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# On range and local time of many-dimensional submartingales

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## Abstract

We consider a discrete-time process adapted to some filtration which lives on a (typically countable) subset of  $\mathbb{R}^d$ ,  $d \geq 2$ . For this process, we assume that it has uniformly bounded jumps, is uniformly elliptic (can advance by at least some fixed amount with respect to any direction, with uniformly positive probability). Also, we assume that the projection of this process on some fixed vector is a submartingale, and that a stronger additional condition on the direction of the drift holds (this condition does not exclude that the drift could be equal to 0 or be arbitrarily small). The main result is that with very high probability the number of visits to any fixed site by time  $n$  is less than  $n^{\frac{1}{2}-\delta}$  for some  $\delta > 0$ . This in its turn implies that the number of different sites visited by the process by time  $n$  should be at least  $n^{\frac{1}{2}+\delta}$ .

**Keywords:** strongly directed submartingale, Lyapunov function, exit probabilities

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# 1 Introduction and results

Let  $\mathfrak{X} \subset \mathbb{R}^d$  be a set of infinite cardinality; the elements of  $\mathfrak{X}$  will be called *sites*. Without restriction of generality, we assume that  $0 \in \mathfrak{X}$ . Throughout this paper we assume that  $d \geq 2$ . We consider a discrete-time process  $X = (X_n, n \geq 0)$  with values in  $\mathfrak{X}$ , adapted to a filtration  $\mathcal{F} = (\mathcal{F}_n, n \geq 0)$ . For the process  $X$ , we suppose that it is uniformly elliptic (can advance in any given direction with uniformly positive probability), has uniformly bounded jumps, and is a *strongly directed* submartingale (see Definition 1.1 below for the precise meaning). In principle, we do not assume homogeneity in space and/or time, or even the fact that the process is Markovian.

In this paper we study two related questions:

- How many different sites can be visited by the process  $X$  by time  $n$ ?
- How large can be the number of visits to a given site?

Of course, in the absence of space/time homogeneity one cannot hope to be able to characterize the precise behavior of the quantities of interest; in this paper we content ourselves in proving that with probability  $1 - \exp(-n^\varepsilon)$  the number of visits to any fixed site by time  $n$  is less than  $n^{\frac{1}{2}-\delta}$  for some  $\delta > 0$ . This in its turn implies that the number of different sites visited by the process by time  $n$  with very high probability will be at least  $n^{\frac{1}{2}+\delta}$ .

Although it is not important for the formulation of our results, while reading the paper one may always assume that  $\mathfrak{X}$  is the vertex set of the integer lattice  $\mathbb{Z}^d$ , the vertex set of some other mosaic, or just any “discrete” (in particular, countable) set. This, of course, is justified by the questions that are of our interest: e.g., if the law of the jump of the process is (in some sense) continuous, then such questions typically do not arise (every site is visited at most once, and the process visits  $n$  different sites by time  $n$ ).

Range (i.e., the cardinality of the set visited sites, or sometimes this set itself) and the local time (i.e., the number of visits to a given site) for space-homogeneous discrete-time random walks were extensively studied in the literature. It is a classical result that the expected range of the simple random walk is  $O(\frac{n}{\ln n})$  for  $d = 2$  and  $O(n)$  for  $d \geq 3$ , see e.g. Section 6.1 of [13]. It is not difficult to obtain from this fact (using an independence argument as e.g. in Lemma 3.1 of [1]) that with very high probability the walk visits at least  $n^{1-\delta}$  distinct sites by time  $n$ . Finer results for the range of homogeneous random walks can be found in a number of papers; see e.g.

in [3, 10, 11] and references therein. For nonhomogeneous random walks these questions, of course, are more difficult; we mention [17] that contains results on the range of simple random walk on supercritical percolation cluster.

The behaviour of the local time (i.e., the number of visits) in a fixed site, or the field of local times in all sites, was much studied in the literature as well. It is quite elementary to obtain that the expected number of visits to the origin by time  $n$  for the simple random walk is  $O(\ln n)$  for  $d = 2$  and  $O(1)$  for  $d \geq 3$ . Also, one can easily obtain for the simple random walk in dimension 2 (using e.g. **E1** of Section III.16 of [18]) that, with stretched-exponentially small probability, the number of visits to the origin is less than  $n^\delta$  for any fixed  $\delta > 0$ . Of course, finer results (for more general random walks as well) are available; see e.g. [6, 7, 8, 9, 15].

As mentioned above, with our assumptions we cannot hope to obtain very “precise” results; however, in some cases it may be important to be able to estimate the range from below. In particular, consider the following process, called *excited random walk*, or sometimes *cookie random walk*. It is a discrete-time stochastic process taking values on  $\mathbb{Z}^d$ ,  $d \geq 2$ , described in the following way: when the particle visits a site for the first time, it has a uniformly positive drift in a given direction  $\ell$ ; when the particle is at a site which was already visited before, it has zero drift (observe that this implies that the cookie random walk is a strongly directed submartingale in the sense of Definition 1.1 below). This process was introduced in a simpler form ( $\ell$  is the first coordinate vector and in already visited sites the process behaves as simple random walk) in [4] and then studied (we mention only the papers that are concerned with dimension  $d \geq 2$ ) in e.g. [5, 12, 16, 19]. The key fact that is usually needed is that such a process typically visits much more than  $n^{1/2}$  different sites by time  $n$ ; this allows to prove bounds on the probability that the process advances in the direction of cookies’ drift by much more than  $Cn^{1/2}$ , and this in turn makes it possible to use regeneration arguments that imply the Law of Large Numbers and the Central Limit Theorem.

It is worth noting that in this paper, differently from what was considered in [16], we allow the process to have a nonzero drift of arbitrarily small absolute value; for such processes, a direct application of methods of [16] fails.

Now, we write formal definitions and state our results. Let  $\|\cdot\|$  be the Euclidean norm in  $\mathbb{R}^d$  and let  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$  be the unit sphere. Let us denote the coordinate vectors of  $\mathbb{R}^d$  by  $\mathbf{e}_1, \dots, \mathbf{e}_d$ . We write  $x \cdot y$  for the usual scalar product of  $x, y \in \mathbb{R}^d$ . For  $A \subset \mathfrak{X}$  we denote by  $|A|$  the

cardinality of  $A$ . Let

$$L_n(x) = \sum_{k=0}^n \mathbf{1}_{\{X_k=x\}} \quad (1.1)$$

be the local time of the process in  $x \in \mathfrak{X}$  by time  $n$ , and we denote by  $\mathcal{R}_n = \{X_0, \dots, X_n\}$  the set of visited sites by time  $n$ . Define the random variable  $D_n \in \mathbb{R}^d$  to be the (conditional) drift of the process  $X$  at time  $n$ :

$$D_n = \mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) - X_n.$$

Next, let  $\mathcal{P}_{\mathcal{L}}$  be the operator of projection on the linear subspace  $\mathcal{L} \subset \mathbb{R}^d$ . Assuming that  $\mathcal{L}$  is a *two-dimensional* subspace of  $\mathbb{R}^d$ ,  $\ell \in \mathbb{S}^{d-1} \cap \mathcal{L}$  and  $u \in \mathbb{R}$ , define

$$H_{\ell, \mathcal{L}}^u = \left\{ x \in \mathbb{R}^d : \mathcal{P}_{\mathcal{L}}x = 0 \text{ or } \frac{\mathcal{P}_{\mathcal{L}}x \cdot \ell}{\|\mathcal{P}_{\mathcal{L}}x\|} \geq u \right\},$$

see Figure 1.

