# Complete Two-Loop Electroweak Fermionic Corrections to the Effective Leptonic Weak Mixing Angle $\sin^2\theta_{\text{eff}}^{\text{lept}}$ and Indirect Determination of the Higgs Boson Mass

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We present a complete calculation of the contributions to the effective leptonic weak mixing angle, $\sin^2\theta_{\text{eff}}^{\text{lept}}$, generated by closed fermion loops at the two-loop level of the electroweak interactions. This quantity is the source of the most stringent bound on the mass $M_H$ of the standard model Higgs boson. The size of the corrections with respect to known partial results varies between $-4 \times 10^{-5}$ and $-8 \times 10^{-5}$ for a realistic range of $M_H$ from 100 to 300 GeV. This translates into a shift of the predicted (from $\sin^2\theta_{\text{eff}}^{\text{lept}}$ alone) central value of $M_H$ by $+19$ GeV, to be compared with the shift induced by a recent change in the measured top quark mass which amounts to $+36$ GeV.

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The search for the standard model (SM) Higgs boson lies among the most important objectives of present elementary particle physics. The experimental discovery will be possible at the CERN Large Hadron Collider within a mass range reaching up to 1 TeV. On the other hand, it is more than desirable to have as stringent indirect bounds on $M_H$ as possible with the help of precision measurements. Should the Higgs boson be discovered, these bounds will serve as a strong test of the model.

In this Letter we study the quantity that has the highest weight in the combined fit to electroweak data as far as $M_H$ prediction is concerned, which is the effective leptonic weak mixing angle, $\sin^2\theta_{\text{eff}}^{\text{lept}}$. It can be defined through the form factors at the $Z$ boson pole of the vertex coupling the $Z$ to leptons ($l$). If this vertex is written as $i\bar{l}\gamma^\mu(g_V - g_A\gamma_5)lZ_\mu$, then

$$\sin^2\theta_{\text{eff}}^{\text{lept}} = 1/4[1 - \text{Re}(g_V/g_A)]. \tag{1}$$

At the tree level this amounts to the sine of the weak mixing angle $\sin^2\theta_{\text{W}} = 1 - M_W^2/M_Z^2$ in the on-shell scheme. In practice, $\sin^2\theta_{\text{eff}}^{\text{lept}}$ is derived from various asymmetries measured around the $Z$ boson peak at $e^+e^-$ colliders after subtraction of QED effects. The current experimental value is $0.231\,50 \pm 0.000\,16$ [1]. The high precision quoted and the expected size of the radiative corrections make the result indispensable for a precise prediction of $M_H$. A lot of effort has been put into the theoretical calculation of $\sin^2\theta_{\text{eff}}^{\text{lept}}$. Besides the one-loop contributions also higher-order QCD corrections [2,3] are known. However, for the electroweak two-loop corrections, only the leading term in the large $M_H$ expansion [4] and the leading [5] and subleading [6] terms in the large top quark mass expansion are available up to now. The goal of the present work is the calculation of the complete two-loop electroweak contributions with one or two closed fermion loops.

The prediction Eq. (1) does not use $M_W$ as the input parameter, but the results are given by using the very precise measurement of the Fermi constant, $G_\mu$, from the muon decay lifetime to derive $M_W$. Consequently, the calculation of $\sin^2\theta_{\text{eff}}^{\text{lept}}$ as a function of $G_\mu$ involves also the computation of the radiative corrections to the relation between $G_\mu$ and $M_W$. For the electroweak two-loop corrections with closed fermion loops considered here, this has been carried out in Ref. [7]. We, therefore, also use the quantity $\Delta\kappa$,

$$\sin^2\theta_{\text{eff}}^{\text{lept}} = (1 - M_W^2/M_Z^2)(1 + \Delta\kappa), \tag{2}$$

which is only weakly sensitive to $M_W$, but encompasses the loop corrections to the $Z$ form factors.

