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# Coloring Graphs Characterized by a Forbidden Subgraph <sup>\*</sup>

Petr A. Golovach<sup>1</sup>, Daniël Paulusma<sup>2</sup>, and Bernard Ries<sup>3</sup>

<sup>1</sup> Department of Informatics, University of Bergen, Norway,  
`petr.golovach@ii.uib.no`

<sup>2</sup> School of Engineering and Computing Sciences, Durham University,  
`daniel.paulusma@durham.ac.uk`

<sup>3</sup> Laboratoire d'Analyse et Modélisation de Systèmes pour l'Aide à la Décision,  
Université Paris Dauphine, `bernard.ries@dauphine.fr`

**Abstract.** The COLORING problem is to test whether a given graph can be colored with at most  $k$  colors for some given  $k$ , such that no two adjacent vertices receive the same color. The complexity of this problem on graphs that do not contain some graph  $H$  as an induced subgraph is known for each fixed graph  $H$ . A natural variant is to forbid a graph  $H$  only as a subgraph. We call such graphs strongly  $H$ -free and initiate a complexity classification of COLORING for strongly  $H$ -free graphs. We show that COLORING is NP-complete for strongly  $H$ -free graphs, even for  $k = 3$ , when  $H$  contains a cycle, has maximum degree at least 5, or contains a connected component with two vertices of degree 4. We also give three conditions on a forest  $H$  of maximum degree at most 4 and with at most one vertex of degree 4 in each of its connected components, such that COLORING is NP-complete for strongly  $H$ -free graphs even for  $k = 3$ . Finally, we classify the computational complexity of COLORING on strongly  $H$ -free graphs for all fixed graphs  $H$  up to seven vertices. In particular, we show that COLORING is polynomial-time solvable when  $H$  is a forest that has at most seven vertices and maximum degree at most 4.

## 1 Introduction

Graph coloring involves the labeling of the vertices of some given graph by integers called colors such that no two adjacent vertices receive the same color. The corresponding COLORING problem is to decide whether a graph can be colored with at most  $k$  colors for some given integer  $k$ . Due to the fact that COLORING is NP-complete for any fixed  $k \geq 3$  [15], there has been considerable interest in studying its complexity when restricted to certain graph classes. One of the most well-known results in this respect is due to Grötschel, Lovász, and Schrijver [9] who show that COLORING is polynomial-time solvable on perfect graphs.

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A well-known structural result that is useful for the design of algorithms for special graph classes is Brooks' Theorem (Theorem 5.2.4 in [6]), which states that any connected graph  $G$  that is neither complete nor an odd cycle can be colored with at most  $\Delta(G)$  colors where  $\Delta(G)$  is the maximum degree of  $G$ . General motivation, background and related work on coloring problems restricted to special graph classes can be found in several surveys [17, 18].

We study the complexity of the COLORING problem restricted to graph classes defined by forbidding a graph  $H$  as a (not necessarily induced) subgraph. So far, COLORING has not been studied in the literature as regards to such graph classes. Before we summarize some related results and present our results, we first state the necessary terminology and notations.

### 1.1 Terminology

We consider finite undirected graphs without loops and multiple edges. We refer to the textbook of Diestel [6] for any undefined graph terminology. Let  $G = (V, E)$  be a graph. The subgraph of  $G$  induced by a subset  $U \subseteq V$  is denoted  $G[U]$ . The graph  $G - u$  is obtained from  $G$  by removing vertex  $u$ . For a vertex  $u$  of  $G$ , its *open neighborhood* is  $N(u) = \{v \mid uv \in E\}$ , its *closed neighborhood* is  $N[u] = N(u) \cup \{u\}$ , and its degree is  $d(u) = |N(u)|$ . The maximum degree of  $G$  is denoted by  $\Delta(G)$  and the minimum degree by  $\delta(G)$ .

The *length* of a path or a cycle is the number of its edges. The *distance*  $\text{dist}(u, v)$  between two vertices  $u$  and  $v$  of  $G$  is the length of a shortest path between them. The *girth*  $g(G)$  is the length of a shortest cycle in  $G$ .

For two graphs  $F$  and  $G$ , we may write  $G \supseteq F$  if  $G$  contains  $F$  as a subgraph. We say that  $G$  is (*strongly*)  $H$ -free for some graph  $H$  if  $G$  has no subgraph isomorphic to  $H$ ; note that this is more restrictive than forbidding  $H$  as an induced subgraph.