**Definition 1.1** *We say that the  $\mathcal{F}$ -adapted process  $X$*

- (a) *has uniformly bounded jumps, if there exists  $K > 0$  such that  $\|X_{n+1} - X_n\| \leq K$  a.s. for all  $n$  (we assume without restriction of generality that  $K \geq 1$ );*
- (b) *is uniformly elliptic (recall that we assume  $d \geq 2$ ), if there exist  $h, r > 0$  such that for all  $\ell \in \mathbb{S}^{d-1}$  we have  $\mathbb{P}[(X_{n+1} - X_n) \cdot \ell > r \mid \mathcal{F}_n] > h$  a.s. (we assume without restriction of generality that  $r \leq 1$ );*
- (c) *is a martingale, if  $D_n = 0$  a.s. for all  $n$ ;*
- (d) *is  $(u, \ell, \mathcal{L})$ -strongly directed submartingale, if  $u > 0$  and  $\mathbb{P}[D_n \in H_{\ell, \mathcal{L}}^u \mid \mathcal{F}_n] = 1$  a.s.*

Observe that for  $(u, \ell, \mathcal{L})$ -strongly directed submartingale it holds that the (conditional on the history) expected projection of the drift to  $\ell$  is always nonnegative (so that it is what one would naturally call a “submartingale in direction  $\ell$ ”).

In this paper we assume that all processes we are considering live in a probability space with probability measure  $\mathbb{P}$ . We also adopt the following notational convention: when it is necessary to assume that the initial state

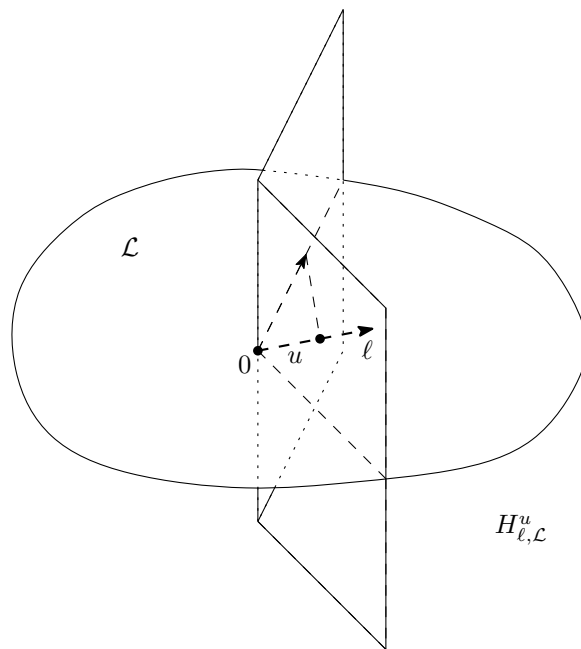


Figure 1: On the definition of  $H_{\ell, \mathcal{L}}^u$ ; observe that  $H_{\ell, \mathcal{L}}^u$  and  $H_{-\ell, \mathcal{L}}^{-u}$  “complement” each other (i.e., their union is  $\mathbb{R}^d$  and they intersect on a set of measure 0).

of the process under consideration is  $x \in \mathfrak{X}$ , we do not write it explicitly but simply add a subscript  $x$  to  $\mathbb{P}$ .

It is known that for the many-dimensional uniformly elliptic martingales with bounded jumps the following result (well hidden in [16] as Lemma 5.3 and a part of argument in Lemma 5.4) holds:

**Theorem 1.2** *Suppose that  $X$  is a uniformly elliptic martingale with uniformly bounded jumps. Then, there exists  $\hat{\gamma} \in (0, \frac{1}{2})$ ,  $\hat{C}_1, \hat{C}_2, \hat{\delta} > 0$  such that for any  $x \in \mathfrak{X}$*

$$\mathbb{P}_x[L_n(x) > n^{\hat{\gamma}}] \leq \hat{C}_1 e^{-\hat{C}_2 n^{\hat{\delta}}} \quad (1.2)$$

and

$$\mathbb{P}[|\mathcal{R}_n| < n^{1-\hat{\gamma}}] \leq \hat{C}_1 n e^{-\hat{C}_2 n^{\hat{\delta}}} \quad (1.3)$$

for all  $n$ .

One can note (we discuss this in detail in Section 3) that (1.3) follows from (1.2) in an elementary way.

The main purpose of this paper is to generalize Theorem 1.2 to (strongly directed) submartingales. Observe that one really needs some additional condition on the submartingale; otherwise it may happen that the typical number of visited sites is of order  $n^{1/2}$ . Indeed, consider the following example:

**Example 1.3** *Let  $X$  be a nearest-neighbor random walk on  $\mathbb{Z}^2$ , with the transition probabilities described in the following way. From the horizontal axis, the particle goes to neighboring sites with equal probabilities. Off the horizontal axis, the particle always goes to the left/right with probabilities  $1/4$ , the absolute value of the second coordinate increases with probability  $1/6$  and decreases with probability  $1/3$ . This process is a submartingale in the direction of the first coordinate vector; on the other hand, it is clear (after some thought) that the number of visited sites by time  $n$  behaves as  $O(n^{1/2})$  (due to the drift towards the horizontal axis, the process is essentially “one-dimensional”, it “lives” in a small neighborhood of the horizontal axis).*

The main result of this paper is the following

**Theorem 1.4** *Suppose that  $d \geq 2$  and  $X$  is a  $(u, \ell, \mathcal{L})$ -strongly directed submartingale, which is uniformly elliptic and has uniformly bounded jumps.*

Then, there exist constants  $\gamma \in (0, \frac{1}{2})$ ,  $C_1, C_2, \delta > 0$  (apart from the dimension, depending only on  $u$  and on  $K, r, h$  from Definition 1.1 (a)–(b)) such that for any  $x \in \mathfrak{X}$

$$\mathbb{P}_x[L_n(x) > n^\gamma] \leq C_1 e^{-C_2 n^\delta} \quad (1.4)$$

and

$$\mathbb{P}[|\mathcal{R}_n| < n^{1-\gamma}] \leq C_1 n e^{-C_2 n^\delta} \quad (1.5)$$

for all  $n$ .

