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Phase transition and anisotropic deformations of neutron star matter

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The Skyrme model is a low energy, effective field theory for QCD that when coupled to a gravitational field provides an ideal semiclassical model to describe neutron stars. We use the Skyrme crystal solution composed of a lattice of α-like particles as a building block to construct minimum energy neutron star configurations, allowing the crystal to be strained anisotropically. We find that below 1.49 solar masses the stars' crystal deforms isotropically and that above this critical mass, it undergoes anisotropic strain. We then find that the maximum mass allowed for a neutron star is 1.90 solar masses, in close agreement with a recent observation of the most massive neutron star yet found. The radii of the computed solutions also match the experimentally estimated values of approximately 10 km.

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I. INTRODUCTION

Neutron stars are stars that have collapsed under intense self-gravitational pressure to the point where all electrons are squeezed into nuclei, hence forming a large cluster of neutrons with a typical radius of about 10 km. A neutron star can thus be seen as a gigantic nuclei that is electrically neutral but is strongly affected by the gravitational field that it generates. For lack of a unified theory of strong interactions and gravity one has to resort to finding an approximate theory that allows us to describe such a system.

One such theory is the Skyrme model. Originally proposed by Skyrme in 1961 [1,2] as a nonlinear theory of pions to describe strong interactions, it was later shown by Witten [3] to be an approximate, low energy, effective field theory for QCD that becomes more exact as the number of quark colors becomes large.

Each solution of the Skyrme model is characterized by an integer valued topological charge that can be identified with the baryon number B. The simplest solution, $B = 1$, is made out of a so-called Skyrmion and corresponds to a proton or neutron. At the semiclassical level, the Skyrme model does not distinguish between a neutron and a proton. Moreover, as the model does not include the electroweak interaction, all Skyrmions are electrically neutral.

The $B = 1$ solution of the Skyrme model is the only exact stable solution that can be computed easily [1]. Solutions with larger values of B can only be computed numerically [4,5] and these solutions have been shown to successfully describe various nuclei and their properties [6].

Moreover, one can also compute crystal-like solutions made out of an infinite number of Skyrmions. In particular, it has been shown that the Skyrme solution with the lowest energy per Skyrmion corresponds to a cubic lattice where each lattice unit has a topological charge $B = 4$ [7]. These solutions can thus be seen as a crystal of α particles.

This lattice of Skyrmions looks thus as the best building block to describe a neutron star as its has the lowest possible energy per baryon. Yet, one must first estimate if a star could instead be made out of a liquid or gas of Skyrmions. The temperature of a neutron star, a few years after its creation, cools down to an approximate temperature of about $100\ \mathrm{eV} \approx 10^6 K$ [8]. While this looks like a very high temperature compared to the binding energy of an electron around a nucleus, this energy is quite small from a nuclear point of view. Indeed, the lowest excited state of an α particle, for example, is 23.3 MeV [9] and the lowest vibration mode of a $B = 4$ Skyrmion is of the order of 100 MeV [10,11]. Even under intense gravitational energy, Walhout [12] showed that the excitation energy of a lattice of $B = 1$ Skyrmions is also of the order of 100 MeV. This points out that the neutron star will be in a solid phase rather than a liquid or a gas and that the thermal energy will only excite acoustic phonon modes. It is thus natural to model a neutron star as a lattice of $B = 4$ Skyrmions.

Before we proceed we must also question the possibility of having an atmosphere around the star, and to estimate its height if it turns out not to be small. At the surface of a neutron star twice the mass of the sun, the gravitational acceleration is $g \approx 2.6 \times 10^{12}\ \mathrm{ms}^{-2}$. It is then easy to compute that the average height that an α particle with a thermal energy of 100 eV will be able to jump is of the order of 1 mm, i.e. much smaller than the radius of the star. We can thus consider that such an atmosphere is extremely thin and assume in our model that the neutron star is fully made out of a solid.

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Having established that we can use a Skyrme crystal as the building block to describe a neutron star we will proceed as follows. First of all, we will use the equations of state computed by Castillejo *et al.* [7] for the $B = 4$ crystal when the lattice is deformed asymmetrically. Following Walhout [12] we will then use a Tolman-Oppenheimer-Volkoff (TOV) equation [13,14], generalizing it to allow for matter to be anisotropic [15]. The TOV equation describes the static equilibrium between matter forces within a solid or fluid and the gravitational forces self-generated by the matter for a spherically symmetric body.

Combining the TOV equation with the equations of state of the Skyrme crystal, we will be able to find configurations that are spherically symmetric distributions of anisotropically deformed matter in static equilibrium and so are suitable to model neutron stars. Solving these equations numerically for large stars we will show that below a critical mass of 1.49 solar masses ($M_\odot = 1.988\,92 \times 10^{30}$ kg) all neutron/Skyrmion stars are made out of an isotropically strained crystal. We will then show that at this critical mass there is a phase transition and that heavier stars are made out of an anisotropically deformed crystal that is less strained radially than tangentially. We will also show that these stars can have a mass of up to $1.90M_\odot$. Finally, we will investigate the impact of adding a mass term to the Skyrme model and describe what happens to a star when its mass is increased above its maximum value.

Using Skyrmions to model neutron stars is not new and has been performed previously in several ways. First of all, Walhout used a lattice of $B = 1$ Skyrmions [12] to describe a neutron star. He then improved his results by considering a lattice of $B = 4$ Skyrmions [16]. In both cases he assumed an isotropic compression of the lattice, assuming a gaslike phase, and he used numerical solutions of the model to estimate the stress tensor. The maximum mass he obtained for the neutron star was $2.57M_\odot$. Later, Jaikumar and Ouyed [17] considered the equation of state for a neutron star based on a Skyrme fluid and obtained a maximum mass of $3.6M_\odot$. The main difference between these two approaches and ours is that they assumed an isotropic fluid of Skyrmions whereas we consider a solid crystal allowed to deform anisotropically, i.e. be compressed differently in the radial and tangential directions of the star. In our previous papers [18,19], we computed minimal energy Skyrmion stars made out of layers of two-dimensional Skyrme lattices. This allowed us to use the rational map ansatz [20] to minimize the energy directly but resulted in relatively small stars with a maximum mass of $0.574M_\odot$. This was mainly due to the fact that the field transition between the different layers in our ansatz overestimated the energy of the configuration and that the energy per baryon in each layer of the ansatz was also larger than that of the crystal of $B = 4$ Skyrmions.

II. SKYRME CRYSTALS

The Skyrme model [1,2] is described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{Sk} &= \frac{F_\pi^2}{16}\,\mathrm{Tr}(\nabla_\mu U \nabla^\mu U^{-1}) \\ &\quad + \frac{1}{32e^2}\,\mathrm{Tr}[(\nabla_\mu U)U^{-1}, (\nabla_\nu U)U^{-1}]^2 \\ &\quad + \frac{m_\pi^2 F_\pi^2}{8}\,\mathrm{Tr}(U-1), \end{aligned} \tag{1}$$

where here U, the Skyrme field, is an $SU(2)$ matrix and F_π, e, and m_π are the pion decay constant, the Skyrme coupling and the pion mass term, respectively. In the Lagrangian (1) the ∇ are ordinary partial derivatives in the absence of a gravitational field and become covarient derivatives when the Skyrme field is coupled to gravity. The results summarized in this section all relate to the pure Skyrme model without gravity.