A *subdivision* of an edge  $uv \in E$  is the operation that removes  $uv$  and adds a new vertex adjacent to  $u$  and  $v$ . A graph  $H$  is a *subdivision* of  $G$  if  $H$  is obtained from  $G$  by a sequence of edge subdivisions.

A *coloring* of  $G$  is a mapping  $c : V \rightarrow \{1, 2, \dots\}$ , such that  $c(u) \neq c(v)$  if  $uv \in E$ . We call  $c(u)$  the *color* of  $u$ . A  $k$ -coloring of  $G$  is a coloring  $c$  of  $G$  with  $1 \leq c(u) \leq k$  for all  $u \in V$ . If  $G$  has a  $k$ -coloring, then  $G$  is called  $k$ -colorable. The *chromatic number*  $\chi(G)$  is the smallest integer  $k$  such that  $G$  is  $k$ -colorable. The  $k$ -COLORING problem is to test whether a graph admits a  $k$ -coloring for some fixed integer  $k$ . If  $k$  is in the input, then we call this problem COLORING.

The graphs  $C_n$ ,  $K_n$ , and  $P_n$  denote the cycle, complete graph and path on  $n$  vertices, respectively.

### 1.2 Related Work

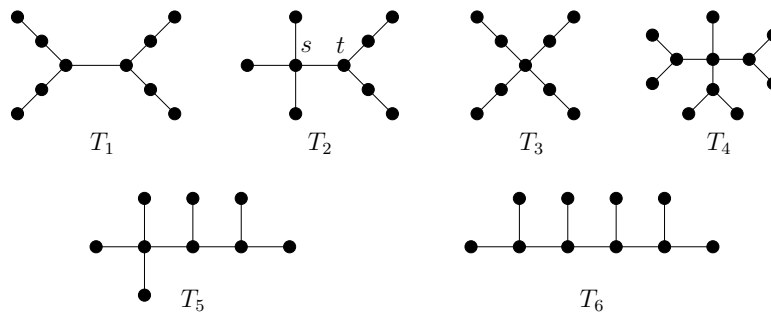
Král', Kratochvíl, Tuza and Woeginger [13] completely determined the computational complexity of COLORING for graph classes characterized by a forbidden *induced* subgraph and achieved the following dichotomy. Here,  $P_1 + P_3$  denotes the disjoint union of  $P_1$  and  $P_3$ .

**Theorem 1 ([13]).** *If some fixed graph  $H$  is a (not necessarily proper) induced subgraph of  $P_4$  or of  $P_1 + P_3$ , then COLORING is polynomial-time solvable on graphs with no induced subgraph isomorphic to  $H$ ; otherwise it is NP-complete on this graph class.*

The complexity classification of the  $k$ -COLORING problem for graphs with no induced subgraphs isomorphic to some fixed graph  $H$  is still open. For  $k = 3$ , it has been classified for graphs  $H$  up to six vertices [3], and for  $k = 4$  for graphs  $H$  up to five vertices [8]. We refer to the latter paper for a survey on the complexity status of  $k$ -COLORING for graph classes characterized by a forbidden induced subgraph and to a recent paper of Huang [10], who showed that 5-COLORING is NP-complete for  $P_6$ -free graphs and that 4-COLORING is NP-complete for  $P_7$ -free graphs.

### 1.3 Our Results

Recall that a strongly  $H$ -free graph denotes a graph with no subgraph isomorphic to some fixed graph  $H$ . Forbidding a graph  $H$  as an induced subgraph is equivalent to forbidding  $H$  as a subgraph if and only if  $H$  is a complete graph (a graph with an edge between any two distinct vertices). Hence, Theorem 1 tells us that COLORING is NP-complete for strongly  $H$ -free graphs if  $H$  is a complete graph on at least three vertices. We extend this result by proving the following two theorems in Sections 2 and 3, respectively; note that the case when  $H$  is a complete graph is covered by condition (a) of Theorem 2. The trees  $T_1, \dots, T_6$  are displayed in Figure 1. For an integer  $p \geq 0$ , the graph  $T_2^p$  is the graph obtained from  $T_2$  after subdividing the edge  $st$   $p$  times; note that  $T_2^0 = T_2$ .



