1 \) for all \( a \in T \). However, the image of \( f \) is contained in \( T \) and since the map \( \phi \) depends on \( x \) the image of \( f \) must contain at least two elements; since \( f \) is a retraction onto its image, this is a contradiction. Furthermore, the new operation \( H \) also has image \( T \). Indeed, we saw above that \( fg(1) = a_1 \); if \( a \in T \) and \( x \) is such that \( f'(x) = a \) then \( H(x, 1) = f'(x) \lor fg(1) = a \lor a_1 = a \). Now
we may apply Lemma 3.5 to construct the operation $\psi(x, y) = \alpha_T(x) \lor g(y)$. By Lemma 3.5, $\psi$ has image equal to $T$ and depends on both variables.

Let $D$ denote the image of $fg$. We’ve seen above that the image of $fg$ contains at least two elements, that it is contained in $T$ and contains $a_1$. Now apply Lemma 3.5 to the operation $\psi$ to obtain that the operation $G(x, y) = \alpha_T(x) \lor \alpha_D(y)$ is in $(M, \phi)$, that it depends on both variables and has image $T$.

\[ \square \]

**Lemma 3.8** Let $\phi \in Pol^\circ$ be an essentially at least binary operation and let $T$ denote its image. Then the operation $J_T$ is in the clone $(M, \phi)$.

**Proof.**

Let $T = \{ a_1 < a_2 < \ldots < a_h \}$. By Lemma 3.7 there exists a subset $D$ of $T$ such that $G(x, y) = \alpha_T(x) \lor \alpha_D(y)$ is in $(M, \phi)$, is essentially binary and has image equal to $T$. Furthermore, $D$ contains at least 2 elements, contains $a_1$ and is contained in $T$. If $D$ is equal to $T$, then $G = J_T$ and we are done. Thus we will assume that $D$ is properly contained in $T$. We shall build an operation $\alpha_T(x) \lor \alpha_D(y)$ where $D'$ is a subset of $T$ that contains $D$ properly.

Let $b_2 < b_3 < \ldots < b_s$ be the elements of $T$ not in $D$; then of course $2 \leq s < h$. Also note that $a_1 < b_2$ since $a_1$ is in $D$.

Define $\sigma \in M$ as follows:

$$
\sigma(t) = \begin{cases} 
  a_1 & \text{if } t \leq a_1, \\
  b_i & \text{if } a_{i-1} < t \leq a_i, \text{ for } 2 \leq i < s \\
  b_s & \text{otherwise.}
\end{cases}
$$

Let $\psi(x, y) = \sigma G(x, y)$ and define

$$
\Delta(x, y, z, w) = G(\psi(x, y), G(z, w)).
$$

Clearly $\Delta$ is in the clone $(M, \phi)$. We have

$$
\Delta(x, y, z, w) = \alpha_T(\sigma \alpha_T(x) \lor \sigma \alpha_D(y)) \lor \alpha_D(\alpha_T(z) \lor \alpha_D(w))
$$

$$
= [\alpha_T \sigma \alpha_T(x) \lor \alpha_D \alpha_T(z)] \lor [\alpha_T \sigma \alpha_D(y) \lor \alpha_D(w)]
$$

Let

$$
\delta(x, z) = \alpha_T \sigma \alpha_T(x) \lor \alpha_D \sigma \alpha_T(z)
$$

15
and

\[ \epsilon(y, w) = \alpha_T \sigma \alpha_D(y) \lor \alpha_D(w). \]

We claim that (1) \( \delta \) has image equal to \( T \) and that (2) the image of \( \epsilon \) contains \( D \) properly. It is immediate that the images of \( \delta \) and \( \epsilon \) are contained in \( T \).

(1) Let \( a \in T \). If \( a \in D \) then

\[ \delta(1, a) = \alpha_T \sigma \alpha_T(1) \lor \alpha_D \alpha_T(a) \]
\[ = \alpha_T \sigma(a_1) \lor \alpha_D(a) \]
\[ = \alpha_T(\alpha_1) \lor a \]
\[ = a_1 \lor a = a. \]

If \( a \notin D \) then \( a = b_i \) for some \( 2 \leq i \leq s \). Then

\[ \delta(a_i, 1) = \alpha_T \sigma \alpha_T(a_i) \lor \alpha_D \alpha_T(1) \]
\[ = \alpha_T \sigma(a_i) \lor \alpha_D(a_1) \]
\[ = \alpha_T(b_i) \lor a_1 \]
\[ = b_i \lor a_1 = b_i. \]

(2) First we show that \( D \) is contained in the image of \( \epsilon \). Let \( d \in D \). Then

\[ \epsilon(1, d) = \alpha_T \sigma \alpha_D(1) \lor \alpha_D(d) \]
\[ = \alpha_T \sigma(a_1) \lor d \]
\[ = \alpha_T(a_1) \lor d \]
\[ = a_1 \lor d = d. \]

Now we show that the image of \( \epsilon \) must contain \( b_i \) for some \( 2 \leq i \leq s \). Suppose first that \( a_i \in D \) for some \( 2 \leq i \leq s \). Then

\[ \epsilon(a_i, 1) = \alpha_T \sigma \alpha_D(a_i) \lor \alpha_D(1) \]
\[ = \alpha_T \sigma(a_i) \lor a_1 \]
\[ = \alpha_T(b_i) \lor a_1 \]
\[ = b_i \lor a_1 = b_i. \]

Otherwise \( D \) must contain \( a_i \) for some \( i > s \). Then

\[ \epsilon(a_i, 1) = \alpha_T \sigma \alpha_D(a_i) \lor \alpha_D(1) \]

16
\[\begin{align*}
&= \alpha_T \sigma(a_i) \lor a_1 \\
&= \alpha_T (b_i) \lor a_1 \\
&= b_i \lor a_1 = b_i.
\end{align*}\]