The key to the proof of Theorem 1.4 is a technical fact about exit probabilities from two-dimensional rectangular domains for martingales. For a simply connected domain  $\mathcal{D} \subset \mathbb{R}^2$ , denote by

$$\tau(\mathcal{D}) = \min\{k \geq 1 : X_k \notin \mathcal{D}\}$$

the exit time from the domain  $\mathcal{D}$ . Let

$$\mathcal{S}(\mathcal{D}) = \{\alpha X_{\tau(\mathcal{D})-1} + (1-\alpha)X_{\tau(\mathcal{D})}, 0 \leq \alpha \leq 1\} \subset \mathbb{R}^2$$

be the segment of the trajectory of the process at the instant when it leaves  $\mathcal{D}$ .

For  $v \in \mathbb{S}^1$  and  $a, b, c, \lambda > 0$ , let us define a rectangular domain

$$R_{v,\lambda}^{a,b,c}(x) = \{y \in \mathbb{R}^2 : |(y-x) \cdot v^\perp| < a\lambda, (y-x) \cdot v \in (-b\lambda, c\lambda)\},$$

where  $v^\perp$  is a unit vector perpendicular to  $v$ , and consider the event

$$G_{v,\lambda}^{a,b,c}(x) = \left\{ \mathcal{S}(R_{v,\lambda}^{a,b,c}(x)) \cap \{y \in \mathbb{R}^2 : |(y-x) \cdot v^\perp| \leq a\lambda, (y-x) \cdot v = -b\lambda\} \neq \emptyset \right\}$$

which means the process goes out of the rectangle through its “left” side, see Figure 2.

Now, the important technical fact in the proof of Theorem 1.4 is

**Theorem 1.5** *Assume that  $X$  is a uniformly elliptic martingale with uniformly bounded jumps. For all  $a, b, c > 0$  there exist  $\varrho = \varrho(a, b, c) > 0$  and  $\lambda_0 = \lambda_0(a, b, c) \geq 1$  such that*

$$\mathbb{P}_x[G_{v,\lambda}^{a,b,c}(x)] \geq \varrho \quad (1.6)$$

for all  $\lambda \geq \lambda_0$  and all  $x \in \mathfrak{X}$ .



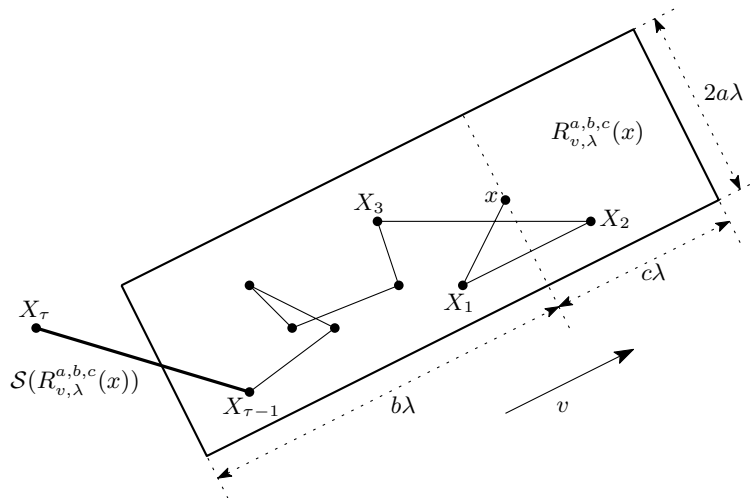


Figure 2: On the definition of the set  $R_{v,\lambda}^{a,b,c}(x)$  and the event  $G_{v,\lambda}^{a,b,c}(x)$ , here we use the abbreviation  $\tau := \tau(R_{v,\lambda}^{a,b,c}(x))$ .

The explicit expressions for  $\varrho(a, b, c)$  and  $\lambda_0(a, b, c)$  can be found at the end of the proof of Theorem 1.5, see (2.11) and (2.12).

In fact, there is only a small distance from the last result to a more general one: let  $\mathcal{D}$  be a “nice” domain such that  $0 \in \mathcal{D}$ , and suppose that  $A \subset \partial\mathcal{D}$  is a connected piece of the boundary of  $\mathcal{D}$  with positive measure. Then there exist  $\varrho = \varrho(\mathcal{D}, A) > 0$  and  $\lambda_0 = \lambda_0(\mathcal{D}, A)$  such that  $\mathbb{P}_x[\mathcal{S}(x + \lambda\mathcal{D}) \cap (x + \lambda A) \neq \emptyset] \geq \varrho$  for all  $\lambda \geq \lambda_0$  and all  $x \in \mathfrak{X}$ . It is not our intention in this paper to investigate, under which (not very restrictive) precise geometric assumptions on  $\mathcal{D}$  and  $A$  the above fact holds; the way how it should follow from Theorem 1.5 is (hopefully) made clear by Figure 3.

The paper is organized in the following way. In Section 2 we prove Theorem 1.5. Then, in Section 3 we prove the main result of this paper, Theorem 1.4. In Section 4, for comparison purposes we give a sketch of the proof of Theorem 1.2, and then discuss some open problems.

## 2 Exit probabilities from rectangles

We begin with the following lemma.

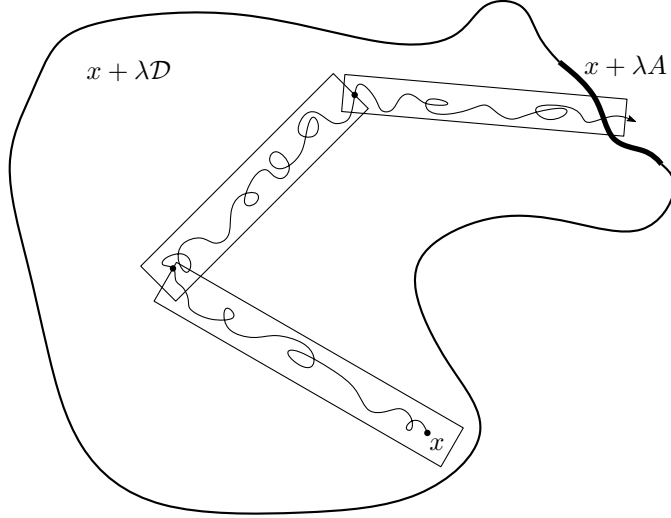


Figure 3: Going out from an arbitrary domain.

**Lemma 2.1** *Let  $X$  be a uniformly elliptic martingale with bounded jumps, and let  $K, r, h$  be the corresponding constants (cf. Definition 1.1 (a)–(b)). Assume that  $b > 0$  and  $a \geq \frac{7K(b+K)}{r\sqrt{h}}$ . Then, for all  $x \in \mathfrak{X}$  and  $v \in \mathbb{S}^1$  we have*

$$\mathbb{P}_x[G_{v,\lambda}^{a,b,b}(x)] \geq \frac{1}{7} \quad (2.1)$$

for all  $\lambda \geq \frac{3K}{b}$ .

It is easy to believe that this result should hold true: it essentially means that if the rectangle is (relatively) thin enough, then with uniformly positive probability the process will exit the rectangle through a fixed long side (in this case, the “left” one), see Figure 4. In fact, the constant  $1/7$  was chosen just for definiteness: one can modify the assumptions to obtain any fixed number less than  $1/2$  on its place.