**Fig. 1.** The trees  $T_1, \dots, T_6$ .

**Theorem 2.** *3-COLORING (and hence COLORING) is NP-complete for strongly  $H$ -free graphs if*

- (a)  $H$  contains a cycle, or

- (b)  $\Delta(H) \geq 5$ , or
- (c)  $H$  has a connected component with at least two vertices of degree 4, or
- (d)  $H$  contains a subdivision of the tree  $T_1$  as a subgraph, or
- (e)  $H$  contains the tree  $T_2^p$  as a subgraph for some  $0 \leq p \leq 9$ , or
- (f)  $H$  contains one of the trees  $T_3, T_4, T_5, T_6$  as a subgraph.

**Theorem 3.** COLORING is polynomial-time solvable for strongly  $H$ -free graphs if

- (a)  $H$  is a forest with  $\Delta(H) \leq 3$ , such that each connected component has at most one vertex of degree 3, or
- (b)  $H$  is a forest with  $\Delta(H) \leq 4$  and  $|V_H| \leq 7$ .

Theorems 1–3 tell us that the COLORING problem behaves differently on graphs characterized by forbidding  $H$  as an induced subgraph or as a subgraph. As a consequence of Theorems 2 and 3(b) we can classify the COLORING problem on strongly  $H$ -free graphs for graphs  $H$  up to 7 vertices. The problem is NP-complete if  $H$  is not a forest or  $\Delta(H) \geq 5$ , and polynomial-time solvable otherwise.

## 2 The Proof of Theorem 2

In the remainder of the paper we write  $H$ -free instead of strongly  $H$ -free as a shorthand notation. Here is the proof of Theorem 2.

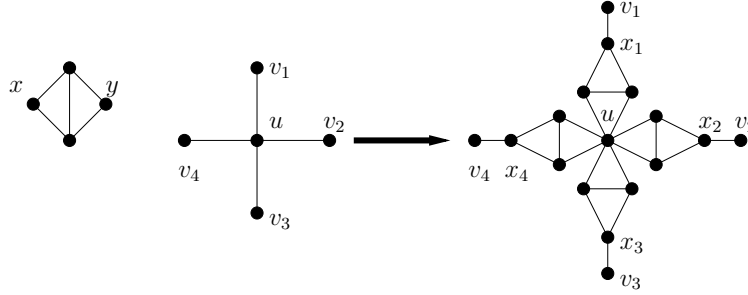
(a) Maffray and Preissmann [16] showed that 3-COLORING is NP-complete for triangle-free graphs. This result has been extended by Kamiński and Lozin [12], who proved that  $k$ -COLORING is NP-complete for the class of graphs of girth at least  $p$  for any fixed  $k \geq 3$  and  $p \geq 3$ . Suppose that  $H$  contains a cycle. Then  $g(H)$  is finite. Let  $p = g(H) + 1$ . It remains to observe that any graph of girth at least  $p$  does not contain  $H$  as a subgraph, and (a) follows.

For the remaining cases, namely cases (b)–(f), we reduce from the 3-COLORING problem restricted to graphs of maximum degree at most 4. It is well known that 3-COLORING is NP-complete for this graph class [7]. It will be readily seen that all our reductions can be carried out in polynomial time.

(b) Let  $G$  be a graph with  $\Delta(G) \leq 4$ . Then  $G$  does not contain a graph  $H$  with  $\Delta(H) \geq 5$  as a subgraph. Hence (b) holds.

(c) Let  $G = (V, E)$  be a graph of maximum degree at most 4. We define a useful graph operation. In order to do this, we need the graph displayed in Figure 2. It has vertex set  $\{x, y, z, t\}$  and edge set  $\{xz, xt, yz, yt, zt\}$  and is called a *diamond* with *poles*  $x, y$ . We observe that in any 3-coloring of a diamond with poles  $x, y$ , the vertices  $x$  and  $y$  are colored alike.

The graph operation that we use is displayed in Figure 2. For a vertex  $u \in V$  with four neighbors  $v_1, \dots, v_4$ , we do as follows. We delete the edges  $uv_i$  for  $i = 1, \dots, 4$ . We then add 4 diamonds with poles  $x_i, y_i$  for  $i = 1, \dots, 4$  and identify  $u$  with each  $y_i$ . Finally, we add the edges  $v_i x_i$  for  $i = 1, \dots, 4$ . We

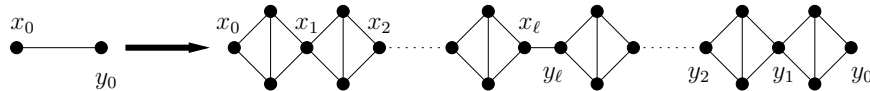


**Fig. 2.** A diamond with poles  $x, y$  and the vertex-diamond operation.

call this operation the *vertex-diamond* operation. Note that this operation is only defined on vertices of degree 4. Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if  $G$  is 3-colorable. We also observe that this operation when applied on a vertex  $u$  increases the distance between  $u$  and any other vertex of  $G$  by 2. Moreover, the new vertices added have degree 3.