The Skyrme field is a map from $\mathbb{R}^3$ to S^3, the group manifold of $SU(2)$, but finite energy considerations imply that the field at spatial infinity should map to the same point, meaning the Skyrme field is a map between two three-spheres. Such maps fall into homotopy classes indexed by an integer, known as the topological charge, which is interpreted as the baryon number, B. The topological soliton solutions, known as Skyrmions, are identified as baryons with an α particle described by a $B = 4$ Skyrmion solution.

Here we will be considering the zero pion mass case where $m_\pi = 0$ with Sec. IV C describing the effects of its inclusion. The two other Skyrme parameters, F_π and e, can be obtained in different ways. Skyrme first evaluated them by taking the experimental value of the pion decay constant $F_\pi = 186$ MeV and then fitting the mass of a Skyrmion to that of a proton and obtained $e = 4.84$. Later Adkins, Nappi, and Witten [21] quantized the $B = 1$ Skyrmion to fit the parameter values to the mass of the nucleon and the delta excitation and obtained $F_\pi = 129$ MeV and $e = 5.45$. These later values were the ones used by Castillejo *et al.* [7] to compute the energy of the deformed $B = 4$ crystal and we will thus use them too.

The solution of the Skyrme model with the lowest energy per baryon has been shown to be a face-centered cubic lattice of Skyrmions [7,22]. Each unit cell is a cube of side length a with a baryon number of $B = 4$ and can therefore be considered as an α particle. In the context of a neutron star, we will be able to interpret each $B = 4$ crystal component as being four neutrons, as the Skyrme model does not distinguish between neutrons and protons.

Castillejo *et al.* [7] also investigated the energy of dense Skyrmion crystals where the configuration was not a face-centered cubic lattice but rather a lattice where the aspect ratio of the unit cell, $B = 4$ α particle, of side a was altered so that it becomes rectangular with aspect ratio r^3. This means that in the x and y directions the lattice size becomes

ra and in the z direction, a/r^2. As in Castillejo *et al.* we use the measure $p = r - 1/r$ to describe the deviation away from the face-centered cubic lattice symmetries that have $p = 0$.

The numerical solutions found in [7] provide an equation for the dependence of the energy of a single Skyrmion, $E(L, p)$, on its size, $L = n^{-1/3}$, where n is the Skyrmion number density, and its aspect ratio measure, p,

$$E(L, p) = E_{p=0}(L) + E_0[\alpha(L)p^2 + \beta(L)p^3 + \gamma(L)p^4 + \delta(L)p^5 + \ldots], \tag{2}$$

where the coefficients are given by

$$E_{p=0}(L) = E_0\left[0.474\left(\frac{L}{L_0} + \frac{L_0}{L}\right) + 0.0515\right], \tag{3}$$

$$\alpha(L) = 0.649 - 0.487\frac{L}{L_0} + 0.089\frac{L_0}{L}, \tag{4}$$

$$\beta(L) = 0.300 + 0.006\frac{L}{L_0} - 0.119\frac{L_0}{L}, \tag{5}$$

$$\gamma(L) = -1.64 + 0.78\frac{L}{L_0} + 0.71\frac{L_0}{L}, \tag{6}$$

$$\delta(L) = 0.53 - 0.55\frac{L}{L_0}. \tag{7}$$

Here $E_0 = 727.4$ MeV and $L_0 = 1.666 \times 10^{-15}$ m. The equation can be extended to include lower densities [7] but they are not of interest here where we are only considering densities higher than the minimal energy crystal. Notice that for any value of L the minimum energy occurs at the face-centered cubic lattice configuration, $p = 0$, and the global minimum is reached for $L = L_0$.

III. TOV EQUATION FOR SKYRMION STARS

Using Eq. (2) relating the energy of a Skyrmion to its size and aspect ratio we will now investigate how one can describe a neutron star using a Skyrme crystal and how this crystal is deformed under the high gravitational field it experiences.

In our numerical work we denote λ_r as the Skyrmion length in the radial direction of the star and λ_t as the Skyrmion length in the tangential direction. These parameters and the parameters L and p used in (2) are related as follows:

$$L = (\lambda_r\lambda_t\lambda_t)^{1/3}, \quad \text{and} \quad p = \left(\frac{\lambda_t}{\lambda_r}\right)^{1/3} - \left(\frac{\lambda_r}{\lambda_t}\right)^{1/3}. \tag{8}$$

To construct a neutron star we consider a spherically symmetric distribution of matter in static equilibrium with a stress tensor that is in general locally anisotropic. Spherical symmetry demands that the stress tensor, T^μ_ν, is diagonal and that all the components are a function of the radial coordinate only. We denote this stress tensor as

$$T^\mu_\nu = \mathrm{diag}(\rho(r), p_r(r), p_\theta(r), p_\phi(r)), \tag{9}$$

and consider that, again due to spherical symmetry, $p_\theta(r) = p_\phi(r)$, which we will denote by $p_t(r) = p_\theta(r) = p_\phi(r)$. The quantities $p_r(r)$ and $p_t(r)$ describe the stresses in the radial and tangential directions, respectively, while the quantity $\rho(r)$ is the mass density.

A generalized TOV equation [13,14] to describe a spherically symmetric star composed of anisotropically deformed matter in static equilibrium has been studied previously [15] and we summarize it now.

The metric for the static spherically symmetric distribution of matter can be written in Schwarzschild coordinates as

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{10}$$

where $e^{\nu(r)}$ and $e^{\lambda(r)}$ are functions of the radial coordinate that need to be determined. The combination of this metric and the matter distribution, described by the stress tensor (9), must be a solution of Einstein's equations

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} = 8\pi T_{ab}, \tag{11}$$

where we have set $G = c = 1$. After calculating the Ricci tensor and Ricci scalar from the metric we find

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\rho, \tag{12}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p_r, \tag{13}$$

$$e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}(\nu')^2 + \frac{(\nu' - \lambda')}{2r}\right) = 8\pi p_t. \tag{14}$$

Equation (12) can be rewritten as

$$(re^{-\lambda})' = 1 - 8\pi\rho r^2 \tag{15}$$

and integrated to give

$$e^{-\lambda} = 1 - \frac{2m}{r}, \tag{16}$$

where $m = m(r)$ is defined as the gravitational mass contained within the radius r and can be calculated by

$$m = \int_0^r 4\pi r^2\rho dr. \tag{17}$$

We can now substitute Eq. (16) for $e^{-\lambda}$ into Eq. (13) to find

$$\frac{1}{2}\nu' = \frac{m + 4\pi r^3 p_r}{r(r - 2m)}. \tag{18}$$

The generalized TOV equation that we will use to find suitable neutron star configurations can now be obtained by differentiating Eq. (13) with respect to r and adding it to Eq. (14) to find

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{\nu'}{2} + \frac{2}{r}(p_t - p_r). \tag{19}$$

Now, substituting (18) into (19), we get

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{m + 4\pi r^3 p_r}{r(r-2m)} + \frac{2}{r}(p_t - p_r). \tag{20}$$

For this generalized TOV equation to be solvable two equations of state need to be specified, $p_r = p_r(\rho)$ and $p_t = p_t(\rho)$, where, as argued above, we are able to use a zero temperature assumption.

We also need to specify appropriate boundary conditions. First, we must require that the solution is regular at the origin and impose that $m(r) \to 0$ as $r \to 0$. Then p_r must be finite at the center of the star implying that $\nu' \to 0$ as $r \to 0$. Moreover, the gradient dp_r/dr must be finite at the origin too and so $(p_t - p_r)$ must vanish at least as rapidly as r when $r \to 0$. This implies that we need to impose the boundary condition $p_t = p_r$ at the center of the star.