Let \( D' \) denote the image of \( \epsilon \). By Lemma 3.3 there exist operations \( f_i, g_i \in M, 1 \leq i \leq 2 \), such that \( P(x) = \delta(f_1(x), f_2(x)) \) has image \( T \) and \( Q(y) = \epsilon(g_1(y), g_2(y)) \) has image \( D' \). Then the operation \( R(x, y) = P(x) \lor Q(y) \) is in the clone \( \langle M, \phi \rangle \) since

\[P(x) \lor Q(y) = \delta(f_1(x), f_2(x)) \lor \epsilon(g_1(y), g_2(y)) = \Delta(f_1(x), g_1(y), f_2(x), g_2(y)).\]

Since \( D' \) contains \( a_1 \) and at least two elements, it is clear that \( R \) has image equal to \( T \) and depends on both variables. We may apply Lemma 3.5 twice to \( R \) to obtain that the clone \( \langle M, \phi \rangle \) contains the operation \( G'(x, y) = \alpha_T(x) \lor \alpha_D(y) \). Repeating the above argument to this operation will eventually yield the operation \( J_T \).

\[\blacksquare\]

**Lemma 3.9** Let \( T \) be any \( h \)-element subset of \( k \) with \( 2 \leq h \leq k \). Then \( \langle M, J_T \rangle = Pol \lor^o \cap P_h \).

**Proof.** It obviously suffices to prove that \( Pol \lor^o \cap P_h \subseteq \langle M, J_T \rangle \). Let \( f \) be an \( n \)-ary operation in \( Pol \lor^o \cap P_h \). We may assume that \( f \) is essentially at least binary. Let \( B = \{b_1 < b_2 < \ldots < b_t\} \) be the image of \( f \), where \( 2 \leq t \leq h \). Let \( T = \{a_1 < a_2 < \ldots < a_h\} \). Define a map \( \sigma \) as follows:

\[\sigma(t) = \begin{cases} 
  b_1 & \text{if } t \leq a_1, \\
  b_i & \text{if } a_{i-1} < t \leq a_i, \text{ for } 2 \leq i < t \\
  b_t & \text{otherwise.}
\end{cases}\]

Consider the operation \( F(x, y) = \sigma(J_T(x, y)) \). Clearly it is in \( \langle M, J_T \rangle \). It has image equal to \( B \) since the image of \( J_T \) is \( T \) and \( \sigma \) maps \( T \) onto \( B \). Furthermore, \( F \) is essentially binary. Indeed, we have that

\[F(1, 1) = \sigma(J_T(1, 1)) = \sigma(\alpha_T(1 \lor 1)) = \sigma(a_1) = b_1,\]

17
and

\[
F(1, k) = \sigma(J_T(1, k)) \\
= \sigma(o_T(1 \lor k)) \\
= \sigma(a_h) = b_k.
\]

and

\[
F(k, 1) = \sigma(J_T(k, 1)) \\
= \sigma(o_T(k \lor 1)) \\
= \sigma(a_h) = b_k.
\]

By Lemma 3.8 we obtain that \( J_B \in \langle M, J_T \rangle \).

By Lemma 2.3 we may write \( f(x_1, \ldots, x_n) = f_1(x_1) \lor \ldots \lor f_n(x_n) \) for some \( f_i \in M \). Since \( o_B \) is a retraction onto the image \( B \) of \( f \) we obtain that

\[
f(x_1, \ldots, x_n) = o_B(f(x_1, \ldots, x_n)) \\
= o_B(f_1(x_1) \lor \ldots \lor f_n(x_n)) \\
= J_B(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)).
\]

Hence \( f \in \langle M, J_B \rangle \subseteq \langle M, J_T \rangle \) and this completes the proof.

\[
\square
\]

We may now prove the result mentioned at the beginning of this section:

**Proof of Theorem 3.1:** Let \( f \) be an essentially binary operation in \( Pol \lor^o \) whose image has \( h \) elements, \( 2 \leq h \leq k \). By Lemma 3.8 the clone \( \langle M, f \rangle \) contains the operation \( J_T \) where \( T \) is the image of \( f \). Hence by Lemma 3.9 we have that \( Pol \lor^o \cap P_h \subseteq \langle M, f \rangle \). The other inclusion is trivial.

\[
\square
\]

**Corollary 3.10** The only clones \( C \) such that \( M \subset C \subseteq Pol \lor^o \) are those of the form \( Pol \lor^o \cap P_h \) with \( 2 \leq h \leq k \). (Mutatis mutandis for \( Pol \land^o \)).
Proof: Let $C$ be a clone that contains $M$ properly and contained in $Pol \lor^\circ$. Then $C$ contains an operation $f$ which is essentially at least binary and has largest image $T$, say $|T| = h$ where $2 \leq h \leq k$. Clearly $C \subseteq Pol \lor^\circ \cap P_h$. By Theorem 3.1 $C$ must contain $Pol \lor^\circ \cap P_h$ and this completes the proof.

\[\]

**Corollary 3.11** Let $f \in Pol \lor^\circ$ be essentially at least binary, and suppose the image of $f$ has $h$ elements, $2 \leq h \leq k$. Let $g \in Pol \land^\circ$ be essentially at least binary, and suppose the image of $g$ has $h$ elements. Then $(M, f, g) = P_h$.

Proof: It follows from Lemmas 2.3 and 2.4 that $Pol \leq (M, \lor, \land)$. In fact, we claim that the $n$-ary operations in $Pol \leq$ are those operations of the form

\[f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) \land f_2(x_1, \ldots, x_n) \land \cdots \land f_s(x_1, \ldots, x_n)\]

where the $f_i$ are $n$-ary operations in $Pol \lor^\circ$. Indeed, it is clear that operations of this form are in $Pol \leq$. It thus suffices to prove that this set of operations is closed under the operations in $M$ (easy) and under the operations $\land$ (obvious) and $\lor$: indeed, just use the distributive law for this last case.