To complete the proof of (c), let  $H$  be a graph that has a connected component  $D$  with at least two vertices of degree 4. Let  $\alpha$  denote the maximum distance between two such vertices in  $D$ . Then we apply  $\alpha$  vertex-diamond operations on each vertex of degree 4 in  $G$ . By our previous observations, the resulting graph  $G^*$  is  $D$ -free, and consequently,  $H$ -free, and in addition,  $G^*$  is 3-colorable if and only if  $G$  is 3-colorable. Hence (c) holds.

(d) Let  $G = (V, E)$  be a graph of maximum degree at most 4. We define the following graph operation displayed in Figure 3. For an edge  $x_0y_0 \in E$ , we do as follows. We delete the edge  $x_0y_0$  (but we keep the vertices  $x_0$  and  $y_0$ ) and add vertices  $x_1, y_1, \dots, x_\ell, y_\ell$ . We then construct diamonds with poles  $x_{i-1}, x_i$  and  $y_{i-1}, y_i$  respectively, for  $i = 1, \dots, \ell$ . Finally, we add the edge  $x_\ell y_\ell$ . We call this operation the *edge-diamond* operation of type  $\ell$ . We let  $G_\ell$  be the graph obtained from  $G$  after applying an edge-diamond operation of type  $\ell$  on each of its edges. Because any 3-coloring gives the poles of a diamond the same color,  $G_\ell$  is 3-colorable for any  $\ell \geq 1$  if and only if  $G$  is 3-colorable.



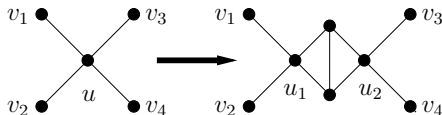
**Fig. 3.** The edge-diamond operation.

To complete the proof of (d), let  $H$  be a graph that contains a subdivision of  $T_1$ , which we will denote by  $T'$ . Let  $u, v$  be the vertices of degree 3 in  $T'$ . We choose  $\ell = \text{dist}_{T'}(u, v)$ . Then  $G_\ell$  is  $H$ -free, and (d) holds.

**subcases  $p = 0$  and  $p = 1$  of (e) and subcase  $\mathbf{H} \supseteq \mathbf{T}_5$  of (f).** Let  $G = (V, E)$  be a graph of maximum degree at most 4. We apply one vertex-diamond operation on each vertex of degree 4 in  $G$ . This results in a graph  $G^*$ . We observe that  $G^*$  is  $T_2^0$ -free,  $T_2^1$ -free and  $T_5$ -free, because every vertex of degree at least 4 in  $G^*$  is obtained by identifying pole vertices of diamonds. Recall that  $G^*$  is 3-colorable if and only if  $G$  is 3-colorable. Hence, the subcases  $p = 0$  and  $p = 1$  of (e) and the subcase  $H \supseteq T_5$  of (f) hold.

**remaining eight subcases of (e) and subcase  $\mathbf{H} \supseteq \mathbf{T}_6$  of (f).** Let  $G = (V, E)$  be a graph of maximum degree at most 4. To complete the proof of (e), let  $H$  be a graph that contains  $T_2^p$  as a subgraph for some  $2 \leq p \leq 9$ . Recall that the graph  $G_\ell$  defined in case (d) is 3-colorable if and only if  $G$  is 3-colorable. We choose  $\ell = \lceil \frac{p-1}{2} \rceil$ . Then  $G_\ell$  is  $H$ -free, and the remaining subcases of (e) hold. As an aside, note that for  $p \geq 10$ , there exists no  $\ell$  such that  $G_\ell$  is  $T_2^p$ -free, because for all  $\ell \geq 1$  we can “map” the degree-3 vertex  $t$  of  $T_2^p$  on a degree-4 vertex in  $G_\ell$  that corresponds to an original degree-4 vertex of  $G$ . Then we will either find in  $G_\ell$  a suitable vertex  $u$  that is in a diamond or that is a degree-4 vertex that corresponds to an original degree-4 vertex of  $G$ , such that we can “map” the degree-4 vertex  $s$  of  $T_2^p$  to  $u$  in order to obtain a subgraph in  $G_\ell$  that is isomorphic to  $T_2^p$ . Hence, the case  $p \geq 10$  is still open.