The radius of the star, R, is determined by the condition $p_r(R) = 0$, as the radial stress for the Skyrmions on the surface of the star will be negligibly small. The equations, however, do not impose that $p_t(R)$ vanishes at the surface. One should also point out that physically relevant solutions will all have $p_r, p_t \geq 0$ for $r \leq R$. We note that an exterior vacuum Schwarzschild metric can always be matched to our metric for the interior of the star across the boundary $r = R$ as long as $p_r(R) = 0$, even though $p_t(R)$ and $\rho_r(R)$ may be discontinuous, implying that the star can have a sharp edge, as expected from a solid rather than gaseous star.

As we are considering Skyrmion matter at zero temperature the equations of state that will be used in finding suitable neutron star configurations can be calculated from Eq. (2), which depends on the lattice scale L, and aspect ratio, p, which are both functions of the radial distance form the center of the star, r. From the theory of elasticity we then find that the radial and the tangential stresses are related to the energy per Skrymion, Eq. (2), as follows:

$$p_r = -\frac{1}{\lambda_t^2}\frac{\partial E}{\partial \lambda_r}, \quad \text{and} \quad p_t = -\frac{1}{\lambda_r}\frac{\partial E}{\partial \lambda_t^2}. \tag{21}$$

Using the generalized TOV Eq. (20) and the two equations of state (21), a minimum energy configuration for various values of the total baryon number can be calculated numerically. The minimum energy configuration is defined as the minimum value of the gravitational mass, M_G,

$$M_G = m(R) = m(\infty) = \int_0^R 4\pi r^2 \rho dr, \tag{22}$$

where R is the total radius of the star and

$$\rho = \frac{E}{\lambda_r \lambda_t^2 c^2}. \tag{23}$$

We now need to minimize M_G as a function of λ_r and λ_t, which both depend on r. To achieve this, we will first assume a profile for $\lambda_t(r)$ and compute M_G for this profile as described below. We will then determine the configuration of the neutron star, with a specific baryon charge, by minimizing M_G over the field λ_t. This can be easily done using the simulated annealing algorithm.

To compute M_G we notice that at the origin, one can use (21) to determine $p_r(0)$ and $p_t(0)$ from the initial values of $\lambda_r(0)$ and $\lambda_t(0)$. Then the integration steps can be performed as follows. Knowing $\lambda_r(r)$ and $\lambda_t(r)$ one computes $\rho(r)$ using (23) and $m(r)$ using (17). Then, knowing $p_r(r)$, $p_t(r)$, $\rho(r)$, and $m(r)$ one can integrate (20) by one step to determine $p_r(r + dr)$. One can then use (21) to determine $\lambda_r(r + dr)$ and as the profile for $\lambda_t(r)$ is fixed, one can proceed with the next integration step.

One then integrates (20) up to the radius R for which $p_r(R) = 0$; this sets the radius of the star. In our integration, we used a radial step of 50 m.

One must then evaluate the total baryon charge of the star using

$$B = \int_0^R \frac{4\pi r^2 n(r)}{(1 - \frac{2Gm}{c^2 r})^{1/2}} dr, \tag{24}$$

where

$$n(r) = \frac{1}{\lambda_r(r)\lambda_t(r)^2} \tag{25}$$

and rescale λ_t to restore the baryon number to the desired value. One then repeats the integration procedure until the baryon charge reaches the correct value without needing any rescaling.

IV. RESULTS

A. Stars made of isotropically deformed Skyrme crystal

We found that up to a baryon number of 2.61×10^{57}, equivalent to $1.49 M_\odot$, the minimum energy configurations are all composed of Skyrme crystals that are isotropically deformed, with $\lambda_t(r) = \lambda_r(r)$ across the whole radius of the star. It can be shown that this indeed has to be the case as we can prove that if it is possible to find an isotropic Skyrme crystal solution then that solution will be the minimum energy configuration. Such isotropic solutions can only be found up to a baryon number of 2.61×10^{57}. Corchero [23] used a similar proof for a quantum model of neutron stars and we adapt this here for our Skyrme crystal model:

If there is a locally isotropic, stable solution to the generalized TOV equation (19) with mass M and total baryon number N, then all locally anisotropic solutions that have the same total baryon number N, in the neighborhood of that stable solution, will have a mass not smaller than M.

To prove this, we first note that stable solutions for a given baryon number at zero temperature, by definition, have a mass that is not greater than that which could be achieved by any variation of the density that preserves the baryon number [24].

We then consider two changes to a stable solution with mass M_1, baryon number N_1, density $\rho_1(r)$, and number density $n_1(r)$. The first involves changing the density from $\rho_1(r)$ to $\rho_2(r)$ while keeping the total baryon number constant and preserving locally isotropy. This will result in a configuration that has a mass M_2 that is greater than or equal to the mass M_1 of our initial configuration as that was defined as the minimum mass solution. This new configuration will have a different number density, $n_2(r)$, but the same total baryon number, N_1, by assumption.

The second change involves introducing local anisotropy while the mass, M_2, remains the same, as does the density, $\rho_2(r)$. In order to keep $\rho_2(r)$ constant when we alter the configuration so that it is now made of anisotropic Skyrme crystal the number density must also be altered. A change from an isotropic to an anisotropic Skyrme crystal involves increasing its energy, and therefore mass, so to keep its mass density constant we need to reduce the number of Skyrmions to the new number density $n_3(r) \leq n_2(r)$, meaning the total baryon number is now $N_3 \leq N_1$.

The two changes described have the effect of first increasing M without changing N and then, second, decreasing N without altering M. We know that M is a monotonically increasing function of N for isotropic Skyrme crystal stable star configurations, so we have proved that moving from isotropic to anisotropic Skyrme crystal configurations increases the energy for a given baryon number so does not produce a minimum energy solution.

The above proof, however, does not rule out the existence of anisotropic Skyrme crystal solutions for those baryon numbers for which there does not exist an isotropic Skyrme crystal solution and such configurations will be discussed in the next section.

To confirm the results obtained for isotropically deformed crystals, we will now determine the properties of these symmetric stars by imposing that symmetry, i.e. $p_t = p_r$. In this case the problem simplifies greatly and the TOV equation (20) reduces to

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{m + 4\pi r^3 p_r}{r(r - 2m)}. \tag{26}$$

Using this standard TOV equation, a central Skyrmion length $\lambda_t(r = 0) = \lambda_r(r = 0) = L(r = 0)$ can be specified at the center of the star. The equation can then be numerically integrated over the radius of the star using the Skyrmion energy Eq. (2) with

$$p_r = -\frac{\partial E}{\partial \lambda_r^3}, \tag{27}$$

where, as we are only considering isotropic Skyrme crystal deformations, $\lambda_t = \lambda_r$ and $p = 0$ in the energy equation. This was done using a fourth order Runge Kutta method over points every 20 m. Notice that this did not require the explicit minimization of M_G. Figure 1 shows a plot of the total baryon number of the star against its Skyrmion length at the center, $L(r = 0)$, calculated using this method.

We found that isotropic Skyrme crystal solutions can be found up to a baryon number of 2.61×10^{57}, which is equivalent to a mass of $1.49M_\odot$. This agrees with the results that we found from our minimization procedure using the generalized TOV equation that allows for anisotropic Skyrme crystal deformations.