*Proof of Lemma 2.1.* Let us suppose without restriction of generality that  $x = 0$  and  $v = \mathbf{e}_1$ . Abbreviate  $R_\lambda^{a,b} := R_{\mathbf{e}_1,\lambda}^{a,b,b}(0)$ . Define

$$\begin{aligned} \tau_{\rightarrow} &= \min\{k : X_k \cdot \mathbf{e}_1 > b\lambda\}, \\ \tau_{\leftarrow} &= \min\{k : X_k \cdot \mathbf{e}_1 < -b\lambda\}, \\ \tau_1 &= \min\{k : |X_k \cdot \mathbf{e}_1| > b\lambda\} = \tau_{\rightarrow} \wedge \tau_{\leftarrow}, \end{aligned}$$

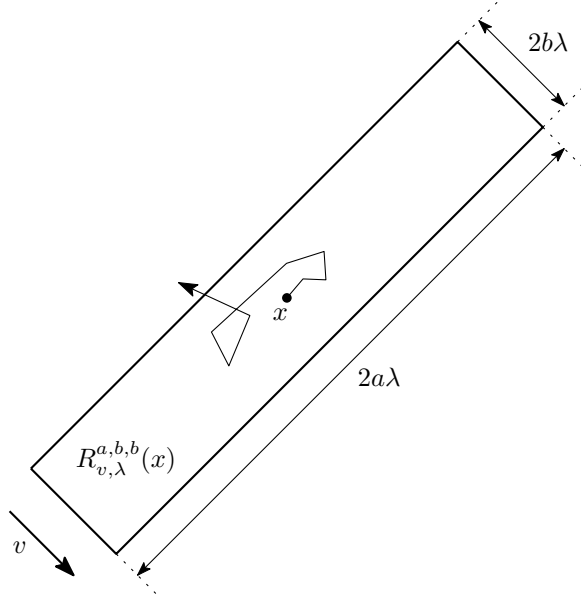


Figure 4: The event that the particle exits the “stretched” rectangle through its “left” long side.

and observe that  $\tau_1 \geq \tau(R_\lambda^{a,b})$ . By uniform ellipticity, it holds that  $\mathbb{E}((X_{m+1} \cdot \mathbf{e}_1 - X_m \cdot \mathbf{e}_1)^2 \mid \mathcal{F}_m) \geq r^2 h$ , and this implies that the process  $((X_m \cdot \mathbf{e}_1)^2 - r^2 h m, m \geq 0)$  is a submartingale with respect to the filtration  $\mathcal{F}$ . So, we have for any  $m$

$$\mathbb{E}_0((X_{m \wedge \tau_1} \cdot \mathbf{e}_1)^2 - r^2 h(m \wedge \tau_1)) \geq 0.$$

Since, by the bounded convergence theorem,

$$\lim_{m \rightarrow \infty} \mathbb{E}_0(X_{m \wedge \tau_1} \cdot \mathbf{e}_1)^2 = \mathbb{E}_0(X_{\tau_1} \cdot \mathbf{e}_1)^2 \leq (b\lambda + K)^2,$$

and, by the monotone convergence theorem,

$$\lim_{m \rightarrow \infty} \mathbb{E}_0(m \wedge \tau_1) = \mathbb{E}_0 \tau_1,$$

we have  $\mathbb{E}_0 \tau_1 \leq \frac{(b\lambda + K)^2}{r^2 h}$ . So, by Chebyshev’s inequality,

$$\mathbb{P}_0 \left[ \tau_1 \geq \frac{7(b\lambda + K)^2}{r^2 h} \right] \leq \frac{1}{7}. \quad (2.2)$$

Next, since  $X_{\cdot \wedge \tau_1} \cdot \mathbf{e}_1$  is a (one-dimensional) bounded martingale, using the Optional Stopping Theorem we write

$$\begin{aligned} 0 &= \mathbb{P}_0[\tau_{\leftarrow} < \tau_{\rightarrow}] \mathbb{E}_0(X_{\tau_1} \cdot \mathbf{e}_1 \mid \tau_{\leftarrow} < \tau_{\rightarrow}) + \mathbb{P}_0[\tau_{\leftarrow} > \tau_{\rightarrow}] \mathbb{E}_0(X_{\tau_1} \cdot \mathbf{e}_1 \mid \tau_{\leftarrow} > \tau_{\rightarrow}) \\ &\geq -(b\lambda + K) \mathbb{P}_0[\tau_{\leftarrow} < \tau_{\rightarrow}] + b\lambda(1 - \mathbb{P}_0[\tau_{\leftarrow} < \tau_{\rightarrow}]) \\ &= b\lambda - (2b\lambda + K) \mathbb{P}_0[\tau_{\leftarrow} < \tau_{\rightarrow}], \end{aligned}$$

so

$$\mathbb{P}_0[\tau_{\leftarrow} < \tau_{\rightarrow}] \geq \frac{b\lambda}{2b\lambda + K} > \frac{3}{7} \quad (2.3)$$

since  $\lambda > \frac{3K}{b}$ .

Then, observe that Doob's inequality together with the fact that the jumps are bounded by  $K$  imply that (abbreviate  $s_\lambda := \frac{7(b\lambda+K)^2}{r^2h}$ )

$$\begin{aligned} \mathbb{P}_0 \left[ \max_{j \leq s_\lambda} |X_j \cdot \mathbf{e}_2| \geq a\lambda \right] &\leq \frac{\mathbb{E}_0(X_{\lfloor s_\lambda \rfloor} \cdot \mathbf{e}_2)^2}{a^2\lambda^2} \\ &\leq \frac{K^2 s_\lambda}{a^2\lambda^2} \\ &\leq \frac{7K^2(b+K)^2}{r^2ha^2} \leq \frac{1}{7}, \end{aligned} \quad (2.4)$$

recall that we assumed that  $a \geq \frac{7K(b+K)}{r\sqrt{h}}$ . The claim of Lemma 2.1 now follows from (2.2), (2.3), and (2.4).  $\square$

Now, we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* For  $x, y \in \mathbb{R}^2 \setminus \{0\}$ , let  $\theta(x, y) \in [0, 2\pi)$  be the angle between  $x$  and  $y$  in the anticlockwise direction; for definiteness, we set  $\theta(x, y) = 0$  if at least one of the vectors  $x, y$  equals 0. Let us abbreviate

$$a_0 := \frac{7K(1+K)}{r\sqrt{h}}, \quad (2.5)$$

$$\alpha_0 := \arctan \frac{1}{3a_0}, \quad (2.6)$$

$$m_0 := \left\lceil \frac{\pi}{\alpha_0} \right\rceil, \quad (2.7)$$

$$s_0 := 2^{m_0} \times 6Ka_0 = 42 \frac{2^{m_0} K^2 (1+K)}{r\sqrt{h}}. \quad (2.8)$$

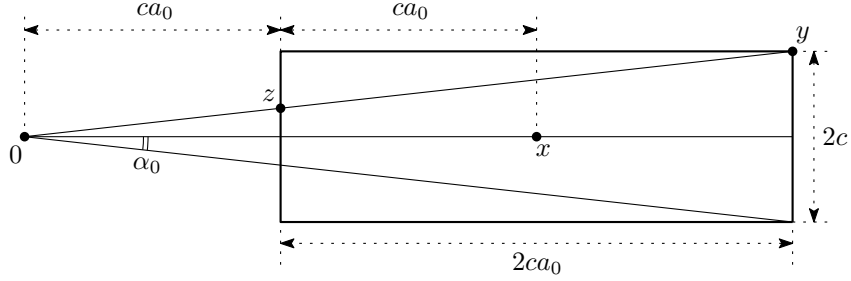


Figure 5: For  $\alpha_0 = \theta(x, y)$  we have  $\alpha_0 = \arctan \frac{1}{3a_0}$ . Observe that  $\|z\| \geq \|x\|/2$ ,  $\|y\| \leq 2\|x\|$ .