Now let  $H$  be a graph that contains  $T_6$  as a subgraph. We choose  $\ell = 1$ . Then  $G_2$  is  $H$ -free, and the corresponding subcase of (f) holds.



**Fig. 4.** The balanced-diamond operation.

**remaining two subcases of (f).** Let  $G = (V, E)$  be a graph of maximum degree at most 4. The last graph operation that we use is displayed in Figure 4. For a vertex  $u \in V$  with four neighbors  $v_1, \dots, v_4$ , we do as follows. We remove  $u$  and add two new vertices  $u_1$  and  $u_2$ . We make  $u_1$  adjacent to  $v_1$  and  $v_2$ , whereas we make  $u_2$  adjacent to  $v_3$  and  $v_4$ . Finally, we add two more vertices that together with  $u_1$  and  $u_2$  form a diamond, in which  $u_1$  and  $u_2$  are the poles. We call this operation the *balanced-diamond* operation. Note that we only define this operation on vertices of degree 4 (we refer to the paper of Kamiński and Lozin [11] for a more general variant called diamond implementation). Because any 3-coloring gives the poles of a diamond the same color, the resulting graph is 3-colorable if and only if  $G$  is 3-colorable.

To complete the proof of (f), let  $H$  be a graph that contains  $T_3$  or  $T_4$  as a subgraph. We apply the balanced-diamond operation on each vertex of degree 4 in  $G$ . The resulting graph  $G'$  is  $H$ -free. Moreover, by our observation,  $G'$  is 3-colorable if and only if  $G$  is 3-colorable. This concludes the proof of Theorem 2.  $\square$

### 3 The Proof of Theorem 3

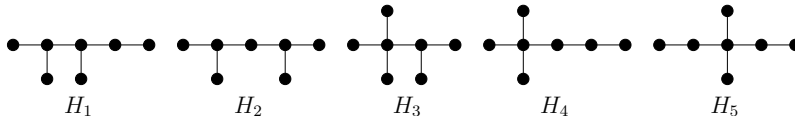
Let  $G$  be a graph. A graph  $H$  is a *minor* of  $G$  if  $H$  can be obtained from a subgraph of  $G$  by a sequence of edge contractions, or equivalently, if  $H$  can be obtained from  $G$  by a sequence of edge deletions, vertex deletions and edge contractions.

We start by proving the following theorem.

**Theorem 3(a).** *Let  $H$  be a fixed forest with  $\Delta(H) \leq 3$ , such that each connected component of  $H$  has at most one vertex of degree 3. Then COLORING can be solved in polynomial time for  $H$ -free graphs.*

*Proof.* Let  $H_1, \dots, H_p$  be the connected components of  $H$ . By our assumption on  $H$ , each  $H_i$  is either a path or a subdivided star, in which the centre vertex has degree 3. As such,  $H_i$  is a subgraph of a graph  $G$  if and only if  $H_i$  is a minor of  $G$ . Consequently,  $H$  is a subgraph of a graph  $G$  if and only if  $H$  is a minor of  $G$ . By a result of Bienstock et al. [2], every graph that does not contain  $H$  as a minor has path-width, and consequently treewidth, at most  $|V_H| - 2$ . Because COLORING can be solved in linear time on graphs of bounded treewidth as shown by Arnborg and Proskurowski [1], the result follows.  $\square$

Theorem 3(a) limits the remaining cases of Theorem 3(b) to those graphs  $H$  that are a forest on at most seven vertices and that contain a vertex of degree 4 or two vertices of degree at least 3. Moreover, our goal is to show polynomial-time solvability for such cases, and a graph is  $H$ -free if it is  $H'$ -free for any subgraph  $H'$  of  $H$ . This narrows down our case analysis to the trees  $H_1, \dots, H_5$  shown in Fig. 5. We consider each such tree, but we first give some auxiliary results.