Table I shows some of the properties of the minimum energy solutions for various baryon numbers obtained from the energy minimization of the generalized TOV equation. The results are in perfect agreement with the results obtained by solving the isotropic TOV equation (26). The quantity $S_{\min}$ is the minimum value, over the radius of the star, of

$$S(r) = e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}, \tag{28}$$

which appears in the static, spherically symmetric metric (10) that we are considering. The zeros of $S(r)$ correspond to singularities in the metric, or in other words, to horizons. Had $S_{\min}$ been negative, we would have concluded that the neutron star would have collapsed into a black hole, but this never occurred.

We note that the solutions are energetically favorable as the energy per baryon decreases when the total baryon number increases, indicating that the solutions are stable. They correspond to the solutions to the right of the maximum in Fig. 1 with solutions to the left being unstable with a higher energy per baryon for a given baryon number and therefore are not found by the energy minimization procedure.

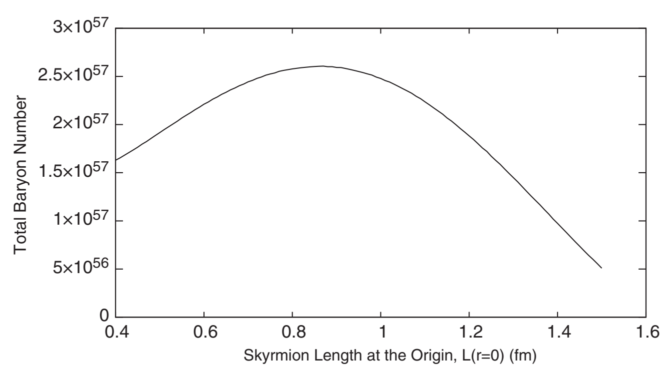

FIG. 1. Total baryon number as a function of the size of the Skyrmions at the center of the star, $L(r = 0)$.

TABLE I. Properties of the isotropic minimum energy neutron star configurations for various baryon numbers.

B	Total energy (J)	Energy/B (J)	Mass/$M_\odot$	R(m)	$S_{\min}$
1.0×10^{55}	1.16210×10^{45}	1.16210×10^{-10}	0.00649160	2219.20	0.991503
1.0×10^{56}	1.15114×10^{46}	1.15114×10^{-10}	0.0643083	4714.35	0.959976
2.0×10^{56}	2.28551×10^{46}	1.14276×10^{-10}	0.127680	5875.04	0.936375
4.0×10^{56}	4.51669×10^{46}	1.12917×10^{-10}	0.252325	7266.13	0.897929
6.0×10^{56}	6.70497×10^{46}	1.11750×10^{-10}	0.374573	8177.42	0.865580
8.0×10^{56}	8.85463×10^{46}	1.10683×10^{-10}	0.494664	8852.67	0.835232
1.0×10^{57}	1.09679×10^{47}	1.09679×10^{-10}	0.612721	9379.47	0.808115
1.2×10^{57}	1.30461×10^{47}	1.08718×10^{-10}	0.728823	9798.86	0.781969
1.4×10^{57}	1.50899×10^{47}	1.07785×10^{-10}	0.842997	10133.2	0.755523
1.6×10^{57}	1.70994×10^{47}	1.06871×10^{-10}	0.955258	10394.6	0.730148
1.8×10^{57}	1.90741×10^{47}	1.05967×10^{-10}	1.065578	10588.7	0.704181
2.0×10^{57}	2.10132×10^{47}	1.05066×10^{-10}	1.173903	10714.6	0.677181
2.2×10^{57}	2.29147×10^{47}	1.04158×10^{-10}	1.280129	10761.8	0.649383
2.4×10^{57}	2.47750×10^{47}	1.032293×10^{-10}	1.38406	10694.5	0.619124
2.6×10^{57}	2.65860×10^{47}	1.022536×10^{-10}	1.48522	10367.5	0.577658

The neutron star solutions that have masses larger than the mass of the Sun have radii of about 10 km, which very much matches the experimental estimates of the radii of observed neutrons stars. Notice also that the largest neutron star, in our model, has a mass of approximately $1.28M_\odot$, and above that value, the radius of the stars decreases with their mass (see Table I and Fig. 2).

We now consider the structures of these isotropic Skyrme crystal stars, in particular, we consider the case of a star with a mass of $1.40M_\odot$, a typical mass for a realistic neutron star, equivalent to a baryon number of 2.44×10^{57}, although all the isotropic Skyrme crystal minimum energy solutions show the same qualitative behavior. Figure 3 shows the size of the Skyrmions, $L(r)$, over the radius of the star. As expected the Skyrmions are deformed more towards the center of the star than at the edge, increasing the Skyrmion mass density by a factor of 4.44. Because of this decrease in the size of the Skyrmions as we reach the center of the star the stress is higher at the center and decreases towards zero at the edge of the star as imposed by the boundary conditions.

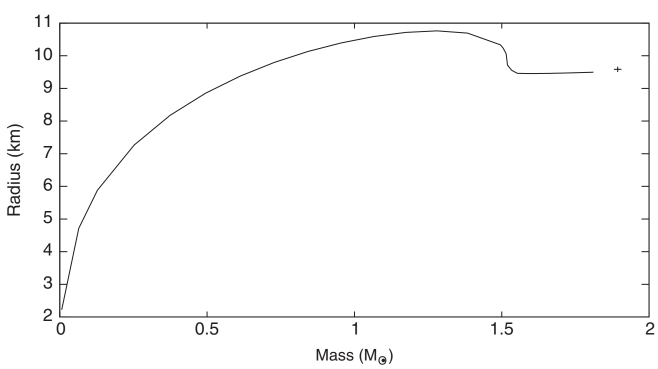

FIG. 2. Radius of the neutron star solutions as a function of their mass (solid line), and that of the maximum mass solution (cross).

The isotropic Skyrme crystal solutions have an $S_{\min}$ that is always greater than zero so the configurations do not collapse into black holes. Figure 3 also shows how the value of $S(r)$ varies over the radius of the star.

B. Stars made of anisotropically deformed Skyrme crystal

Having shown in the previous section that no isotropic Skyrme crystal solutions exist for baryon numbers larger than 2.61×10^{57}, we will now show that anisotropic solutions do exist.

Table II shows some of the properties of the anisotropic minimum energy Skyrme crystal solutions for various baryon numbers obtained using the generalized TOV equation. We found solutions in this way up to a baryon number of 3.25×10^{57}, corresponding to $1.81M_\odot$, after which the numerical energy minimization procedure became difficult to implement. However by using a similar simulated annealing process to maximize the baryon number, rather than minimize the energy for a particular baryon number,

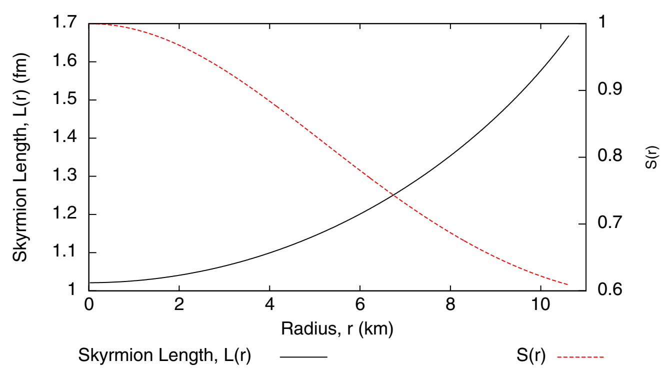

FIG. 3 (color online). Variation of the size of the isotropic Skyrmions, $L(r)$ (solid line) and of the metric function $S(r)$ (dotted line) over the radius of a star of mass $1.40M_\odot$.