We focus in this Letter on the discussion of our main results. A detailed description of the calculation will be given in a forthcoming publication. Here, we note only that the contributions to the form factors can be divided into two major parts. The first one comprises the terms from renormalization. We use the on-shell renormalization scheme, similarly to the previous calculation of $M_W$ [7]. The second one consists of the bare two-loop vertex diagrams, the total number of which approaches 500. Upon restriction to those containing a closed fermion loop we count only a few tens, which can be cast into four topologies as shown in Fig. 1. There is no dependence on the Higgs boson mass in the pure two-loop vertex diagrams, since $CP$ conservation makes Fig. 1(d) vanish. It is convenient to subdivide the remaining diagrams into those containing a top quark line and those containing only light fermion lines. The former can be evaluated with the large top quark mass expansion, using the smallness of the ratio $M_Z^2/m_t^2 \approx 1/4$. We have convinced our-

  

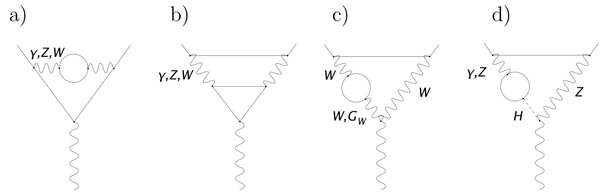


FIG. 1. Two-loop vertex diagrams entering the calculation.

selves that an expansion up to $(M_Z^2/m_t^2)^5$ is sufficient to obtain an intrinsic precision of the order of $10^{-7}$. The diagrams with only light fermion lines introduce also an important simplification: they have just two scales at most, $M_W$ and $M_Z$, since at the level we are considering, light fermion masses can be safely neglected. The problem thus contains only one variable and lends itself naturally to the approach of differential equations [8]. A prerequisite for this method is the complete reduction of the integrals to a small set of independent masters. This has been achieved with the C++ library DIAGEN/IDSOLVER [9] and has been checked for a number of diagrams by an independent calculation. At the end we obtained analytic expressions for all of the integrals but one. The latter, corresponding to Fig. 1(b), has been evaluated by a one-dimensional integral representation. All integrals have been checked by different expansions in physical and unphysical regimes and by numerical integrations based on dispersion relations [10] and Feynman parametrizations [11].

An interesting problem connected to two-loop vertex diagrams is the treatment of the $\gamma_5$ matrix in triangle fermion loops. We used the naive dimensional regularization with a four-dimensional treatment of resulting epsilon tensors as already explained in [7]. We observed, however, that the contributions are divergent due to the soft-collinear behavior of the diagrams with external on-shell massless fermions. This would undermine the correctness of the approach if the dimension of space-time were the only regulator. We decided to use a finite photon mass as the regulator at the expense of a subsequent difficult expansion corresponding to a mixed Sudakov-threshold regime. The difference between the full result and the result which would be obtained if all traces containing a single $\gamma_5$ were set to zero is denoted in what follows with a $\mathrm{tr}\gamma_5$ subscript.

We now discuss the numerical effect of the new two-loop result for the effective weak mixing angle. We focus on the contributions to $\Delta\kappa$, Eq. (2), taking the current experimental value for $M_W$ as input. The associated error is not relevant for the analysis, since the final prediction uses $G_\mu$ as input, combining the radiative corrections to $M_W$ and $\Delta\kappa$. We use the parameter values given in Table I. Note that the experimentally determined $W$ and $Z$ boson masses correspond to a Breit-Wigner parametrization with a running width and have to be translated to the pole mass scheme used in our calculation [7], resulting in a downward shift [12]. For $M_W$ and $M_Z$, this shift amounts to about 27 and 34 MeV, respectively.