Now, we define inductively two sequences of stopping times:  $\sigma(0) = T(0) = 0$ , and

$$\begin{aligned} \sigma(m+1) &= \min \{k > \sigma(m) : \theta(X_{\sigma(m)}, X_k) \in [\alpha_0, \pi]\} \\ T(m+1) &= \min \left\{ k > \sigma(m) : X_k \cdot \frac{X_{\sigma(m)}}{\|X_{\sigma(m)}\|} \notin \left[ \frac{\|X_{\sigma(m)}\|}{2}, \frac{3\|X_{\sigma(m)}\|}{2} \right] \right\}, \end{aligned}$$

for  $m \geq 1$ .

Define the sequence of events  $A_k = \{\sigma(k) < T(k)\}$ ,  $k \geq 1$ . Lemma 2.1 then implies (see Figure 5 and recall that this lemma holds for rectangles with smaller side at least  $3K$ ) that

$$\begin{aligned} \mathbb{P}[A_{k+1} \mid \mathcal{F}_{\sigma(k)}, \|X_{\sigma(k)}\| \geq 6Ka_0] &\geq \mathbb{P}[G_{v_k, \lambda_k}^{a_0, 1, 1}(X_{\sigma(k)}) \mid \mathcal{F}_{\sigma(k)}, \|X_{\sigma(k)}\| \geq 6Ka_0] \\ &\geq \frac{1}{7}, \end{aligned} \tag{2.9}$$

where  $v_k = \left(\frac{X_{\sigma(k)}}{\|X_{\sigma(k)}\|}\right)^\perp$ ,  $\lambda_k = \frac{\|X_{\sigma(k)}\|}{2a_0}$ .

Now, let us define

$$\hat{\sigma} = \min\{k \geq 1 : X_k \cdot \mathbf{e}_1 < 0, 0 \leq X_k \cdot \mathbf{e}_2 < K\},$$

and abbreviate as  $\mathcal{P}_1$  the projector on the linear subspace spanned by  $\mathbf{e}_1$ . Then, (2.9) implies (see Figure 6) that for any  $x_0$  such that  $\|x_0\| \geq s_0$  (recall (2.8)),  $x_0 \cdot \mathbf{e}_1 > 0$ ,  $|x_0 \cdot \mathbf{e}_2| \leq K$  we have

$$\mathbb{P}_{x_0} \left[ X_{\hat{\sigma}} \cdot \mathbf{e}_1 \in [-2^{m_0}\|x_0\|, -2^{-m_0}\|x_0\|], \max_{k \leq \hat{\sigma}} \|X_k - \mathcal{P}_1 x_0\| < 2^{m_0}\|x_0\| \right] \geq \frac{1}{7^{m_0}}. \tag{2.10}$$

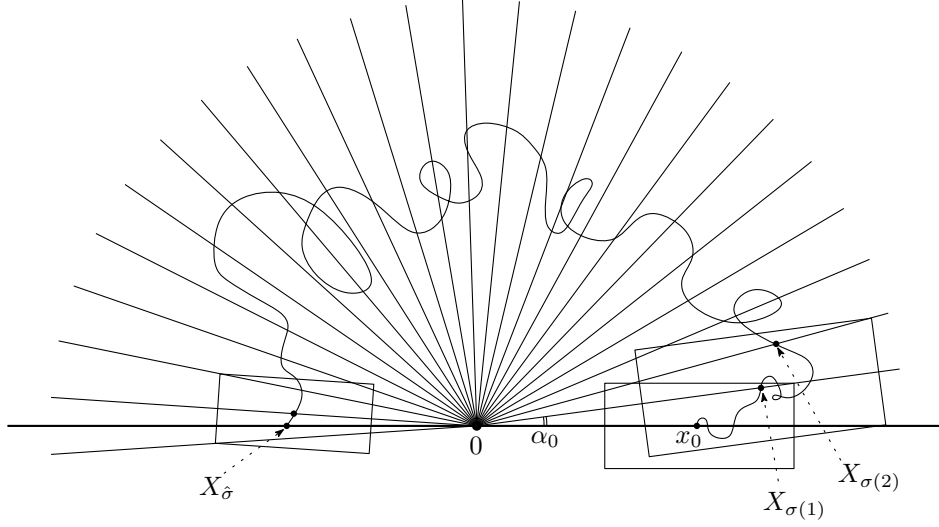


Figure 6: On the proof of (2.10). For this picture, we assume that the scale is such that  $K$  is not visible. The ratio of the longer side to the shorter side of the rectangles is  $a_0$ .

Now, we finish the proof of Theorem 1.5. Again, without loss of generality we assume that  $x = 0$  and  $v = \mathbf{e}_1$ . Also, let us assume that  $\lambda 2^{-m_0}(a \wedge c) > s_0$ .

For  $n \geq 1$ , let us define the sequence of stopping times, starting with  $\tau_0 = 0$ , by

$$\tau_n = \min \left\{ t \geq \tau_{n-1} : (X_t - X_{\tau_{n-1}}) \cdot \mathbf{e}_1 < -\lambda 2^{-m_0}(a \wedge c), \right. \\ \left. X_t \cdot \mathbf{e}_2 \in [0, K], \max_{k \in [\tau_{n-1}, t]} \|X_k - \mathcal{P}_1 X_{\tau_{n-1}}\| < \lambda(a \wedge c) \right\}$$

(we set formally  $\min \emptyset = +\infty$ ), and define the events  $M_n = \{\tau_n < \infty\}$ . From (2.10) we obtain that  $\mathbb{P}[M_n | \mathcal{F}_{\tau_{n-1}}] \geq 7^{-m_0}$ , and, clearly, it holds that

$$G_{\mathbf{e}_1, \lambda}^{a, b, c}(0) \supset \bigcup_{n=1}^{\lceil \frac{2^{m_0} b}{a \wedge c} \rceil} M_n$$