TABLE II. Properties of the anisotropic minimum energy neutron star configurations for various baryon numbers.

B	Total energy (J)	Energy/B (J)	Mass/$M_\odot$	R(m)	S_{min}
2.65×10^{57}	2.70277×10^{47}	1.01991×10^{-10}	1.50990	10091.8	0.559060
2.70×10^{57}	2.74605×10^{47}	1.01706×10^{-10}	1.53408	9555.51	0.526465
2.75×10^{57}	2.78943×10^{47}	1.01434×10^{-10}	1.55832	9460.46	0.514207
2.80×10^{57}	2.83310×10^{47}	1.01182×10^{-10}	1.58271	9456.89	0.506402
2.85×10^{57}	2.87706×10^{47}	1.00949×10^{-10}	1.60727	9456.46	0.498735
2.90×10^{57}	2.92133×10^{47}	1.00735×10^{-10}	1.63200	9457.92	0.491152
2.95×10^{57}	2.96592×10^{47}	1.00540×10^{-10}	1.65691	9460.65	0.483633
3.00×10^{57}	3.01087×10^{47}	1.00362×10^{-10}	1.68202	9465.06	0.476231
3.05×10^{57}	3.05619×10^{47}	1.00203×10^{-10}	1.70734	9469.97	0.468880
3.10×10^{57}	3.10191×10^{47}	1.00062×10^{-10}	1.73288	9475.76	0.461631
3.15×10^{57}	3.14807×10^{47}	9.99388×10^{-11}	1.75867	9481.95	0.454438
3.20×10^{57}	3.19472×10^{47}	9.98351×10^{-11}	1.78473	9489.04	0.447382
3.25×10^{57}	3.24191×10^{47}	9.97510×10^{-11}	1.81109	9496.62	0.440435

we found anisotropic Skyrme crystal solutions up to a baryon number of 3.41×10^{57}, equivalent to $1.90 M_\odot$. At this maximum baryon number solution there is only one possible configuration of the Skyrmions, as any modification to it results in a decrease in the baryon number, hence it is the minimum energy solution. Above this baryon number, solutions do not exist.

As in the case of isotropic Skyrme crystal deformations we find that the solutions are energetically favorable as the energy per baryon decreases as the total baryon number increases, indicating stable solutions. As the baryon number is increased towards its maximum value of 3.41×10^{57} the energy per baryon begins to level off, and we find that the maximum baryon number has the lowest energy per baryon, as in the isotropic case.

We can see that the configurations we have constructed do not collapse into a black hole by noticing that the values of S_{min} are always positive, as shown in Fig. 4.

Figure 2 shows a plot of the mass radius curve for both the isotropic and anisotropic Skyrme crystal cases, with the mass in units of $M_\odot$. As stated above, large isotropic crystal neutron stars have a radius that decreases as the mass

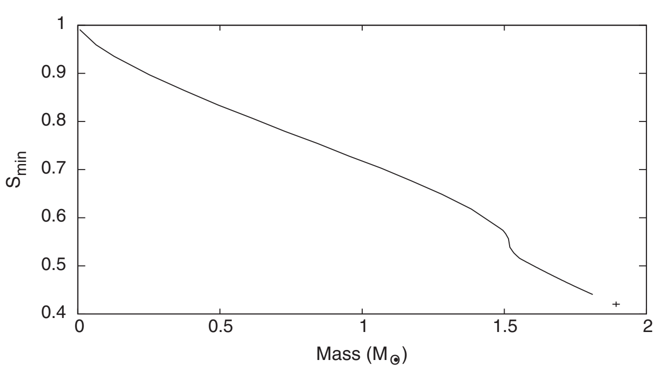

FIG. 4. S_{min} of the neutron star solutions as a function of their mass. The maximum mass solution is shown as a cross.

increases. We can clearly see in Fig. 2, that at the critical mass of $1.49 M_\odot$, the radius keeps decreasing as the mass of the star increases. Moreover, we also observe a sharp drop of radius just over $1.5 M_\odot$ followed by a plateau at about 9.5 km.

By considering anisotropic as well as isotropic Skyrme crystal solutions we have extended the mass range over which solutions can be found, finding masses up to 28% above the maximum mass of the isotropic case. This is an interesting finding because isotropy of matter is often taken as an assumption when studying neutron star models, including the Skyrme crystal case considered in [12,16,17], and a maximum mass is then derived. We have shown that by not assuming isotropy and instead allowing anisotropic matter configurations, the maximum mass can be increased by a significant amount. In this simple Skyrme crystal model the maximum mass found is equivalent to $1.90 M_\odot$, and the recent discovery of a $1.97 \pm 0.04 M_\odot$ neutron star [25], the highest neutron star mass ever determined, makes this an encouraging finding, especially when we consider that including the effects of rotation into our model will increase the maximum mass found, by up to 2% for a star with a typical 3.15 ms spin period [26].

Figure 5 shows a selection of plots of the Skyrmion lengths λ_r and λ_t and the Skyrmion size L, Eq. (8), over the radius of the star for four special stars: the largest star, with radius $R = 10.8$ km and mass $M = 1.28 M_\odot$ [Fig. 5(a)]; the heaviest isotropically deformed star $M = 1.49 M_\odot$ [Fig. 5(b)]; the densest neutron star, $M = 1.54 M_\odot$ [Fig. 5(c)]; and the heaviest neutron star, $M = 1.90 M_\odot$ [Fig. 5(d)]. The first two are made out of an isotropically deformed crystal, while the last two are anisotropically deformed and one notices that the amount of anisotropy increases as the mass increases (the divergence between λ_r and λ_t increases). Throughout this paper, we will use these four special stars as examples to illustrate various properties of the neutron stars.