Let $F$ be an $n$-ary operation in $P_h$ and denote its image by $T$. Let $C = (M, f, g)$ where the operations $f$ and $g$ are as in the statement of the corollary. By Theorem 3.1 (and its dual) $C$ contains $Pol \lor^\circ \cap P_h$ and $Pol \land^\circ \cap P_h$. In particular, $C$ contains $M_T$.

Write

\[F(\bar{x}) = f_1(\bar{x}) \land f_2(\bar{x}) \land \cdots \land f_s(\bar{x})\]

where $\bar{x} = (x_1, \ldots, x_n)$, and $f_i \in Pol \lor^\circ$. Since the image of $F$ is $T$, we have that

\[F(\bar{x}) = \alpha_T(F(\bar{x})) = \alpha_T^2(F(\bar{x})) = \alpha_T^2(f_1(\bar{x}) \land f_2(\bar{x}) \land \cdots \land f_s(\bar{x})) = \alpha_T(\alpha_T(f_1(\bar{x})) \land \alpha_T(f_2(\bar{x})) \land \cdots \land \alpha_T(f_s(\bar{x}))) = M_T(\alpha_T(f_1(\bar{x})), \alpha_T(f_2(\bar{x})), \ldots, \alpha_T(f_s(\bar{x})))\]

where each $\alpha_T(f_i(\bar{x}))$ is in $Pol \lor^\circ \cap P_h$. Hence $F$ is in the clone $C$ and we are done.
The following result improves on Corollary 3.11. It states that, if a clone $C$ above $M$ contains non-trivial (i.e. non-unary) operations in both $\text{Pol} \lor^o$ and $\text{Pol} \land^o$, then it contains $P_h$ where $h$ is the maximum value for which either $\text{Pol} \lor^o \cap P_h \subseteq C$ or $\text{Pol} \land^o \cap P_h \subseteq C$.

**Theorem 3.12** Let $f \in \text{Pol} \lor^o$ be essentially at least binary and assume its image contains $h$ elements, $2 \leq h \leq k$. Let $g \in \text{Pol} \land^o$ be essentially at least binary and assume its image contains $r$ elements, $2 \leq r \leq k$. Then the clone $(M, f, g)$ contains $P_t$ where $t = \max\{h, r\}$.

**Proof.** We shall prove the result for $r \leq h$ (the other case follows easily by dualising the argument). By Corollary 3.11 we may assume without loss of generality that $r < h$. Let $C = \langle M, f, g \rangle$. Let $U = \{1, 2, \ldots, h\}$ and let $V = \{1, 2, \ldots, r\}$. By Theorem 3.1 the clone $C$ contains the operations $J_U$ and $M_V$. By Corollary 3.11 the clone $C$ contains $P_2$, and hence contains the operation

$$f(x, y) = \begin{cases} 1 & \text{if } x \leq r \text{ or } y \leq r, \\ r + 1 & \text{otherwise.} \end{cases}$$

Then $C$ contains the operation

$$\phi(x, y) = J_U(M_V(x, y), f(x, y)).$$

We claim that $\phi = M_D$ where $D = \{1, 2, \ldots, r + 1\}$. Indeed, we have by definition that

$$M_V(x, y) = \begin{cases} x \land y & \text{if } x \leq r \text{ or } y \leq r, \\ r & \text{otherwise.} \end{cases}$$

On the other hand, it easy to see that

$$M_D(x, y) = \begin{cases} x \land y & \text{if } x \leq r \text{ or } y \leq r, \\ r + 1 & \text{otherwise.} \end{cases}$$

Suppose that $x \leq r$ or $y \leq r$. Then $M_V(x, y) = x \land y$ and $f(x, y) = 1$. Thus $\phi(x, y) = J_U(x \land y, 1) = x \land y$. Otherwise we have that $M_V(x, y) = r$ and $f(x, y) = r + 1$ so $\phi(x, y) = J_U(r, r + 1) = r + 1$.

Thus the clone $C$ contains $M_D$ where $D$ contains $r + 1$ elements. If $r + 1 < h$ then repeat the above construction until the operation $M_U$ is shown to be in $C$. By Corollary 3.11 we conclude that $C$ contains $P_h$. 

20
Lemma 3.13 Let \( f \) be an isotone operation not in \( \text{Pol} \lor^0 \). Then \( \text{Pol} \land^0 \cap P_2 \subseteq (M, f) \). (Mutatis mutandis for the dual).

**Proof.** Let \( f \) satisfy the hypothesis of the lemma. Then permuting variables if necessary, we may assume that there exist \( a_i \leq b_i \) in \( k \), \( 1 \leq i \leq n \) such that

\[
\begin{align*}
f(a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n) &= u \\
f(b_1, b_2, \ldots, b_k, a_{k+1}, \ldots, a_n) &= v \\
f(b_1, b_2, \ldots, b_k, b_{k+1}, \ldots, b_n) &= w
\end{align*}
\]

where \( u \lor v \neq w \). Since \( f \) is isotone we actually have that \( u \lor v < w \). For \( 1 \leq i \leq n \) define

\[
f_i(t) = \begin{cases} 
a_i & \text{if } t < k, \\
b_i & \text{otherwise.} \end{cases}
\]

and define

\[
h(t) = \begin{cases} 
1 & \text{if } t \leq u \lor v, \\
k & \text{otherwise.} \end{cases}
\]

Consider the operation defined by

\[
\phi(x, y) = h(f(f_1(x), \ldots, f_k(x), f_{k+1}(y), \ldots, f_n(y))).
\]

Clearly \( \phi \) is in \( (M, f) \). Let \( x = y = k \). Then \( \phi(x, y) = h(f(b_1, \ldots, b_n)) = h(w) = k \). If \( x = k \) and \( y < k \) then \( \phi(x, y) = h(f(b_1, \ldots, b_k, a_{k+1}, \ldots, a_n)) = h(v) = 1 \). If \( x < k \) and \( y = k \) then \( \phi(x, y) = h(f(a_1, \ldots, a_k, b_{k+1}, \ldots, b_n)) = h(u) = 1 \). Finally if \( x < k \) and \( y < k \) then \( \phi(x, y) = h(f(a_1, \ldots, a_n)) \leq h(v) = 1 \). Hence \( \phi(x, y) = k \) if \( x = y = k \) and \( \phi(x, y) = 1 \) otherwise. This is obviously an essentially binary operation in \( \text{Pol} \land^0 \), so by Theorem 3.1 we are done.

