The Continuing Story of Zeta

Graham Everest, Christian Röttger and Tom Ward

February 2, 2008

1. TAKING THE LOW ROAD. Riemann’s Zeta Function $\zeta(s)$ is defined for complex $s = \sigma + it$ with $\Re(s) = \sigma > 1$ by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

There are many ways to obtain the analytic continuation of $\zeta(s)$ to the left hand half-plane. The high road, Riemann’s own [10], uses contour integration at an early stage, and leads directly to the functional equation. Many authors ([1], [3], [4], [8], [9], [12], and [13]) use this method, or variants of it, often at a more leisurely pace. Other methods are known (Chapter 2 of [12] lists seven) but a toll seems inevitable on any route ending with the functional equation.

There are lower roads which give both the continuation to the whole plane and the evaluation at non-positive integers but stop short of proving the functional equation. If these are rigorous, yet quick and simple, there must surely be a case for using them as well. The point of this article to draw wider attention to these, often very scenic, roads. In his beautiful article [2, Sect. 7], Ayoub comments upon Euler’s paper of 1740 in which he boldly evaluates divergent series to obtain $\zeta(-k)$ for integers $k \geq 0$, thereby predicting the functional equation. Recently, Sondow [11] has noted one way in which Euler’s argument can be made rigorous. Simultaneously, Minač [6] showed how to evaluate $\zeta(-k)$ in an extremely simple and elegant way, by integrating a polynomial on $[0, 1]$. More recently, Murty and Reece [7] have shown how the continuation and evaluation of the Hurwitz zeta function can be obtained in a simple down-to-earth way and this is applicable to $\zeta(s)$ and many $L$-functions. The point of this note is to highlight just how easily the continuation and evaluation of $\zeta(s)$ can be obtained. All that we say can be found in the articles cited. For example, our work-horse [10] is the truncation of Landau’s formula [5, p. 274].

2. A JOURNEY OF A THOUSAND MILES... Notice that for $\sigma > 1$,

$$\int_1^{\infty} x^{-s} \, dx = \frac{-1}{1-s} = \frac{1}{s-1},$$

which yields at once the continuation to the whole complex plane of the function represented by the integral for $\sigma > 1$. Obviously the continuation is analytic
everywhere apart from a simple pole at \( s = 1 \). For \( \sigma > 1 \),

\[
\frac{1}{s - 1} = \int_1^\infty x^{-s} \, dx = \sum_{n=1}^{\infty} \int_n^{n+1} x^{-s} \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_0^1 (n+x)^{-s} \, dx = \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^1 \left(1 + \frac{x}{n}\right)^{-s} \, dx.
\] (1)

All the sums converge absolutely for \( \sigma > 1 \). In what follows we assume that \( \sigma > 1 \) and that \( |s| \) is bounded by \( K \), a fixed (although arbitrary) constant. Now begin the binomial expansion of the bracketed term, noting that the higher binomial coefficients all include a factor \( s \):

\[
\left(1 + \frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + sE_1(s, x, n),
\] (2)

where the function \( E_1(s, x, n) \) satisfies

\[
|E_1(s, x, n)| \leq \frac{C_1 x^2}{n^2} \leq \frac{C_1}{n^2},
\] (3)

for all \( x \in [0, 1] \) and all \( n \geq 1 \), with \( C_1 = C_1(K) \) (since \( E_1 \) is just the error term of a Taylor series in \( x/n \)). Substitute Equation (2) into the sum (1) and perform the integration with respect to \( x \). We find that

\[
\frac{1}{s - 1} = \zeta(s) - \frac{s}{2} \zeta(s+1) + sA_1(s),
\] (4)

where \( A_1(s) \) is analytic for \( \sigma > -1 \) by (3). Thus Equation (4) may be used to extend \( \zeta(s) \) to the half-plane \( \sigma > 0 \). It even shows that the extended function will be analytic there apart from a simple pole at \( s = 1 \) with residue 1. In other words, Equation (4) implies that

\[
\lim_{s \to 1^-} (s - 1) \zeta(s) = 1.
\] (5)

Equation (5) can also be written \( \lim_{s \to 0^+} s \zeta(s+1) = 1 \). Using this fact, and letting \( s \to 0^+ \) in Equation (4), we obtain

\[
-1 = \zeta(0) - \frac{1}{2},
\]

which yields the known value \( \zeta(0) = -1/2 \).

The preceding argument begins with the binomial estimate (2), finds the analytic continuation of the zeta function to the half-plane \( \sigma > 0 \) and evaluates \( \zeta(0) \). What happens if more terms of the binomial expansion are included? An additional term in the binomial expansion gives

\[
\left(1 + \frac{x}{n}\right)^{-s} = 1 - \frac{sx}{n} + \frac{s(s+1)x^2}{2n^2} + (s+1)E_2(s, x, n);
\]
notice that the higher binomial coefficients all include a factor \((s + 1)\). Here, \(E_2\)
is a function which satisfies

\[ |E_2(s, x, n)| \leq \frac{C_2}{n^3} \leq \frac{C_2}{n^3}, \]

for all \(x \in [0, 1]\) and all \(n\), where \(C_2 = C_2(K)\). Substituting this into \((1)\) and integrating as before gives

\[ \frac{1}{s - 1} = \zeta(s) - \frac{s}{2} \zeta(s + 1) + \frac{s(s + 1)}{6} \zeta(s + 2) + (s + 1)A_2(s), \]

where \(A_2(s)\) is analytic for \(\sigma > -2\). Thus, Equation \((6)\) may be used to continue \(\zeta(s)\) to the half-plane \(\sigma > -1\). As before, letting \(s \to -1^+\) and using Equation \((5)\), we obtain

\[ -\frac{1}{2} = \zeta(-1) + \frac{1}{2} \zeta(0) - \frac{1}{6} = \zeta(-1) - \frac{1}{4} - \frac{1}{6} \]
yielding the known value \(\zeta(-1) = -1/12\).