Table II contains the values of the one- and two-loop electroweak corrections in comparison with different components with a single fermion loop for different values of the Higgs boson mass. The full two-loop result in the third column corresponds to the sum of the fourth, fifth and sixth columns plus the contributions with two fermion loops as well as the effect induced by the running, $\Delta\alpha$, of the fine structure constant. In the one-loop result we have kept a finite $b$ quark mass, which has an impact of the order of $-4.5 \times 10^{-5}$. The perturbative expansion is performed in the fine structure constant, $\alpha$, and not in $G_\mu$, since we want to avoid any uncontrolled "resummed" terms. The first observation is that the third quark family contributions are very large, which is expected, since they include the leading top-bottom mass splitting effects in the $\rho$ parameter, $\Delta\rho$ [13]. We have convinced ourselves that the result has the correct behavior for large $m_t$ [5]. It is interesting that even though the light fermion contributions in Table II do not contain the running, $\Delta\alpha$, of the fine structure constant, they are sizable. The last column gives the values of the $\mathrm{tr}\gamma_5$ contribution. It has to vanish for vanishing mass splittings in the fermion families and can be at most logarithmic for large top quark masses, which explains its smallness. For small $M_H$, the total two-loop result is rather small, but we note that this is due to a fragile cancellation strongly dependent on $m_t$. With the older value $m_t = 174.3$ GeV, the result would be of the order of $5 \times 10^{-4}$ for all values of $M_H$.

In order to provide the most precise prediction for $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$ in the SM we must use the muon decay constant, $G_\mu$, as the input parameter. The procedure to derive $M_W$ from $G_\mu$ is described in detail in [14]. In analogy to that work, we do not want to perform any "resummations." Instead, we include both in $\Delta r$ and $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$ all known effects in expanded form. Besides the electroweak two-loop terms presented above, these effects encompass

TABLE I. Input parameters with errors where relevant for the present analysis.

| Input | Value | Input | Value |
|---|---|---|---|
| $M_W$ | 80.426 GeV | $m_b$ | 4.85 GeV |
| $M_Z$ | $91.1876 \pm 0.0021$ GeV | $\Delta\alpha(M_Z^2)$ | $0.05907 \pm 0.00036$ |
| $\Gamma_Z$ | 2.4952 GeV | $\alpha_s(M_Z)$ | $0.117 \pm 0.002$ |
| $m_t$ | $178.0 \pm 4.3$ GeV | $G_\mu$ | $1.16637 \times 10^{-5}$ GeV$^{-2}$ |

TABLE II. One-loop and fermionic two-loop electroweak contributions to $\Delta\kappa$ with $M_W$ as input parameter. The subscripts "tb," "lf," and "tr$\gamma_5$" correspond to the contributions of single loops of the third quark family, of the light fermions (without the running of the fine structure constant), and of the tr$\gamma_5$ effects in the triangle fermion subloops, respectively (see the text).

| $M_H$ (GeV) | $\mathcal{O}(\alpha) \times 10^{-4}$ | $\mathcal{O}(\alpha^2)_{\rm ferm} \times 10^{-4}$ | $\mathcal{O}(\alpha^2)_{\rm tb} \times 10^{-4}$ | $\mathcal{O}(\alpha^2)_{\rm lf} \times 10^{-4}$ | $\mathcal{O}(\alpha^2)_{{\rm tr}\gamma_5} \times 10^{-4}$ |
|---|---|---|---|---|---|
| 100 | 438.94 | −0.63 | −16.96 | −2.84 | 0.27 |
| 200 | 419.60 | −2.16 | −17.10 | −3.08 | 0.27 |
| 600 | 379.56 | −5.01 | −16.89 | −3.77 | 0.27 |
| 1000 | 358.62 | −4.73 | −14.90 | −4.25 | 0.27 |

QCD corrections to the one-loop prediction at the two- [2] and three-loop levels [3] and also the recently obtained $\mathcal{O}(\alpha^2\alpha_s m_t^4)$ and $\mathcal{O}(\alpha^3 m_t^6)$ corrections to $\Delta\rho$ [15]. We kept again a finite $b$ quark mass in the $\mathcal{O}(\alpha\alpha_s)$ correction, which has an impact of $4.5 \times 10^{-5}$, almost completely canceling the similar effect in the $\mathcal{O}(\alpha)$ prediction. Consistency requires that we also take leading reducible effects at $\mathcal{O}(\alpha^2\alpha_s)$ and $\mathcal{O}(\alpha^3)$ into account. It turns out that separate terms as, e.g., $c_W^2/s_W^2\Delta\rho\Delta\alpha^2$ are quite sizable, but when summed cancel each other as seen in Table III. We stress once more at this point that the same effects have been included in $\Delta r$ and in $\sin^2\theta_{\rm eff}^{\rm lept}$. This means, in particular, that, contrary to [14], we do not take the bosonic corrections [16] to $\Delta r$ into account. Such precautions are enforced by the sensitivity of $\sin^2\theta_{\rm eff}^{\rm lept}$ to $M_W$, since a 1 MeV shift in the latter causes a shift of about $-2 \times 10^{-5}$ in the former.