**Theorem 3.14** Let \( C \) be a clone containing \( M \) and contained in \( \text{Pol} \leq \). Suppose that \( C \) is contained neither in \( \text{Pol} \lor^0 \) nor in \( \text{Pol} \land^0 \). Let \( 3 \leq h \leq k \). If \( C \) is not contained in \( \text{Pol} \mu_h \) then \( C \) contains \( P_h \).
Proof. Let $C$ be a clone containing $M$ and contained in $Pol \leq$, and suppose that $C$ is contained neither in $Pol \vee^{\circ}$ nor in $Pol \wedge^{\circ}$. To prove the theorem, it will suffice to prove the following equivalent statement:

\[
\text{for all } 3 \leq h \leq k, \text{ if } C \text{ contains } P_{h-1} \text{ and is not } (*) \\
\text{contained in } Pol \mu_h \text{ then } P_h \text{ is contained in } C.
\]

We first prove by induction on $h$ that statement (*) implies our result. Assume that (*) holds for all $3 \leq h \leq k$. Let $h = 3$. By Lemma 3.13 (and its dual) $C$ must contain $Pol \vee^{\circ} \cap P_2$ and $Pol \wedge^{\circ} \cap P_2$. Hence by Theorem 3.12 $C$ contains $P_2$ and we conclude from (*) that $C$ contains $P_3$. Now assume the result holds for $h - 1$. If $C$ is not contained in $Pol \mu_h$ then by Lemma 2.5 (1) $C$ is not contained in $Pol \mu_{h-1}$. By induction hypothesis we then have that $P_{h-1} \subseteq C$. We then conclude from (*) that $C$ contains $P_h$ and we are done.

We now proceed to prove statement (*). Since $C$ contains $P_2$, it will suffice by Theorem 3.12 to find an essentially at least binary operation $\phi \in C$ such that $\phi$ is in $Pol \wedge^{\circ}$ and whose image contains (at least) $h$ elements. There exists an $n$-ary operation $f \in C$ that does not preserve $\mu_h$, i.e. there are elements $a_{ij} \in k$, $1 \leq i \leq n$, $1 \leq j \leq h$ such that $(a_{i1}, \ldots, a_{ih}) \in \mu_h$ for all $i$ and such that $(u_1, \ldots, u_h) = (f(a_{i1}, \ldots, a_{n1}), \ldots, f(a_{ih}, \ldots, a_{nh}))$ is not in $\mu_h$. Notice that by definition of $\mu_h$ we have that $a_{ij} \leq a_{i(j+1)}$ for all $i$ and $j$. But $f \in Pol \leq$ so it follows that $u_1 < u_2 < \ldots < u_h$. Since $C$ contains $M$, we may assume that $u_i = i$ for all $1 \leq i \leq h$ (simply compose $f$ with an operation $g \in M$ that maps $u_i$ to $i$). For each $1 \leq i \leq n$ define an operation $g_i \in M$ as follows:

\[
g_i(j) = \begin{cases} 
 a_{ij} & \text{if } i < h, \\
 a_{ih} & \text{if } i \geq h.
\end{cases}
\]

Let $T = \{1, 2, \ldots, h\}$ and for convenience let $\mathbf{x}$ stand for $(x_1, \ldots, x_n)$. We claim that the following operation is the one we seek:

\[
\phi(\mathbf{x}) = f(g_1 M_T(\mathbf{x}), \ldots, g_n M_T(\mathbf{x}))
\]

where $M_T$ is the ‘partial meet’ operation defined earlier. We will prove that (1) $\phi$ is in $C$, (2) $\phi$ depends on all its variables, (3) the image of $\phi$ contains $T$ and (4) $\phi$ is in $Pol \wedge^{\circ}$.
(1) By definition of $\mu_h$ the set \{\(a_{i_1}, \ldots, a_{i_h}\)\} contains at most \(h - 1\) elements, hence the operation \(g_i M_T\) is in \(P_{h-1}\) for all \(i\). It follows that \(\phi \in C\).

(2) For any \(1 \leq i \leq n\) we have that
\[
\phi(2, 2, \ldots, 2, 1, 2, \ldots, 2) = f(g_1(1), \ldots, g_n(1))
= f(a_{11}, \ldots, a_{n1})
= 1
\]
(where the lone 1 appears in the \(i\)-th place) and
\[
\phi(2, \ldots, 2) = f(g_1(2), \ldots, g_n(2))
= f(a_{12}, \ldots, a_{n2})
= 2.
\]

(3) Let \(1 \leq j \leq h\). Then
\[
\phi(j, \ldots, j) = f(g_1(j), \ldots, g_n(j))
= f(a_{1j}, \ldots, a_{nj})
= j.
\]

Hence the image of \(\phi\) contains \(T\).

(4) We start with a simple observation: for any \(\bar{x} \in \mathbb{K}^n\), there exists \(1 \leq j \leq n\) such that
\[
(g_1 M_T(\bar{x}), \ldots, g_n M_T(\bar{x})) = (a_{1j}, \ldots, a_{nj}).
\]
Notice also that the tuples \((a_{1j}, \ldots, a_{nj})\), \(1 \leq j \leq n\) form a chain in \(\mathbb{K}^n\) (this follows from the definition of \(\mu_h\)).