3. GENERAL METHOD. This method can be repeated in order to continue \(\zeta(s)\) further and further to the left of the complex plane. Moreover, it yields the explicit evaluation at the non-positive integers in terms of the Bernoulli numbers. The sequence of Bernoulli numbers \((B_n)\) is defined via the generating function

\[ \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \]

from which it is clear that all the \(B_n\) are rational numbers. We need two well-known properties of this fascinating sequence which are stated in the following lemma.

Lemma 3.1. With \(B_n\) defined by \((7)\),

\[ \sum_{n=0}^{N-1} \binom{N}{n} B_n = 0 \quad \text{for all } N > 1, \]

and

\[ B_n = 0 \quad \text{for all odd } n \geq 3. \]

Proof. The relation \((7)\) can be written

\[ (e^x - 1) \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = x. \]

For \(N > 1\) the coefficient of \(x^N\) in the left-hand side is

\[ \sum_{m=0}^{N-1} \frac{1}{(N-m)!m!} B_m, \]
which gives (8) after multiplying by $N!$. The second statement follows from the fact that
\[
\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x(1 + e^x)}{e^x - 1}
\]
is an even function.

The recurrence relation (7) can be used to calculate $B_n$ inductively. The first few Bernoulli numbers are given below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>1</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>0</td>
<td>$-\frac{1}{30}$</td>
<td>0</td>
<td>$\frac{1}{42}$</td>
<td>0</td>
<td>$-\frac{1}{30}$</td>
<td>0</td>
<td>$\frac{5}{66}$</td>
</tr>
</tbody>
</table>

**Theorem 3.2.** There is an analytic continuation of $\zeta(s)$ to the entire complex plane where it is analytic apart from a simple pole at $s = 1$ with residue 1. For all $k \geq 1$,
\[
\zeta(-k) = -\frac{B_{k+1}}{k+1}.
\]

(9)

Note that Equation (9) is not true for $k = 0$ but our method has already given us the special value $\zeta(0) = -1/2$.

**Proof of Theorem 3.2.** The analytic continuation of the zeta function to the half-plane $\sigma > -k$ arises in exactly the same way as before, by extracting an appropriate number of terms of the binomial expansion and using induction. For integral $k \geq 0$ and $\sigma > 1$, this gives the relation
\[
\frac{1}{s - 1} = \zeta(s) + \sum_{r=0}^{k} \frac{(-1)^{r+1}s(s+1)\ldots(s+r)}{(r+2)!} \zeta(s + r + 1) + (s+k)A_{k+1}(s)
\]
(10)

where $A_{k+1}(s)$ is analytic in $\sigma > -(k+1)$, again because all higher binomial coefficients include a factor $(s+k)$. Notice that $k = 0$ gives Equation (1) and $k = 1$ gives Equation (6).

By induction, we may assume that $\zeta(s)$ has already been extended to the half-plane $\sigma > 1-k$ so Equation (10) is valid there, because the singularities at $s = 0, -1, \ldots$ are removable. All the functions in Equation (10) except $\zeta(s)$ are defined at least for $\sigma > -k$, which gives the analytic continuation of $\zeta(s)$ to that half-plane. Let $s \to -k^+$ in (10) and use Equation (5) for the term with $r = k$ to obtain
\[
-\frac{1}{k+1} = \zeta(-k) + \sum_{r=0}^{k-1} \binom{k}{r+1} \frac{\zeta(-k+r+1)}{r+2} - \frac{1}{(k+1)(k+2)}.
\]

Writing $r$ for every $r+1$ simplifies this to
\[
0 = \zeta(-k) + \frac{1}{k+2} + \sum_{r=1}^{k} \binom{k}{r} \frac{\zeta(-k+r)}{r+1}.
\]
The term with \( r = k \) is known. Using induction on the others gives

\[
0 = \zeta(-k) + \frac{1}{k + 2} - \sum_{r=1}^{k-1} \binom{k}{r} \frac{B_{k-r+1}}{(r+1)(k-r+1)} - \frac{1}{2(k+1)}, \tag{11}
\]

A simple manipulation of factorials gives

\[
\frac{(k+1)(k+2)}{(r+1)(k-r+1)} \binom{k}{r} = \binom{k+2}{r+1} = \binom{k+2}{k-r+1},
\]

which transforms Equation (11) to

\[
0 = \zeta(-k) + \frac{k}{2(k+1)(k+2)} - \frac{1}{(k+1)(k+2)} \sum_{r=1}^{k-1} \binom{k+2}{k-r+1} B_{k-r+1}. \tag{12}
\]

Multiply by \((k+1)(k+2)\) and apply Equation (7) with \( N = k+2 \). Only the terms for \( r = 0, k, k+1 \), missing in Equation (12) survive, yielding

\[
0 = (k+1)(k+2)\zeta(-k) + \frac{k}{2} + (k+2)B_{k+1} + (k+2)B_1 + B_0
= (k+1)(k+2)\zeta(-k) + (k+2)B_{k+1}
\]

and this completes the induction argument.

\[\square\]

**ACKNOWLEDGEMENTS.** Our thanks go to Ján Mináč and Robin Kronenberg for helpful comments.

**References**