(since on each of the events  $M_n$  the process advances to the left by at least  $\lambda 2^{-m_0}(a \wedge c)$ , while staying inside the rectangle  $R_{\mathbf{e}_1, \lambda}^{a, b, c}(0)$ ). This concludes the

proof of Theorem 1.5; the explicit expressions for  $\varrho$  and  $\lambda_0$  are then given by

$$\lambda_0 = \frac{s_0}{a \wedge c} \quad (2.11)$$

and

$$\varrho = \exp\left(-m_0 \left\lceil \frac{2^{m_0} b}{a \wedge c} \right\rceil \ln 7\right), \quad (2.12)$$

with  $m_0, s_0$  defined by (2.5)–(2.8).  $\square$

### 3 Proof of Theorem 1.4

We consider first the case  $d = 2$ . For this case, abbreviate  $H_\ell^u := H_{\ell, \mathbb{R}^2}^u$ . Define  $Y_0 = X_0$ ,

$$Y_n = X_n - \sum_{k=0}^{n-1} D_k$$

for  $n \geq 1$ . Clearly,  $Y$  is a martingale with jumps uniformly bounded by  $K' = 2K$ . To prove that  $Y$  is uniformly elliptic, let us define

$$\tilde{D}_n = \begin{cases} \frac{D_n}{\|D_n\|}, & \text{on } \{D_n \neq 0\}, \\ \mathbf{e}_1, & \text{on } \{D_n = 0\}. \end{cases}$$

Then, observe that  $Y_{n+1} - Y_n = X_{n+1} - X_n - D_n$ , so we have

$$\mathbb{P}[(Y_{n+1} - Y_n) \cdot Z \geq r \mid \mathcal{F}_n] = \mathbb{P}[(X_{n+1} - X_n) \cdot Z \geq r \mid \mathcal{F}_n] \geq h \quad (3.1)$$

for  $Z \in \{\tilde{D}_n^\perp, -\tilde{D}_n^\perp\}$ , and

$$\mathbb{P}[(Y_{n+1} - Y_n) \cdot (-\tilde{D}_n) \geq r \mid \mathcal{F}_n] \geq \mathbb{P}[(X_{n+1} - X_n) \cdot (-\tilde{D}_n) \geq r \mid \mathcal{F}_n] \geq h. \quad (3.2)$$

Using that  $Y$  is a martingale and  $\tilde{D}_n$  is  $\mathcal{F}_n$ -measurable, we have  $\mathbb{E}((Y_{n+1} - Y_n) \cdot \tilde{D}_n \mid \mathcal{F}_n) = 0$ . Since (3.2) implies that  $\mathbb{E}((Y_{n+1} - Y_n) \cdot \tilde{D}_n \mid \mathcal{F}_n)^- = \mathbb{E}((Y_{n+1} - Y_n) \cdot \tilde{D}_n \mid \mathcal{F}_n)^+ \geq rh$  and we have also  $((Y_{n+1} - Y_n) \cdot \tilde{D}_n)^+ \leq 2K$ , it holds that<sup>1</sup>

$$\mathbb{P}[(Y_{n+1} - Y_n) \cdot \tilde{D}_n \geq rh/2] \geq \frac{rh}{4K}. \quad (3.3)$$

---

<sup>1</sup>it is elementary to obtain that for any random variable  $\xi$  with  $0 \leq \xi \leq a$  a.s. and  $\mathbb{E}\xi \geq b$ , it is true that  $\mathbb{P}[\xi \geq b/2] \geq b/(2a)$

Then, (3.1)–(3.3) imply that the process  $Y$  is uniformly elliptic with  $r' = \frac{rh}{2\sqrt{2}}$  and  $h' = \frac{rh}{4K}$  (recall that we assume without restricting generality that  $K \geq 1$ ).

Next, let us define

$$\begin{aligned}\tilde{\sigma} &= \min\{j \geq 1 : Y_j \in H_{-\ell}^u\}, \\ \sigma_0 &= \min\{j \geq 1 : Y_j \cdot \ell \leq 0\},\end{aligned}$$

and, for  $k \geq 1$

$$\sigma_k = \min\{j \geq 1 : Y_j \cdot \ell \geq k\}.$$

Now, our goal is to find a lower bound on the probability that, starting from  $0 \in \mathfrak{X}$ , the process avoids the origin for the next  $k$  steps. Clearly, since  $X_k - Y_k \in H_\ell^u$  a.s., for any  $k$  it holds that

$$\mathbb{P}_0[Y_1 \notin H_{-\ell}^u, \dots, Y_k \notin H_{-\ell}^u] \leq \mathbb{P}_0[X_1 \neq 0, \dots, X_k \neq 0], \quad (3.4)$$

so we concentrate on finding a lower bound for the term in the left-hand side of the above display.

For that, let us prove that there exists  $\beta > \frac{1}{2}$  and  $m_1$  such that for all  $m \geq m_1$

$$\mathbb{P}_y[\sigma_{2m} < \tilde{\sigma}] \geq \beta \quad (3.5)$$

for all  $y \in \mathfrak{X}$  such that  $y \cdot \ell \in [m, m + K)$ . First, note that the Optional Stopping Theorem implies analogously to (2.3) that

$$\mathbb{P}_y[\sigma_{2m} < \sigma_0] \geq \frac{m}{2m + K} \quad (3.6)$$

for all  $y$  such that  $y \cdot \ell \in [m, m + K)$ .

Abbreviate  $W_0^\ell = \{x \in \mathbb{R}^2 : x \cdot \ell \in (-K, 0]\}$  and consider two vectors  $v_1, v_2 \in \mathbb{S}^1$  such that  $v_j \cdot \ell > 0$  for  $j = 1, 2$  and  $v_1 + v_2$  is collinear to  $\ell$ . Then, there exist  $a_1, b_1, c_1, a_2, b_2, c_2 > 0$  such that

(i) we have

$$\begin{aligned}b_1 |\cos \theta(v_1, \ell)| - a_1 |\sin \theta(v_1, \ell)| &> 1, \\ a_1 |\sin \theta(v_1, \ell)| + c_1 |\cos \theta(v_1, \ell)| &< 1\end{aligned}$$

(ii)  $b_2 > 1$ ,



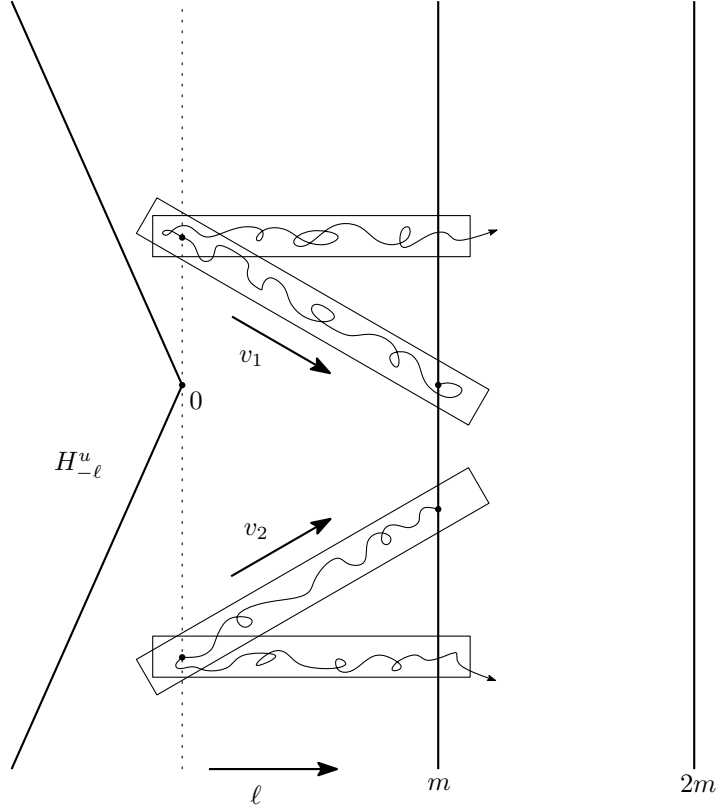


Figure 7: On the proof of (3.5)