Suppose for a contradiction that there exist \(\bar{x} = (x_1, \ldots, x_n)\) and \(\bar{y} = (y_1, \ldots, y_n)\) such that \(\phi(\bar{x}) \wedge \phi(\bar{y}) \neq \phi(\bar{x} \wedge \bar{y})\). Since \(\phi\) is isotone it implies that \(\phi(\bar{x} \wedge \bar{y})\) is distinct from \(\phi(\bar{x})\) and \(\phi(\bar{y})\). However, there exist \(j\) and \(r\) such that
\[
(g_1 M_T(\bar{x}), \ldots, g_n M_T(\bar{x})) = (a_{1j}, \ldots, a_{nj})
\]
and
\[
(g_1 M_T(\bar{y}), \ldots, g_n M_T(\bar{y})) = (a_{1r}, \ldots, a_{nr}).
\]
Since these \(n\)-tuples are comparable, assume without loss of generality that
\[
(g_1 M_T(\bar{x}), \ldots, g_n M_T(\bar{x})) \leq (g_1 M_T(\bar{y}), \ldots, g_n M_T(\bar{y})).
\]

23
Hence
\[
\phi(\overline{x} \land \overline{y}) = f(g_1 M_T(\overline{x} \land \overline{y}), \ldots, g_n M_T(\overline{x} \land \overline{y})) \\
= f(g_1 M_T(\overline{x}) \land g_1 M_T(\overline{y}), \ldots, g_n M_T(\overline{x}) \land g_n M_T(\overline{y})) \\
= f(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x})) \\
= \phi(\overline{x}),
\]
and this is a contradiction. Hence \(\phi\) preserves the meet and we are done.

\[\square\]

**Theorem 3.15** Let \(C\) be a clone in the interval \([M, Pol \leq]\). Suppose that \(C\) is not one of \(M, Pol \leq, P_h, Pol \land^c \cap P_h, Pol \land^c \cap P_h, Pol \mu_h\), for any \(h\). Then \(C\) is contained in an interval \([P_h, Pol \mu_{h+1}]\) for some \(3 \leq h \leq k - 1\).

**Proof.** By Corollary 3.10 \(C\) can be contained neither in \(Pol \land^c\) nor in \(Pol \land^c\). Hence \(C\) contains \(Pol \land^c \cap P_2\) and \(Pol \land^c \cap P_2\), by Lemma 3.13. Then by Theorem 3.12 \(C\) contains \(P_2\), which is equal to \(Pol \mu_3\) by Lemma 2.5 (4). Since \(C\) is not equal to \(Pol \mu_3\), it follows by Theorem 3.14 that \(C\) contains \(P_3\). Now let \(h\) be the largest integer such that \(P_h \subseteq C\). Clearly \(h \geq 3\). Since \(C\) does not contain \(P_{h+1}\), we conclude from Theorem 3.14 again that \(C\) is contained in \(Pol \mu_{h+1}\), which concludes the proof.

\[\square\]

4 The case \(k = 4\)

(In the following we shall assume throughout that \(k = 4\).) We shall now prove Theorem 1.1. By Theorem 3.15 it will suffice to prove that \(Pol \mu_4\) covers \(P_3\) (Lemma 4.5). We start with a few basic remarks concerning relations \(\theta\) such that \(P_3 \subseteq Pol \theta\).

Let \(\theta\) be an irredundant relation of arity \(r \geq 2\) such that \(M \subseteq Pol \theta\). By Lemma 2.1, there exists a partial ordering \(\langle \underline{r}, \sqsubseteq \rangle\) of the indices \(\{1, 2, \ldots, r\}\) such that \(i \sqsubseteq j\) iff \(\theta_{ij} \leq \). By permuting the indices of \(\theta\) (this does not affect the clone \(Pol \theta\)) we may assume that the natural ordering \(\underline{r}\) is a linear extension of \(\langle \underline{r}, \sqsubseteq \rangle\). We shall say that an \(r\)-tuple \(\overline{a} = (a_1, \ldots, a_r)\) respects the ordering of \(\theta\) if \(a_i \leq a_j\) whenever \(i \sqsubseteq j\).

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24
Lemma 4.1 Let $M \subseteq \text{Pol } \theta$, where $\theta$ is an irredundant $r$-ary relation. Then $P_h \subseteq \text{Pol } \theta$ if and only if $\theta$ contains every $\overline{a}$ which respects the ordering of $\theta$ and $|\{a_1, \ldots, a_r\}| \leq h$.

Proof. ($\Rightarrow$) For $i \subseteq j$ ($i \neq j$) we may find an element $w \in \theta$ such that $w_i < w_j$. For every pair of incomparable elements $i$ and $j$ in $\langle \overline{r}, \sqsubseteq \rangle$, we may find elements $u$ and $v$ in $\theta$ such that $u_i < v_j$ and $v_i > v_j$. Consider the matrix $X$ whose columns are all these tuples, say of size $r \times m$. Certainly the rows of $X$ form a subposet of $k^m$ isomorphic to $\langle \overline{r}, \sqsubseteq \rangle$.

Let $\overline{a} = (a_1, \ldots, a_r)$ be an $r$-tuple that respects the ordering of $\theta$ and such that $|\{a_1, \ldots, a_r\}| \leq h$. Then the map $f$ which sends row $i$ of matrix $X$ to $a_i$ is isotone. By the extension lemma, there is an isotone map $g$ that extends $f$ and whose image contains at most $h$ elements. Hence this map is in $P_h$.

Since the columns of $X$ are in $\theta$, it follows that $\overline{a}$ must also be in $\theta$.

($\Leftarrow$) Let $f \in P_h$. If $f$ is unary then we are done. Otherwise let $Y$ be a matrix whose columns are in $\theta$. We must show that $f(Y) \in \theta$. Since $f$ is isotone, $f(Y)$ respects the ordering of $\theta$, and since $f$ is not essentially unary $f(Y)$ contains at most $h$ distinct entries. Hence $f(Y) \in \theta$ and we are done.