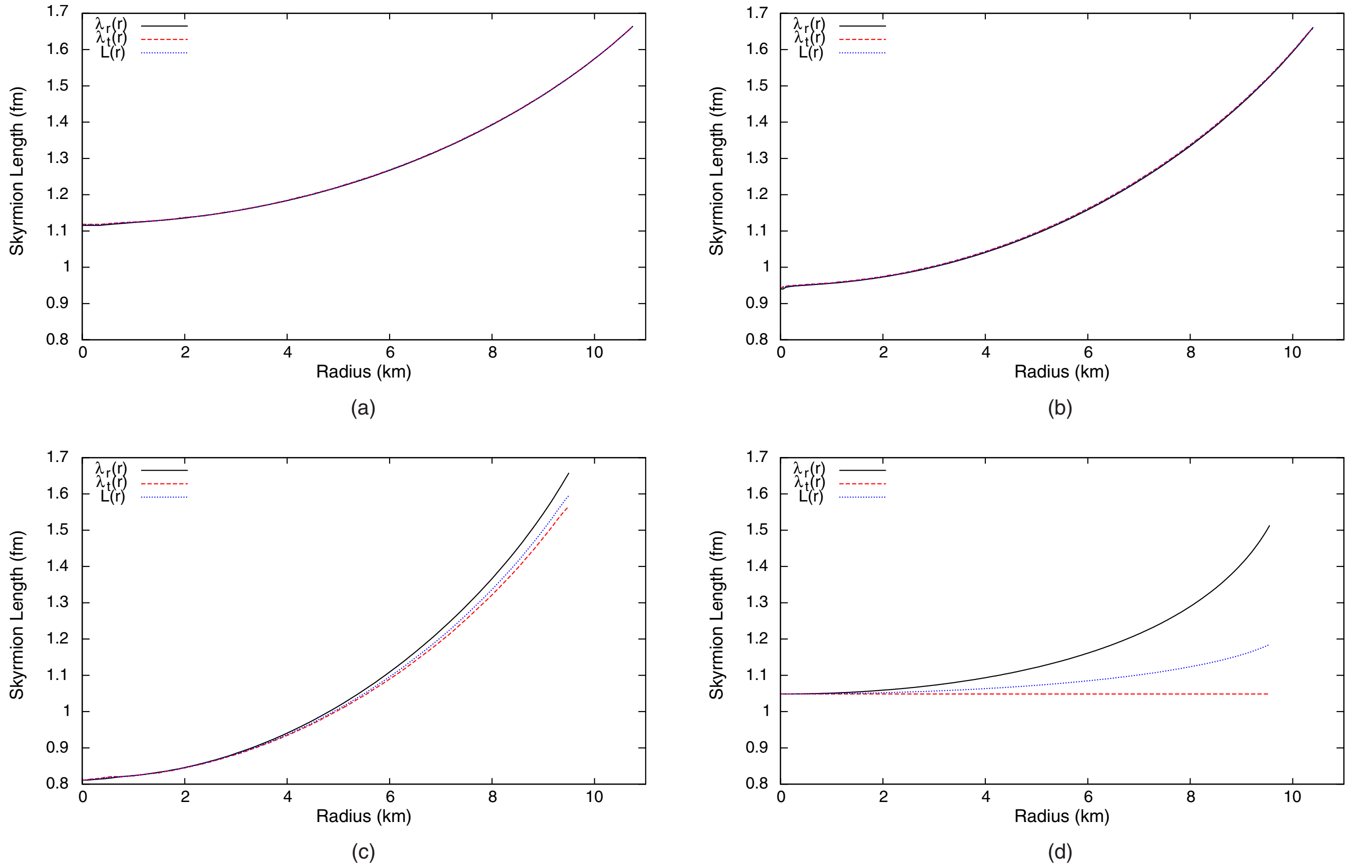

FIG. 5 (color online). Skyrmion lengths $\lambda_r(r)$ (solid line), $\lambda_t(r)$ (dashed line), and $L(r)$ (dotted line) for (a) largest neutron star ($R = 10.8$ km): $M = 1.28M_\odot$; (b) heaviest isotropic neutron star: $M = 1.49M_\odot$ (all lengths coincide as they are made of isotropically deformed crystal); (c) densest neutron star: $M = 1.54M_\odot$; (d) heaviest neutron star: $M = 1.90M_\odot$.

As the maximum mass is approached the gradient of the profile of tangential Skyrmion lengths over the radius of the star becomes smaller, and we note that physically meaningful stars composed of anisotropically deformed crystal should have $d\lambda_t/dr \geq 0$ [27]. This confirms that the minimum energy solution for the maximum mass found, $1.90M_\odot$, for anisotropic Skyrme crystal solutions is the configuration with a constant tangential Skyrmion length as illustrated in Fig. 5(d).

The generalized TOV equation imposes that the sizes of the Skyrmions are equal in all directions at the center of the star, but away from the center, for all the anisotropic Skyrme crystal solutions, we find that the amount of Skyrmion anisotropy increases as we move towards the edge of the star, reaching the maximum at the edge. The Skyrmions are deformed to a greater extent in the tangential direction in agreement with the value of the aspect ratio, p, being negative over the values where $\lambda_r \neq \lambda_t$.

As expected, the profiles for λ_r and λ_t show that the mass density at the center of the star is higher than at the edge, decreasing monotonically as the radial distance increases. This is shown by Fig. 6 for the largest, heaviest isotropic, densest, and maximum mass solutions.

In Fig. 7 one can see how the lengths of the Skyrme crystal λ_r and λ_t vary with the mass of the star both at the center ($r = 0$) and the edge of the star ($r = R$). For isotropically deformed stars, $\lambda_r(R) = \lambda_t(R)$ is constant and corresponds to the minimum energy Skyrme crystal in the

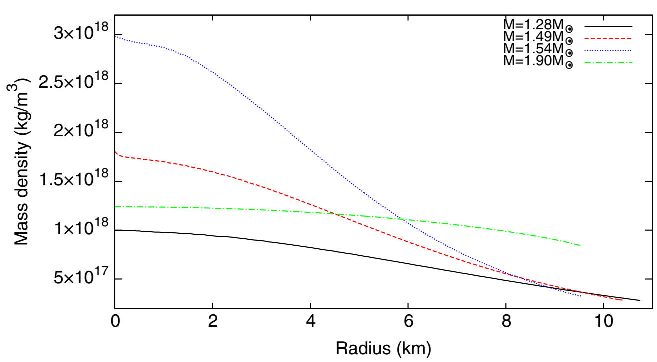

FIG. 6 (color online). Mass density $\rho(r)$ for (a) largest neutron star ($R = 10.8$ km): $M = 1.28M_\odot$ (solid line); (b) heaviest isotropic neutron star: $M = 1.49M_\odot$ (dashed line); (c) densest neutron star: $M = 1.54M_\odot$ (dotted line); (d) heaviest neutron star: $M = 1.90M_\odot$ (dashed-dotted line).

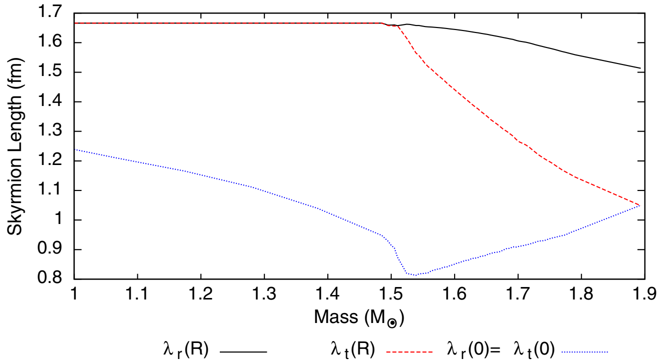

FIG. 7 (color online). Skyrmion lengths at the edge of the star, $\lambda_r(R)$ (solid line) and $\lambda_t(R)$ (dashed line), and at the center of the star, $\lambda_r(0) = \lambda_t(0)$ (dotted line), as a function of the star mass.

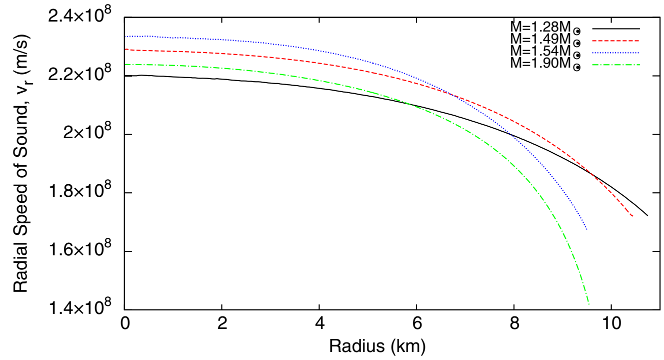

FIG. 8 (color online). Radial speed of sound, $v_r(r)$ for (a) largest neutron star ($R = 10.8$ km): $M = 1.28M_\odot$ (solid line); (b) heaviest isotropic neutron star: $M = 1.49M_\odot$ (dashed line); (c) densest neutron star: $M = 1.54M_\odot$ (dotted line); (d) heaviest neutron star: $M = 1.90M_\odot$ (dash dotted line).

absence of gravity. Not surprisingly, $\lambda_r(0) = \lambda_t(0)$ decreases steadily as the mass of the star increases, showing that the density at the center of the star increases. Once the phase transition has taken place and the star is too heavy to remain isotropically deformed, we observe that $\lambda_r(0) = \lambda_t(0)$ drops sharply to a local minimum, reached for $M \approx 1.54M_\odot$. Meanwhile, $\lambda_r(R)$ and $\lambda_t(R)$ remain nearly identical. Beyond the minimum of $\lambda_{r,t}(0)$, $\lambda_r(R)$ and $\lambda_t(R)$ start to diverge sharply; $\lambda_r(R)$ decreases slightly in value while $\lambda_t(R)$ decreases rapidly. These stars are thus much more compressed in the tangential direction than in the radial one. As seen on Fig. 5(d), $\lambda_t(R) = \lambda_t(0)$ for the maximum mass neutron star.