**Fig. 5.** The trees  $H_1, \dots, H_5$ .

**Observation 1** *Let  $G$  be a graph with  $|V_G| \geq 2$ . Let  $u \in V_G$  with  $d_G(u) < k$  for some integer  $k \geq 1$ . Then  $G$  is  $k$ -colorable if and only if  $G - u$  is  $k$ -colorable.*



We say that a vertex  $u$  of a graph  $G$  is *universal* if  $G = G[N_G[u]]$ , that is, if  $u$  is adjacent to all other vertices of  $G$ .

**Observation 2** *Let  $u$  be a universal vertex of a graph  $G$  with  $|V_G| \geq 2$ . Let  $k \geq 2$  be an integer. Then  $G$  is  $k$ -colorable if and only if  $G - u$  is  $(k - 1)$ -colorable.*

A vertex  $u$  of a connected graph  $G$  with at least two vertices is a *cut-vertex* if  $G - u$  is disconnected. A maximal connected subgraph of  $G$  with no cut-vertices is called a *block* of  $G$ .

**Observation 3** *Let  $G$  be a connected graph, and let  $k$  be a positive integer. Then  $G$  is  $k$ -colorable if and only if each block of  $G$  is  $k$ -colorable.*

Let  $(G, k)$  be an instance of COLORING. We apply the following preprocessing rules *exhaustively*, which in our context means recursively and as long as possible; in particular, if after the application of some rule we can apply some other rule with a smaller index, then we will do this.

**Rule 1.** Find all connected components of  $G$  and consider each of them.

**Rule 2.** Check if  $G$  is 1-colorable or 2-colorable. If so, then stop considering  $G$ .

**Rule 3.** If  $|V_G| \geq 2$ ,  $k \geq 3$ , and  $G$  has a vertex  $u$  with  $d_G(u) \leq 2$ , take  $(G - u, k)$ .

**Rule 4.** If  $|V_G| \geq 2$ ,  $k \geq 3$ , and  $G$  has a universal vertex  $u$ , take  $(G - u, k - 1)$ .

**Rule 5.** If  $G$  is connected, then find all blocks of  $G$  and consider each of them.

We obtain the following lemma.

**Lemma 1.** *Let  $(G, k)$  be an instance of COLORING with  $k \geq 3$ . Exhaustively applying Rules 1–5 takes polynomial time and yields a set  $I$  of at most  $|V_G|$  instances, such that  $(G, k)$  is a yes-instance if and only if every instance of  $I$  is a yes-instance. Moreover, each  $(G', k') \in I$  has the following properties:*

- (i)  $|V_{G'}| \leq |V_G|$ ;
- (ii) if  $k' \geq 3$ , then  $\delta(G') \geq 3$ ;
- (iii) if  $k' \geq 3$ , then  $G'$  has no universal vertices;
- (iv)  $G'$  is 2-connected;
- (v)  $k' \leq k$ ;
- (vi) if  $G$  is  $H$ -free for some graph  $H$ , then  $G'$  is  $H$ -free as well.

*Proof.* Let  $(G, k)$  be an instance of COLORING with  $k \geq 3$ . We denote the number of vertices of  $G$  by  $n$ .

We first show that applying Rules 1–5 exhaustively takes polynomial time. Rule 1 takes linear time, because we only have to find the connected components of  $G$ . Rule 2 takes linear time, because  $G$  is 1-colorable if and only if  $G$  has no edges, and  $G$  is 2-colorable if and only if  $G$  is bipartite. Rules 3 and 4 take linear time, because we only need to check the degree of each vertex. Rule 5 takes linear time, because we only need to find the set of blocks of  $G$ . Because the size of  $G$  decreases after applying Rule 3 or Rule 4, our procedure terminates.

We now show that Rules 1–5 are *correct*, that is, applying them yields a set of one or more new instances such that the original instance is a yes-instance of COLORING if and only if each newly created instance is a yes-instance. It is readily seen that  $G$  is  $k$ -colorable if and only if each connected component of  $G$  is  $k$ -colorable. Hence, Rule 1 is correct. Clearly, Rule 2 is correct as well. Rule 3 is correct due to Observation 1. Rule 4 is correct due to Observation 2. Rule 5 is correct due to Observation 3. Hence, our procedure creates a set  $I$  of instances, such that  $(G, k)$  is a yes-instance if and only if each instance of  $I$  is a yes-instance. In particular, we note that  $(G, k)$  is a yes-instance if  $I = \emptyset$ , as in that case  $G$  is 2-colorable, and consequently,  $k$ -colorable, due to one or more applications of Rule 2.