Our complete result is summarized by the following fitting formula, which reproduces the exact calculation with maximal and average deviations of $4.5 \times 10^{-6}$ and $1.2 \times 10^{-6}$, respectively, as long as the input parameters stay within their $2\sigma$ ranges and the Higgs boson mass in the range 10 GeV $\leq M_H \leq$ 1 TeV:

$$\sin^2\theta_{\rm eff}^{\rm lept} = s_0 + d_1 L_H + d_2 L_H^2 + d_3 L_H^4 + d_4(\Delta_H^2 - 1) + d_5\Delta_\alpha + d_6\Delta_t + d_7\Delta_t^2 + d_8\Delta_t(\Delta_H - 1) + d_9\Delta_{\alpha_s} + d_{10}\Delta_Z,$$

$$L_H = \log\left(\frac{M_H}{100\ \text{GeV}}\right), \quad \Delta_H = \frac{M_H}{100\ \text{GeV}}, \quad \Delta_\alpha = \frac{\Delta\alpha}{0.05907} - 1, \quad \Delta_t = \left(\frac{m_t}{178.0\ \text{GeV}}\right)^2 - 1, \tag{3}$$

$$\Delta_{\alpha_s} = \frac{\alpha_s(M_Z)}{0.117} - 1, \qquad \Delta_Z = \frac{M_Z}{91.1876\ \text{GeV}} - 1,$$

with

$$s_0 = 0.2312527, \quad d_1 = 4.729 \times 10^{-4}, \quad d_2 = 2.07 \times 10^{-5}, \quad d_3 = 3.85 \times 10^{-6}, \quad d_4 = -1.85 \times 10^{-6}, \quad d_5 = 0.0207,$$

$$d_6 = -0.002851, \quad d_7 = 1.82 \times 10^{-4}, \quad d_8 = -9.74 \times 10^{-6}, \quad d_9 = 3.98 \times 10^{-4}, \quad d_{10} = -0.655. \tag{4}$$

The impact of this result is shown in Table IV, where we compare our prediction with the previous result as given in the fitting formula in [17] and implemented in ZFITTER [18]. The difference varies from roughly $-4 \times 10^{-5}$ to $-8 \times 10^{-5}$ for the $M_H$ range from 100 to 300 GeV, which is the preferred mass region inferred from precision electroweak data. These values reach half of the experimental error and induce an important shift in the central value of $M_H$ derived from $\sin^2\theta_{\rm eff}^{\rm lept}$ alone. With the most recent value of the top quark mass given in Table I the result shifts the central value from 149 to 168 GeV, to be compared with the shift induced by the new $m_t$ measurement which gives a jump from 132 to 168 GeV. The formula Eq. (3) has been implemented in the most recent version of ZFITTER, version 6.40 [19].

Besides providing an up-to-date fitting formula, it is necessary to discuss the error on the theoretical prediction connected with the unknown higher-order contributions. Here one has to incorporate the treatment of the error of the $M_W$ prediction, since the final prediction for $\sin^2\theta_{\rm eff}^{\rm lept}$ takes $G_\mu$ as input. In particular, there are some cancellations between the radiative corrections to $M_W$ and the $Z$ decay form factors that go into $\sin^2\theta_{\rm eff}^{\rm lept}$. We take the point of view that these cancellations are natural

TABLE III. Various QCD corrections to $\Delta\kappa$ and the only known pure three-loop electroweak irreducible contribution, stemming from $\Delta\rho$, in comparison with three-loop reducible effects. The input parameter is $M_W$.