The following lemma follows from a more general result [5] we shall discuss in the next section. At any rate, the proof of this very special case is not difficult. Let $Q = \langle \overline{4}, \sqsubseteq \rangle$ be an ordering of $\{1, 2, 3, 4\}$. Consider the 4-ary relation $\mu_Q$ consisting of all $(a_1, a_2, a_3, a_4)$ that satisfy (i) $a_i \leq a_j$ if $i \subseteq j$ and (ii) $|\{a_1, a_2, a_3, a_4\}| \leq 3$. Notice that $\mu_4 = \mu_Q$ when $Q$ is the usual ordering of $4$. The next lemma states that an operation preserves $\mu_Q$ if and only if $f$ is unary or the image of any copy of $Q$ under $f$ contains at most 3 elements.

Lemma 4.2 An $n$-ary operation $f$ is in $\text{Pol } \mu_Q$ if and only if either (i) $f$ is unary or (ii) $|f(\epsilon(Q))| \leq 3$ for any isotope map $\epsilon : Q \rightarrow 2^n$.

Let $\theta$ be an $r$-ary relation such that $P_3 \subseteq \text{Pol } \theta \subseteq \text{Pol } \mu_4$. Let $\langle \overline{r}, \sqsubseteq \rangle$ denote the ordering of the indices described above. Let $\overline{a}$ be an $r$-tuple. We shall say that $\overline{a}$ is fine for $\theta$ if it satisfies the following condition: if $a_i = 2$ and $a_j = 3$ then $i$ and $j$ are incomparable in $\langle \overline{r}, \sqsubseteq \rangle$. 25
Lemma 4.3 Let $\theta$ be an irredundant $r$-ary relation such that $P_{3} \subseteq \text{Pol} \theta$. Then $\text{Pol} \mu_4 \subseteq \text{Pol} \theta$ if and only if $\theta$ contains every $\bar{\alpha}$ which respects the ordering of $\theta$ and is fine for $\theta$.

Proof. $(\Rightarrow)$ Suppose that $\text{Pol} \mu_4 \subseteq \text{Pol} \theta$. Notice that $M \subseteq \text{Pol} \theta$. Proceeding just as in the proof of Lemma 4.1 we may find a matrix $X$ whose columns are in $\theta$ and whose rows $\{\bar{\pi}_1, \ldots, \bar{\pi}_r\}$ form a subposet of $k^n$ isomorphic to $\langle \mathcal{L}, \sqsubseteq \rangle$, the ordering of $\theta$. Let $\bar{\pi}$ be an $r$-tuple which respects the ordering of $\theta$ and which is fine for $\theta$. Define an operation as follows:

$$f(\bar{\gamma}) = \begin{cases} a_i & \text{if } \bar{\gamma} = \bar{\pi}_i, \\
4 & \text{if } \bar{\gamma} > \bar{\pi}_i \text{ with } a_i > 1, \\
1 & \text{otherwise.} \end{cases}$$

Clearly this map is isotone and $f(X) = \bar{\pi}$. Since $\bar{\pi}$ is fine for $\theta$, $f$ maps chains to at most 3 elements and hence by Lemma 4.2 it belongs to $\text{Pol} \mu_4$. Since the columns of $X$ are in $\theta$ it follows that $\bar{\pi} \in \theta$.

$(\Leftarrow)$ Suppose that $\theta$ contains all tuples which satisfy the desired conditions. By Lemma 4.1 $\theta$ also contains every $\bar{\pi}$ which respects its ordering and such that $|\{a_1, \ldots, a_r\}| \leq 3$. Let $f \in \text{Pol} \mu_4$. If $f$ is unary then we are done. Otherwise we may suppose by Lemma 4.2 that $f$ maps every chain to at most 3 elements. Let $X$ be a matrix with columns in $\theta$. We must show that $\bar{\pi} = f(X) \in \theta$. Clearly $\bar{\pi}$ respects the ordering of $\theta$ since $f$ is isotone. If $|\{a_1, \ldots, a_r\}| \leq 3$ then we are done, so we may suppose that $f$ is onto. In particular, it is clear that $f(1, \ldots, 1) = 1$ and $f(4, \ldots, 4) = 4$. Then $\bar{\pi}$ must be fine for $\theta$; indeed, suppose the contrary so that $a_i = 2$ and $a_j = 3$ where $i$ and $j$ are comparable in $\langle \mathcal{L}, \sqsubseteq \rangle$. Since $f$ is isotone this implies that $i \sqsubseteq j$, which means that $\bar{\pi}_i \leq \bar{\pi}_j$ where $\bar{\pi}_l$ denotes the $l$-th row of $X$. But then $f$ maps the chain $\{(1, \ldots, 1), \bar{\pi}_i, \bar{\pi}_j, (4, \ldots, 4)\}$ onto 4 elements, a contradiction.

We define two relations of arity 4 on $k$: let $\xi$ consist of all 4-tuples $(a_1, a_2, a_3, a_4)$ such that (i) $a_1 \leq a_i \leq a_4$ for every $i$ and (ii) $|\{a_1, a_2, a_3, a_4\}| \leq 3$. Let $\beta = \xi \cup \{(1, 3, 2, 4)\}$. (Note that $\xi = \mu_Q$ where $Q$ is described by $1 \sqsubseteq i \sqsubseteq 4$ for all $i$).

Lemma 4.4 $\text{Pol} \xi = \text{Pol} \beta = P_3$. 

26
**Proof.** By Lemma 4.1 we have that $P_3 \subseteq Pol\xi$ and $P_3 \subseteq Pol\beta$. Next we show that $Pol\xi \subseteq P_3$ using Lemma 4.2. Let $f \in Pol\xi$; if $f$ is unary we are done. Otherwise, suppose for a contradiction that $f$ is onto. Then certainly $f(1, \ldots, 1) = 1$ and $f(4, \ldots, 4) = 4$ and it follows that $f$ will either map a chain or a copy of $Q$ onto 4 elements, which is impossible.