The number of instances created only increases after applying Rule 1 or Rule 5. Because the total number of blocks of all connected components is at most  $n$ , the set  $I$  has size at most  $n$ .

Let  $(G', k')$  be an instance of  $I$ . Then  $|V_{G'}| \leq |V_G|$  because we only decreased the size of  $G$ . This proves (i). By Rule 3,  $G'$  has minimum degree at least 3 if  $k' \geq 3$ . This proves (ii). By Rule 4,  $G'$  has no universal vertices if  $k' \geq 3$ . This proves (iii). By Rule 5,  $G'$  is 2-connected. This proves (iv). By our assumption,  $k \geq 3$ . We have  $k' \leq k$ , because we do not increase  $k$  when applying Rules 1–5. This proves (v). Because we only removed vertices from  $G$ , we find that  $G'$  is a subgraph of  $G$ . Hence, if  $G$  is  $H$ -free for some graph  $H$ , then  $G'$  is  $H$ -free. This proves (vi).  $\square$

### 3.1 The Cases $H = H_1$ and $H = H_2$

We first give some extra terminology. Let  $G = (V, E)$  be a graph. We let  $\omega(G)$  denote the size of a maximum clique in  $G$ . The *complement* of  $G$  is the graph  $\overline{G}$  with vertex set  $V$ , such that any two distinct vertices are adjacent in  $\overline{G}$  if and only if they are not adjacent in  $G$ . If  $\chi(F) = \omega(F)$  for any induced subgraph  $F$  of  $G$ , then  $G$  is called *perfect*. We will use the Strong Perfect Graph Theorem proved by Chudnovsky et al. [5]. This theorem tells us that a graph is perfect if and only if it does not contain  $C_r$  or  $\overline{C}_r$  as an induced subgraph for any odd integer  $r \geq 5$ .

**Lemma 2.** *Let  $G$  be a 2-connected graph with  $\delta(G) \geq 3$  that has no universal vertices. If  $G$  is  $H_1$ -free or  $H_2$ -free, then  $G$  is perfect.*

*Proof.* Note that  $H_1$  and  $H_2$  are both subgraphs of  $\overline{C}_r$  for any  $r \geq 7$ . Moreover,  $C_5 = \overline{C}_5$ . Then, by the Strong Perfect Graph Theorem [5], we are left to prove that  $G$  contains no induced cycle  $C_r$  for any odd integer  $r \geq 5$ . To obtain a contradiction, assume that  $G$  does contain an induced cycle  $C = v_0v_1 \cdots v_{r-1}v_{r-1}v_0$  for some odd integer  $r \geq 5$ .

First suppose that  $G$  is  $H_1$ -free. Let  $0 \leq i \leq r-1$  and consider the path  $v_i v_{i+1} \cdots v_{i+3} v_{i+4}$ , where the indices are taken modulo  $r$ . Since  $\delta(G) \geq 3$ ,  $v_{i+1}$  and  $v_{i+2}$  each have at least one neighbor in  $V' = V \setminus \{v_0, \dots, v_{r-1}\}$ , say  $v_{i+1}$  is adjacent to some vertex  $u$  and  $v_{i+2}$  is adjacent to some vertex  $v$ . Because  $G$

is  $H_1$ -free,  $u = v$ , and moreover,  $|N(v_{i+1}) \cap V'| = |N(v_{i+2}) \cap V'| = 1$ . Because  $0 \leq i \leq r-1$  was taken arbitrarily, we deduce that the vertices  $v_0, \dots, v_{r-1}$  are all adjacent to the same vertex  $u \in V'$  and to no other vertices in  $V'$ . Because  $G$  is 2-connected,  $u$  is not a cut-vertex. Hence,  $V' = \{u\}$ . However, then  $u$  is a universal vertex. This is a contradiction.

Now suppose that  $G$  is  $H_2$ -free. By the same arguments and the fact that  $r$  is odd, we conclude again that there exists a universal vertex  $u \in V'$ . This is a contradiction.  $\square$

We are now ready to prove that COLORING is polynomial-time solvable for  $H_1$ -free and for  $H_2$ -free graphs. Let  $G$  be a graph, and let  $k \geq 1$  be an integer. If  $k \leq 2$ , then COLORING is even polynomial-time solvable for general graphs. Suppose that  $k \geq 3$ . Then, by Lemma 1, we may assume without loss of generality that  $G$  is 2-connected, has  $\delta(G) \geq 3$  and does not contain any universal vertices. Lemma 2 then tells us that  $G$  is perfect. Because Grötschel et al. [9] showed that COLORING is polynomial-time solvable for perfect graphs, our result follows.