| $M_H$ (GeV) | $\mathcal{O}(\alpha\alpha_s) \times 10^{-4}$ | $\mathcal{O}(\alpha\alpha_s^2) \times 10^{-4}$ | $\mathcal{O}(\alpha^2\alpha_s m_t^4) \times 10^{-4}$ | $\mathcal{O}(\alpha^3 m_t^6) \times 10^{-4}$ | Reducible $\times 10^{-4}$ |
|---|---|---|---|---|---|
| 100 | −36.83 | −7.32 | 1.25 | 0.17 | 0.92 |
| 200 | −36.83 | −7.32 | 2.08 | 0.09 | 0.94 |
| 600 | −36.83 | −7.32 | 4.07 | 0.07 | 0.97 |
| 1000 | −36.83 | −7.32 | 5.01 | 0.99 | 0.98 |

TABLE IV. Difference between the result Eq. (3) and the previous result including terms of $\mathcal{O}(\alpha^2 m_t^2)$ from [6], obtained from the ZFITTER implementation (left column) or from the fitting formula from [17].

| $M_H$ (GeV) | $(\Delta\sin^2\theta_{\rm eff}^{\rm lept})_{\rm ZFITTER} \times 10^{-4}$ | $(\Delta\sin^2\theta_{\rm eff}^{\rm lept}) \times 10^{-4}$ [17] |
|---|---|---|
| 100 | −0.45 | −0.40 |
| 200 | −0.69 | −0.72 |
| 300 | −0.85 | −0.83 |
| 600 | −1.17 | −0.94 |
| 1000 | −1.60 | −1.28 |

and discuss both quantities in conjunction. To this end, we use geometric progression from lower orders to estimate the missing higher-order contributions and add them quadratically at the end. In units of $10^{-5}$ we assign the following errors: corrections of $\mathcal{O}(\alpha^2\alpha_s)$ beyond $m_t^4$ vary between 2.3 and 2.0 for $M_H$ between 10 GeV and 1 TeV, corrections of $\mathcal{O}(\alpha^3)$ between 1.8 and 2.5, $\mathcal{O}(\alpha\alpha_s^3)$ between 1.1 and 1.0, $\mathcal{O}(\alpha^2\alpha_s^2)$ between 1.7 and 2.4, and finally the bosonic corrections at $\mathcal{O}(\alpha^2)$ are expected to be of the order of 1.2. This gives an error varying between 3.3 and 3.5, and we take as our estimate the latter, largest, value. To account for possible deviations from the geometric progression of the perturbation series, we included an additional overall factor of $\sqrt{2}$, giving a final error of $4.9 \times 10^{-5}$.

If we take into account all of the input parameter errors, our prediction can be compared with the experimental value as shown in Fig. 2.

In conclusion, we have calculated the complete fermionic corrections to $\sin^2\theta_{\rm eff}^{\rm lept}$ at the two-loop level and obtained a sizable contribution when compared to the previously known leading and subleading terms in the top quark mass expansion. Together with our result for the $W$ boson mass and recently obtained three-loop terms, we are able to give the most up-to-date prediction to be used in the global fit to electroweak data. Furthermore, we implemented the result in the program ZFITTER, widely used for this purpose.

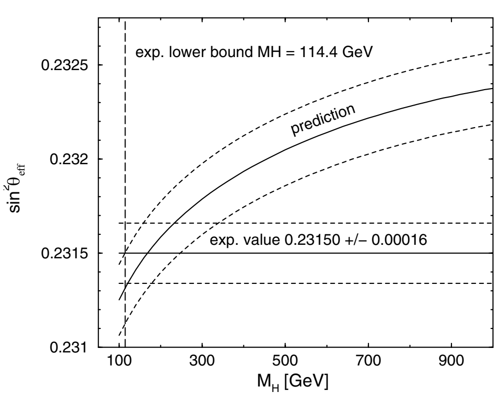


FIG. 2. The $\sin^2\theta_{\rm eff}^{\rm lept}$ prediction against the current experimental value, with $1\sigma$ bands from the experimental input.

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