Now suppose that there is some $f$ in $Pol\beta$ which is not in $Pol\xi$. This means there exists a matrix $X$ with columns in $\xi$ such that $f(X)$ is not in $\xi$. Since $f \in Pol\beta$ and $\beta$ contains $\xi$ it follows that $f(X) = (1, 3, 2, 4)^T$. Now consider the matrix $Y$ obtained from $X$ by exchanging the two middle rows. Clearly the columns of $Y$ are in $\xi$ and hence in $\beta$; however, $f(Y) = (1, 2, 3, 4)^T$ which is not in $\beta$, a contradiction.

\[\]

**Lemma 4.5** Let $C$ be a clone such that $P_3 \subseteq C \subseteq Pol\mu_4$. Then $C = P_3$ or $C = Pol\mu_4$.

**Proof.** We may write $C = \cap_{\theta_i \in \theta} Pol\theta_i$ where each $\theta_i$ is irredundant. If $C \neq Pol\mu_4$ then there is some $i$ such that $Pol\mu_4 \not\subseteq Pol\theta_i$. For convenience let $\theta = \theta_i$. We shall show that $Pol(\theta, \leq) = P_3$, from which $C = P_3$ follows. Let $r$ denote the arity of $\theta$ and let $\langle [\xi, \sqsubseteq'] \rangle$ denote the partial ordering of the indices of $\theta$. By Lemma 4.1 $\theta$ must contain every $\theta$ which respects this ordering and such that $|\{b_1, \ldots, b_r\}| \leq 3$. In particular $r \geq 4$. By Lemma 4.3 there exists a tuple $\vec{a} = (a_1, a_2, \ldots, a_r)$ which respects the ordering of $\theta$ and which is fine for $\theta$ such that $\vec{a} \not\in \theta$. We construct a 4-ary relation as follows: let $\rho$ consist of all tuples $\vec{x} = (x_1, x_2, x_3, x_4)$ such that $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) \in \theta$. It is clear that $Pol\theta \subseteq Pol\rho$.

**Claim 1.** $(1, 2, 3, 4) \not\in \rho$.

Indeed, if $x_i = i$ for all $i$ then $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) = (a_1, a_2, \ldots, a_r)$ which is not in $\theta$.

Let $\langle [\xi, \sqsubseteq'] \rangle$ denote the partial ordering of the indices of $\rho$. Also, let $Q = \langle [\xi, \sqsubseteq] \rangle$ denote the partial ordering defined by $1 \sqsubseteq i \sqsubseteq 4$ for all $i$ (i.e. this is the ordering of the relation $\xi$ defined earlier).

**Claim 2.** $\langle [\xi, \sqsubseteq'] \rangle$ admits $\langle [\xi, \sqsubseteq] \rangle$ as an extension, i.e. if $i \sqsubseteq' j$ then $i \sqsubseteq j$.

It is easy to see it suffices to show that $(1, 2, 3, 3)$ and $(1, 3, 2, 3)$ belong to $\rho$. By the definition of $\rho$, if $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) \in \theta$ then $\vec{x} = (1, 2, 3, 3) \in \rho$. Since there are only three distinct entries, it suffices to prove that $\vec{x}$ respects

27
the ordering of $\theta$. Now clearly $x_j = a_j$ if $j = 1, 2, 3$ and $x_4 = 3$ implies that $\pi$ is obtained from $\pi'$ by replacing occurrences of 4 by 3. If $i \preceq j$ then $a_i \leq a_j$ and hence $x_i \leq x_j$. Now consider the case of $(1, 3, 2, 3)$. As above it suffices to show that $\pi$ respects the ordering of $\theta$. Now $\pi$ is obtained from $\pi'$ as follows: replace all occurrences of 2 by 3 and occurrences of 3 by 2, then replace all occurrences of 4 by 3. Let $i \preceq j$. Then $a_i \leq a_j$ and since $\pi'$ is fine for $\theta$, either $a_i \neq 2$ or $a_j \neq 3$. It is easy to see that $x_i \leq x_j$ (the correspondence $a_i \mapsto x_i$ is order-preserving except for the pair $(2, 3)$).

We construct a 4-ary relation as follows: let $\gamma$ consist of all $(x_1, x_2, x_3, x_4)$ in $\rho$ such that $x_1 \leq x_i \leq x_4$ for all $i$. Clearly $\Pol(\theta, \leq) \subseteq \Pol \gamma$. Hence to finish our proof it will suffice to prove $\Pol \gamma = P_3$. To do this, we prove that $\gamma$ is one of $\xi$ or $\beta$ and invoke Lemma 4.4.

**Claim 3.** $\gamma = \xi$ or $\gamma = \beta$.

Indeed: by Claim 2 and its proof, it is easy to see that the ordering of $\gamma$ is $Q$. By Lemma 4.1 $\gamma$ contains every tuple that respects $Q$ and has at most 3 entries. The only other tuples that can be in $\gamma$ are $(1, 2, 3, 4)$ and $(1, 3, 2, 4)$. By Claim 1, $(1, 2, 3, 4) \notin \gamma$. Hence $\gamma = \xi$ if it does not contain $(1, 3, 2, 4)$ and $\gamma = \beta$ otherwise.

## 5 Comments on the structure of the interval for $k \geq 5$

It appears that the structure of the interval $[M, \Pol \leq]$ is much more complicated for $k \geq 5$ than the cases $k = 3$ and $k = 4$ would let us believe. Indeed, consider the following generalisation of the relation $\mu_h$: let $3 \leq r$ and $h \geq 2$. Let $Q = \langle L, \sqsubseteq \rangle$ be a partial ordering and define $\mu_{Q,h}$ as the set of all $r$-tuples $\pi$ that respect the ordering $Q$ and such that $|\{a_1, \ldots, a_r\}| \leq h$. It is clear that we may suppose that $h < \max\{r, k\}$, otherwise $\Pol \leq$ is contained in $\Pol \mu_{Q,h}$. If $Q$ is an $h + 1$-element chain then of course we find $\mu_{Q,h} = \mu_{h+1}$ and if $Q$ is an antichain then $\Pol \mu_{Q,h}$ is a Burle clone. From now on we shall assume without loss of generality that there is always at least some comparability in $Q$. 