(iii) for  $j = 1, 2$  the set

$$R_{v_j, m}^{a_1, b_1, c_1}(m\ell) \cup \bigcup_{y \in W_0 \cap R_{v_j, m}^{a_1, b_1, c_1}(m\ell)} R_{-\ell, m}^{a_2, b_2, c_2}(y)$$

does not intersect  $H_{-\ell}^u$ ,

see Figure 7. In words, (i) means that (for large enough  $m$ ) the left side of the inclined rectangles lies to the left of the dotted line (level 0) and the right side does not reach the level  $2m$ , (ii) means that the right side of the horizontal rectangle (relative to a point close to where the inclined rectangle intersect with the dotted line) is to the right of the level  $m$ , and (iii) means that the two rectangles cannot touch the set  $H_{-\ell}^u$ . Then, it is clear that Theorem 1.5 and (3.6) imply that there exist  $m_1, \varepsilon > 0$  such that for all  $m \geq m_1$

$$\mathbb{P}_y[\sigma_0 < \sigma_{2m}, \sigma_{2m} < \tilde{\sigma}] \geq \varepsilon \quad (3.7)$$

for any  $y$  such that  $y \cdot \ell \in [m, m + K)$ . Since

$$\mathbb{P}_y[\sigma_{2m} < \tilde{\sigma}] = \mathbb{P}_y[\sigma_{2m} < \sigma_0] + \mathbb{P}_y[\sigma_0 < \sigma_{2m}, \sigma_{2m} < \tilde{\sigma}],$$

we obtain (3.5) from (3.6) and (3.7).

Next, denote

$$k_1 = \left\lceil \log_2 \frac{K\sqrt{2n}}{m_1} \right\rceil;$$

observe that  $m_1 2^{k_1} \geq K\sqrt{2n}$ . Analogously to (2.4), applying Doob's inequality we obtain that

$$\mathbb{P}_y[\sigma_0 > n] \geq \frac{1}{2} \quad (3.8)$$

for all  $y$  such that  $y \cdot \ell \geq K\sqrt{2n}$ . Then, (3.5) implies that for any  $y$  such that  $y \cdot \ell \geq m_1$

$$\mathbb{P}_y[\sigma_{K\sqrt{2n}} < \tilde{\sigma}] \geq \beta^{k_1}. \quad (3.9)$$

So, using (3.8)–(3.9) and uniform ellipticity (to assure that the process can initially advance to level  $m_1$ ), we obtain for some  $\tilde{c} > 0$ ,  $\varepsilon > 0$ ,

$$\mathbb{P}_0[Y_1 \notin H_{-\ell}^u, \dots, Y_n \notin H_{-\ell}^u] = \mathbb{P}_0[\tilde{\sigma} > n] \geq \frac{(h')^{\lceil m_1/r' \rceil}}{2} \beta^{k_1} \geq \tilde{c} n^{-\frac{1}{2} + \varepsilon} \quad (3.10)$$

since  $\beta > \frac{1}{2}$ . Using (3.4), we obtain (1.4) for  $d = 2$ ; for  $d \geq 3$  it then follows if one considers the projection on  $\mathcal{L}$ .

To prove (1.5), define  $\hat{\tau}_0 = 0$ ,

$$\hat{\tau}_{k+1} = \min \{m > \hat{\tau}_k : X_m \notin \{X_0, \dots, X_{m-1}\}\};$$

i.e.,  $(\hat{\tau}_k, k \geq 0)$  is the sequence of times when the process enters previously unvisited sites. Then, to obtain (1.5), we use the fact that the process  $X^{(k)} = X_{+\hat{\tau}_k}$  satisfies conditions of the theorem, and apply the union bound (again, the projection argument implies that (1.5) holds for all  $d \geq 2$ ). This concludes the proof of Theorem 1.4.  $\square$

## 4 Final remarks and open problems

First, let us briefly sketch the proof of Theorem 1.2. So, suppose that  $X$  is a martingale in dimension  $d \geq 2$ , with bounded jumps and uniform ellipticity.

To begin, we show that there exist  $b \in (0, 1)$  close enough to 1 and  $\gamma' > 0$  (depending only on  $K, h, r$  — the constants in Definition 1.1 (a)–(b)) such that

$$\mathbb{E}(\|X_{n+1}\|^b \mid \mathcal{F}_n) \geq \|X_n\|^b \mathbf{1}_{\{\|Y_n\| > \gamma'\}}. \quad (4.1)$$

To see that (4.1) holds, first observe that for a fixed  $y \in \mathbb{R}^d$  we have

$$\begin{aligned} \|x + y\|^b &= (\|x\|^2 + 2x \cdot y + \|y\|^2)^{b/2} \\ &= \|x\|^b \left( 1 + b \frac{x \cdot y}{\|x\|^2} + \frac{b\|y\|^2}{2\|x\|^2} - \frac{1}{2}b(2-b) \frac{(x \cdot y)^2}{\|x\|^4} + o(\|x\|^{-2}) \right), \end{aligned} \quad (4.2)$$

as  $x \rightarrow \infty$ . So, denoting by  $\varphi_n$  the angle between  $x$  and  $\Delta_n := X_{n+1} - x$ , we have

$$\begin{aligned} &\mathbb{E}(\|X_{n+1}\|^b - \|X_n\|^b \mid \mathcal{F}_n, X_n = x) \\ &= \frac{b}{2\|x\|^{2-b}} (\mathbb{E}(\|\Delta_n\|^2 \mid \mathcal{F}_n) - (2-b)\mathbb{E}(\|\Delta_n\|^2 \cos^2 \varphi_n \mid \mathcal{F}_n) + o(\|x\|^{-2})). \end{aligned} \quad (4.3)$$

Using the uniform ellipticity and the boundedness of jumps, one can obtain that

$$\mathbb{E}(\|\Delta_n\|^2 \cos^2 \varphi_n \mid \mathcal{F}_n) < (1 - \varepsilon') \mathbb{E}(\|\Delta_n\|^2 \mid \mathcal{F}_n)$$

for some  $\varepsilon' > 0$ , so if  $b < 1$  is close enough to 1, the right-hand side of (4.3) is positive for all large enough  $x$  (see more details in the proof of Lemma 5.2 in [16]).