Another property of a neutron star worth considering is the speed of sound. To compute it one needs to know how the energy of the crystal varies when it is deformed in the direction of wave propagation. Using (2) we can thus compute the speed of sound in the z direction. To compute the speed of sound in the x and y directions when the crystal is deformed we need to know how the energy of the crystal varies when the crystal is deformed in all three directions independently, an expression we do not have.

We are thus only able to compute the radial speed of sound inside a neutron star and it is given by

$$v_r = \left(\frac{dp_r}{d\lambda_r}\left(\frac{d\rho}{d\lambda_r}\right)^{-1}\right)^{1/2}, \tag{29}$$

where both p_r and ρ are functions of λ_r and λ_t given, respectively, by (21) and (23). Obviously, when the crystal inside the star is isotropically deformed, the speed of sound is the same in all three directions.

First of all it is interesting to notice that the speed of sound in the minimum energy Skyrme crystal, in the absence of a gravitational field, is amazingly large: $v = 0.57c$. This is the speed of sound at the surface of a neutron star when it is deformed isotropically. From Fig. 8 one sees that v_r increases as one moves towards the center of the star. As v_r is directly related to the density of the star, it is not surprising to find that the maximum radial speed, $v_r = 0.78c$, is reached at the center of the densest neutron star, i.e. the one with $M = 1.54M_\odot$. As expected, $v_r < c$ everywhere.

Figure 9 shows how the value of $S(r)$ varies over the radius of the star for, again, the largest, heaviest isotropic, densest, and maximum mass solutions, showing how the metric is altered as r varies. The minimum value of $S(r)$ is always located at the edge of the star, i.e. $S_{\min} = S(R)$, and it is presented in Fig. 4 as a function of the star masses. One sees that $S_{\min}$ decreases monotonically as the mass increases, and exhibits a sharp decrease just over $1.5M_\odot$, i.e. just above the critical mass. However $S_{\min}$ always remains positive, indicating that no black hole is formed.

Figure 10 shows how the total baryon number and the mass of all the solutions found are related. As the baryon number increases the effects of gravitational attraction increase, resulting in a slightly lower gravitational mass per baryon than expected from a linear relation.

We note that the minimum value of the aspect ratio, p, for the minimum energy configurations found is -0.283

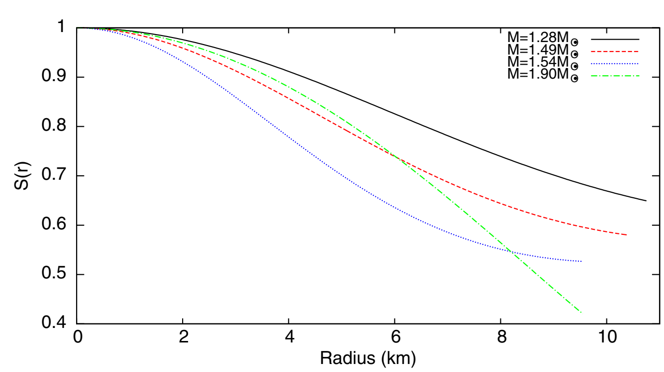

FIG. 9 (color online). The function $S(r)$ for (a) largest neutron star ($R = 10.8$ km): $M = 1.28M_\odot$ (solid line); (b) heaviest isotropic neutron star: $M = 1.49M_\odot$ (dashed line); (c) densest neutron star: $M = 1.54M_\odot$ (dotted line); (d) heaviest neutron star: $M = 1.90M_\odot$ (dashed-dotted line).

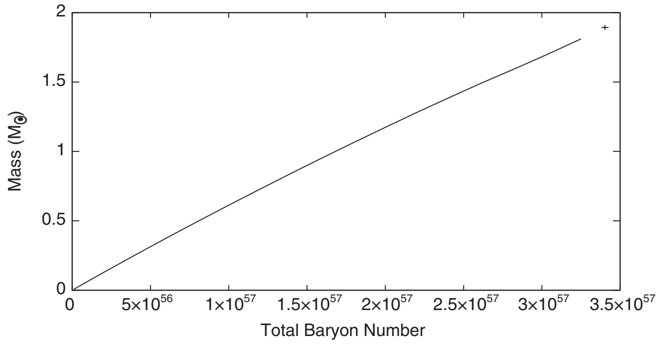

FIG. 10. Mass of the neutron star solutions as a function of their baryon number. The maximum mass solution is shown as a cross.

and the minimum value of L is 8.11×10^{-16}, both of which are within the valid range of values for Eq. (2) [7].

C. Inclusion of the pion mass

Throughout the work described we have assumed a zero pion mass. The inclusion of a nonzero pion mass can be considered by including the pion mass term,

$$\int \frac{m_\pi^2 F_\pi^2}{8} \mathrm{Tr}(U - 1) d^3x, \tag{30}$$

in the static Skyrme Lagrangian (1), where U is the Skyrme field, F_π is the pion decay constant, and m_π is the pion mass. Using the cubic lattice of α-like Skyrmions that has been considered above one finds that $\mathrm{Tr}(U - 1) = -2$, meaning that the energy E_π arising from the pion mass term reduces to

$$E_\pi = \tfrac{1}{4} m_\pi^2 F_\pi^2 L^3, \tag{31}$$

an energy term proportional to the volume of the Skyrmions.

It can be seen in Fig. 11 that including a pion mass of $m = 138$ MeV decreases the maximum mass of the star by a very small amount from 1.49 to $1.47 M_\odot$ while also slightly decreasing the central density at which this occurs.

Including a pion mass of $m = 138$ MeV in the simulated annealing process used to find the maximum baryon number for the anisotropic Skyrme crystal solutions results in a maximum baryon number of 3.34×10^{57}, equivalent to $1.88 M_\odot$, a decrease of $0.02 M_\odot$ from the maximum mass found in the case without a pion mass.

This gives an indication as to how the pion mass affects the structures of the neutron star configurations that can be constructed, and a similar reduction in the maximum mass is expected for all the anisotropic crystal solutions; however, when the pion mass is included it also has the effect of driving the Skyrme crystal lattice away from the half-Skyrmion symmetry [7]. This will be a small effect for the dense Skyrme crystals that we are considering because while the pion mass term is the dominant term in the

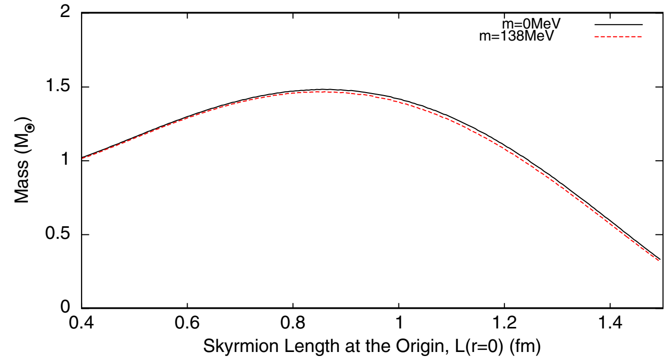

FIG. 11 (color online). Mass of the star as a function of the size of the Skyrmions at the center, L_0, for zero pion mass (solid line) and $m = 138$ MeV (dashed line).