28
Lemma 5.1 Let $\mu_{Q,h} = \cap_{\alpha \in A} \text{Pol} \, \alpha$ where $A$ is the set of all restrictions of $\mu_{Q,h}$ to $h + 1$ indices. Moreover, each $\alpha \in A$ is of the form $\alpha = \mu_{Q,h}$ for some partial ordering $Q'$.

Proof. The inclusion $\subseteq$ is trivial. Now let $f$ be an $n$-ary operation that preserves every $\alpha \in A$ and let $X$ be an $r \times n$ matrix whose columns are in $\mu_{Q,h}$. Since $Q$ is non-trivial $f$ is isotone. Hence $f(X)$ respects $Q$. If $|f(X)| > h$ then there must be a subset $I$ of $r$ with $h + 1$ elements such that $|f(X')| > h$ where $X'$ is the matrix obtained from $X$ by deleting rows whose index is not in $I$. Hence $f$ does not preserve $\alpha$, the restriction of $\mu_{Q,h}$ to $I$, and this is a contradiction.

For the second statement: Let $I$ be a subset of $\mathbb{r}$ with $h + 1$ elements. We prove that $(\mu_{Q,h})^I = \mu_{Q',h}$ where $Q'$ is the restriction of $Q$ to $I$. The inclusion $\subseteq$ is easy. Now let $\overline{b}$ respect the ordering $Q'$ and $|\{b_1, \ldots, b_{h+1}\}| \leq h$. Consider the partial map $i \mapsto b_i$ from $\mathbb{r}$ to $\mathbb{k}$. By the extension lemma, there exists an isotone map $i \mapsto a_i$ from $\mathbb{r}$ to $\mathbb{k}$ that extends $\overline{b}$ and such that $\pi \in \mu_{Q,h}$.

If $Q$ is an ordering of $h + 1$ then we denote $\mu_{Q,h}$ simply by $\mu_Q$.

Lemma 5.2 Let $Q$ be an ordering of $h + 1$. Then $\mu_Q = \cap_{Q' \in \mathbb{B}} \text{Pol} \, \mu_{Q'}$ where $\mathbb{B}$ is the set of all bounded extensions $Q'$ of $Q$.

Proof. As in the previous result we need only prove that if $f$ is an $n$-ary operation that preserves $\mu_Q$ for every $Q' \in \mathbb{B}$ then $f$ preserves $\mu_Q$. Certainly $f$ is isotone. Let $X$ be an $(h + 1) \times n$ matrix whose columns are in $\mu_Q$. Then $f(X)$ respects $Q$. Now suppose that $|f(X)| = h + 1$. Let $\pi_i$ and $\pi_j$ be the rows of $X$ such that $f(\pi_i) = \min \{f(X)\}$ and $f(\pi_j) = \max \{f(X)\}$. Consider the new matrix $X'$ obtained from $X$ by replacing $\pi_i$ by the tuple $(u_1, \ldots, u_n)$ where $u_l$ is the least element appearing in column $l$, and replacing $\pi_j$ by the tuple $(v_1, \ldots, v_n)$ where $v_l$ is the greatest element appearing in column $l$. We claim that the columns of $X'$ are in $\mu_Q$. Since $f$ is isotone and one-to-one on $X$ it is clear by definition of $i$ and $j$ that $l \subset i$ for no $l$ and $j \subset l$ for no $l$. It follows that each column respects $Q$. If column $l$ of $X'$ is equal to column $l$ of $X$ then of course it is in $\mu_Q$; otherwise it means that column $l$ of $X'$ must contain a repetition and hence is in $\mu_Q$. Now consider the ordering $Q'$

29
obtained from $Q$ by adding the comparibilities $i \sqsubseteq m \sqsubseteq j$ for all $m$. This is obviously a bounded extension of $Q$, and it is clear that the columns of $X'$ are all in $\mu_{Q'}$. But since $f$ is isotone it is clear that $|f(X')| = h + 1$ so $f(X') \notin \mu_Q$, a contradiction.

There is a nice characterisation of the operations in $Pol \mu_{Q,h}$ which helps in comparing these clones. It is a generalisation of a result of Jablonskii [2] (see also [10], p. 152) which we mentioned before Lemma 4.2. Notice that the two previous lemmas allow us to reduce the proof of this result to the case $Pol \mu_{Q'}$ where $Q'$ is bounded.

**Lemma 5.3** [5] An $n$-ary operation $f$ is in $Pol \mu_{Q,h}$ if and only if either (i) $f$ is unary or (ii) $|f(e(Q))| \leq h$ for any isotone map $e: Q \to (h + 1)^n$.

These results show that it suffices to consider clones of the form $Pol \mu_{Q'}$ where $Q'$ is a bounded ordering of $\mathbb{R}$ if we want to classify the clones $Pol \mu_{Q,h}$. Moreover, notice that as a result, there are only finitely many clones $Pol \mu_{Q,h}$. On the other hand, it would appear that these are not the only clones in the interval $[M, Pol \leq]$. Furthermore, for large $k$, even the poset of clones $Pol \mu_{Q'}$ seems difficult to characterize. As a simple example, consider, for any $k \geq 6$, the partial ordering $Q$ of $\{1, 2, 3, 4, 5\}$ given by $1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq 4$. It is a simple exercise to verify that $P_4 \subset Pol \mu_Q \subset Pol \mu_5$, and that in fact the clones $Pol \mu_Q$ and $P_5 \cap Pol \mu_5$ are incomparable elements of $[M, Pol \leq]$.

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