Denote by  $\mathcal{B}(x, s) = \{y \in \mathbb{R}^d : \|x - y\| \leq s\}$  the closed ball of radius  $s$  centered in  $x$ . The proof of Theorem 1.2 is now quite straightforward:

- the Optional Stopping Theorem implies that, starting from  $x_0 \in \mathbb{R}^d$ , the process  $X$  will reach  $\mathbb{R}^d \setminus \mathcal{B}(x_0, \alpha)$  (without coming back to 0) with probability at least  $O(\alpha^{-b})$  (to apply the Optional Stopping Theorem, one has first to force the process a bit away from the origin, which happens with positive probability by uniform ellipticity).
- Doob's inequality implies that from any place in  $\mathbb{R}^d \setminus \mathcal{B}(x_0, \alpha)$  with probability bounded away from 0 the process will not return to  $x_0$  after additional  $\tilde{c}_1 \alpha^2$  steps, where  $\tilde{c}_1 > 0$  is a small enough constant.

- Then, consider  $\alpha = \tilde{c}_2 n^{1/2}$  for large enough  $\tilde{c}_2$ . By the previous discussion, each time the process is in  $x_0$ , independently of the past it has probability at least of order  $n^{-b/2}$  of not returning to  $x_0$  during next  $n$  steps.
- Take  $\varepsilon > 0$  such that  $b + \varepsilon < 1$ . Then, by an obvious coin-tossing argument, the number of visits to  $x_0$  by time  $n$  will not exceed  $n^{\frac{b+\varepsilon}{2}}$  with probability at least  $1 - \tilde{c}_3 e^{-\tilde{c}_4 n^{\varepsilon/2}}$ .
- So, with probability at least  $1 - \tilde{c}_3 n e^{-\tilde{c}_4 n^{\varepsilon/2}}$  the process  $X$  will have to visit at least  $n^{1-\frac{b+\varepsilon}{2}}$  different sites.

Now, we see that this proof is in sharp contrast with the proof of Theorem 1.4. Originally, our intention was to find a proof for Theorem 1.4 that would use Lyapunov functions in a similar way; this amounts to finding a function  $f : \mathbb{R}^2 \mapsto \mathbb{R}$  with the following properties:  $f \equiv 0$  on  $\partial H_{\mathbf{e}_1}^{-u}$ ,

$$\sup_{x \in \partial H_{\mathbf{e}_1}^u + n^{1/2} \mathbf{e}_1} f(x) = O(n^{\frac{1}{2}-\varepsilon})$$

for some  $\varepsilon > 0$ , and  $f(X_{\cdot \wedge \tau})$  is a submartingale, where  $\tau$  is the hitting time of  $H_{-\mathbf{e}_1}^u$ . A possible idea would be to modify somehow the function  $f_w$  of formula (3.5) of [14], but we did not succeed in developing it properly.

A natural question is if the results of Theorems 1.2 and 1.4 can be improved. As for the results for the local time ((1.2) and (1.4)), it is not the case, as the following example shows:

**Example 4.1** *Consider a zero-drift random walk on  $\mathbb{Z}^2$ , defined in the way indicated on Figure 8. More specifically, we first divide the plane in sectors with (small enough) angle  $\alpha$ , and then define the transition probabilities in each sector in such a way, roughly speaking, that the walk “prefers” the radial direction to the transversal one. For any fixed  $b < 1$ , it is clear that one can define the parameters of the model so that the expression in the parentheses in the right-hand side of (4.3) is negative for all large enough  $x$  (because the absolute value of the cosine in (4.3) will typically be close to 1). Applying Theorem 1 of [2], we obtain that  $\mathbb{E}_0 \tau_0^p < \infty$  for any  $p < \frac{b}{2}$ , where  $\tau_0$  is the hitting time of the origin. But this implies that there typically will be  $n^p$  visits to the origin by time  $n$  (and that, with probability stretched-exponentially close to 1, the number of visits will be at least  $n^{p-\delta}$ , where  $\delta$  is an arbitrarily small positive number).*

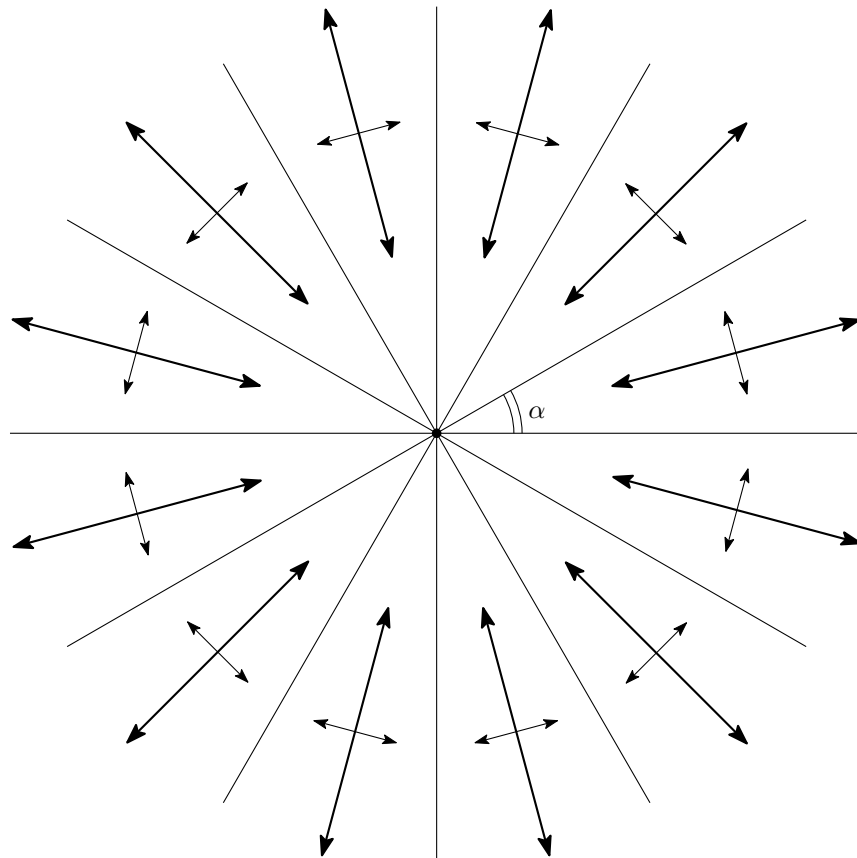


Figure 8: Example of a recurrent random walk with many visits to the origin.

However, the situation with (1.3) and (1.5) is less clear: it is an open problem to find out if (1.3) (respectively, (1.5)) should be valid for all  $\hat{\gamma} > 0$  (respectively, for all  $\gamma > 0$ ). In fact, the authors were unable to find any examples of uniformly elliptic martingales with uniformly bounded jumps for which the expected range is of order less than that of the simple random walk (as mentioned in the introduction, it is  $O(\frac{n}{\ln n})$  for  $d = 2$  and  $O(n)$  for  $d \geq 3$ ).

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