Lagrangian far away from the centers of the Skyrmions when they are well separated, in the dense Skyrme crystal there is no space away from the centers of the Skyrmions so it becomes less important in affecting the field distributions. Its effect will be to reduce the pion mass term, Eq. (31), by a small amount.

D. Stars above the maximum mass

As in other studies of neutron starts based on the Skyrme model, we found a critical mass above which solutions do not exist. In other words, when the star is too massive, the crystal of which it is made is not capable of counterbalancing the gravitation pull and the star then collapses into a black hole. This is indeed what we observed when trying to construct solutions above the critical mass: the energy of the configuration kept decreasing as the radius of the star decreased and the S_{min} function became negative, indicating the formation of a horizon, and hence a black hole.

Throughout this work we have assumed a spherically symmetric metric and stress tensor; however, these assumptions could be removed and it may be that higher mass solutions could be found. We could instead consider an axially symmetric metric, the most general form [28] being

$$ds^2 = \alpha^2(d\rho^2 + dz^2) + \beta^2 d\phi^2 - \gamma^2 dt^2, \tag{32}$$

when written in cylindrical coordinates. The stress tensor,

$$T^\mu_\nu = \mathrm{diag}(\rho, p_1, p_2, p_3), \tag{33}$$

could then be completely anisotropic with $p_1 \neq p_2 \neq p_3$. Minimum energy solutions to Einstein's equations for such a metric and stress tensor could be found by direct minimization of the action of the Skyrme model coupled to gravity or by using an, as yet undetermined, axisymmetric form of the TOV equation. Another approach to investigate such solutions would be to perturb the spherically symmetric solutions that we have found. Following the procedure for doing so described in [28] the exterior metric for an axially symmetric solution can be written in

Schwarzschild coordinates and, after comparing the exterior spherically symmetric Schwarzschild solution to our solutions for the interior metric of the star and finding the substitutions necessary to move from one to the other, we can make the same substitutions to the axially symmetric exterior metric. This allows us to then describe approximately both the metric and the stress energy tensor of the axially symmetric solution. To carry out such investigations into axially symmetric static configurations, an equation analogous to (2) which would relate the energy of the Skyrme crystal to its size and deformation in all three directions independently would need to be considered.

We have also assumed that the stress tensor, $T^{\mu}_{\nu} = \mathrm{diag}(\rho, p_r, p_\theta, p_\phi)$, is diagonal; however, if shear strains are included in our model off diagonal components would have to be introduced. This would also remove the assumption of spherical symmetry altering the configurations found.

Spherical symmetry also needs to be removed to consider rotating stars. This will result in configurations above the maximum mass found in this work, by up to 2% for a star with a typical 3.15 ms spin period [26], and as neutron stars are known to be rotating, this is an important effect to consider.

E. Stability and oscillations of anisotropic stars

Having found the minimum energy solutions to a Skyrme crystal neutron star model, we now briefly consider their stability. First, as our procedure to determine the solution minimized the energy for a fixed baryon number, we know that our solutions are stable under small perturbations, i.e. they correspond to local minima. We must also consider the effect that a nonzero temperature could have on the solutions. In our case, as we are considering a solid, thermal excitations correspond to the excitation of phonon modes which, from an energy point of view, is, in our model, equivalent to increasing the mass term discussed above. We thus see that the effect of raising the star temperature will be to lower the value of the maximum mass marginally and so stars that are not close to the maximum mass will remain stable when their temperature increases.

It is now relevant to consider the oscillations of anisotropic neutron stars as these oscillations are expected to generate gravitational waves which, once they have been experimentally observed, might help determine the structure of neutron stars. To compute the small amplitude fluctuations of our solutions we would need to know a generalization of (2) for the energy of the Skyrme crystal when it is deformed and sheared in all three directions independently. The determination of such vibrations is thus well beyond the scope of this article, but using the results of Doneva *et al.* [29], we can hint at what properties the vibrations of the neutron stars considered in this paper would have. Doneva *et al.* have computed the spectrum of an anisotropic neutron star using an equation of state where the anisotropy was put in by hand and controlled by a parameter they called λ. The isotropic star corresponds to the case $\lambda = 0$ while stars with a radial stress smaller than the tangential one correspond to $\lambda < 0$. We would like to stress here that the anisotropy in our model comes from the energy of the Skyrme lattice, which has cubic symmetry at rest, and is fully determined by the minimization of the neutron star energy; there is no anisotropy parameter to tune in. To compare our model with that of [29] we have computed $\lambda = r(p_r - p_t)/(p_r 2m)$, which is a function of r in our model, for our solutions. We found that, except near the origin and the edge of the star, its value is nearly constant, $\lambda \approx -2$, for the maximum mass neutron star and for less massive stars λ is larger but always negative. Moreover it is 0 for the isotropic stars. From the spectrum computed by Doneva *et al.* we see that the anisotropy of the star will affect its oscillation spectrum and that the frequencies will be a few percent larger than for an isotropic star of the same mass.

V. CONCLUSIONS

Neutron stars are large bodies of matter where the electrons, instead of circling atoms, are forced to merge with the nuclei, resulting in extremely dense stars made entirely of neutrons. Their temperature, from a nuclear point of view, is very low and this means nuclear matter must be considered as a solid rather than a fluid. Moreover, the gravitational pull of the star is so strong that the "atmospheric" fluid one might expect at the surface is of negligible height.

In this context, the Skyrme model, known to be a low energy effective field theory for QCD [3], is an ideal candidate to describe neutron stars once the model is coupled to gravity. The minimum energy configuration of large numbers of Skyrmions is a cubic crystal made of $B = 4$ Skyrmions that correspond to a crystal of α-like particles. We have thus used these solutions as a building block to describe the neutron star by combining the deformation energy computed in [7] and a generalized version of the TOV equation [13–15], which describes the static equilibrium between matter forces, within a solid or fluid, and the gravitational forces self-generated by the matter for a spherically symmetric body.

The key feature of our approach to the problem was to consider the star as a solid that could potentially deform itself anisotropically. We then found that below $1.49M_\odot$, all stars were made of a crystal deformed isotropically, i.e. the radial strain was identical to the tangential one. Above that critical value, the neutron star undergoes a critical phase transition and the lattice of Skyrmions compresses anisotropically: the Skyrmions are more compressed tangentially than radially. Stars were shown to exist up to a critical mass of $1.90M_\odot$, a result that closely matches the recent discovery of Demorest *et al.*. [25] who measured the mass of the heaviest neutron star found to date, PSR J1614-2230, to be $1.97M_{\odot}$. We also observed that the maximum radius for a Skyrmion star was approximately 11 km, a figure that matches well the experimental estimations.

In our model we did not consider the rotational energy of the star that is approximated at about 2% of its total energy. If we included that extra energy, our upper bound would thus just fit above the mass of PSR J1614-2230.

Finally we have also shown that if the mass of a neutron star was to be raised to cross the critical mass threshold, it would collapse into a black hole.

ACKNOWLEDGMENTS

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