### 3.2 The Case $H = H_3$

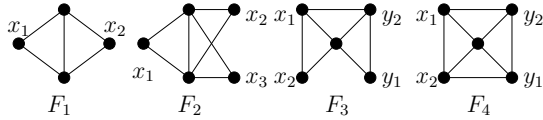
We start with showing the following useful lemma that gives an upper bound on the maximum degree of connected  $H_3$ -free graphs with no universal vertices and with minimum degree at least 3. We may impose the latter two conditions, because our polynomial-time algorithm for solving COLORING on  $H_3$ -free graphs will apply Rules 1–5 exhaustively.

**Lemma 3.** *Let  $G$  be a connected  $H_3$ -free graph with no universal vertices. If  $\delta(G) \geq 3$ , then  $\Delta(G) \leq 4$ .*

*Proof.* Let  $G = (V, E)$  be an  $H_3$ -free graph with no universal vertices. Suppose that  $\delta(G) \geq 3$ . To obtain a contradiction assume that  $d_G(u) \geq 5$  for some vertex  $u \in V$ . Because  $G$  has no universal vertices and because  $G$  is connected, there is a vertex  $v \in N_G(u)$  such that  $v$  has a neighbor  $x \in V \setminus N_G[u]$ . Because  $d_G(v) \geq \delta(G) \geq 3$ , we deduce that  $v$  has another neighbor  $y \notin \{u, x\}$ . Because  $d_G(u) \geq 5$ , we also deduce that  $u$  has three neighbors  $z_1, z_2, z_3$  neither equal to  $v$  nor to  $y$ . However, the subgraph of  $G$  with vertices  $u, v, x, y, z_1, z_2, z_3$  and edges  $uz_1, uz_2, uz_3, uv, vx, vy$  is isomorphic to  $H_3$ . This is a contradiction, because  $G$  is  $H_3$ -free.  $\square$

We now state some additional terminology. We say that we *identify* two distinct vertices  $u, v \in V_G$  if we first remove  $u, v$  and then add a new vertex  $w$  by making it (only) adjacent to the vertices of  $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$ .

Consider the graphs  $F_1, \dots, F_4$  shown in Fig. 6. Vertices  $x_1, x_2$  of  $F_1$ , vertices  $x_1, x_2, x_3$  of  $F_2$  and vertices  $x_1, x_2, y_1, y_2$  of  $F_3$  and  $F_4$  are called the *pole vertices* of the corresponding graph  $F_i$ , whereas the other vertices of  $F_i$  are called *centre vertices*. We say that a graph  $G$  *properly contains*  $F_i$  for some  $1 \leq i \leq 4$  if  $G$  contains  $F_i$  as an induced subgraph, in such a way that centre vertices of  $F_i$  are



**Fig. 6.** The graphs  $F_1, F_2, F_3, F_4$ .

only adjacent to vertices of  $F_i$ , that is, the subgraph  $F_i$  is connected to other vertices of  $G$  only via its poles.

Our polynomial-time algorithm for solving COLORING on  $H_3$ -free graphs will try to apply Rules 1–5 and one additional rule.

**Rule 6.** If  $G$  properly contains  $F_i$  for some  $1 \leq i \leq 4$ , then remove the centre vertices of  $F_i$  from  $G$  and identify the pole vertices of  $F_i$  as follows:

- if  $i = 1$ , then identify  $x_1$  and  $x_2$ ;
- if  $i = 2$ , then identify  $x_1, x_2$ , and  $x_3$ ;
- if  $i = 3$  or  $i = 4$ , then identify  $x_1$  and  $y_1$ , and also identify  $x_2$  and  $y_2$ .

The next lemma shows that we may safely apply Rule 6 on an  $H_3$ -free graph  $G$  with  $\delta(G) \geq 3$  and  $\Delta(G) \leq 4$ .

**Lemma 4.** *Let  $G$  be an  $H_3$ -free graph with  $\delta(G) \geq 3$  and  $\Delta(G) \leq 4$ . Let  $G'$  be the graph obtained from  $G$  after one application of Rule 6. Then  $G'$  is 3-colorable if and only if  $G$  is 3-colorable. Moreover,  $G'$  is  $H_3$ -free.*

*Proof.* Let  $G$  be an  $H_3$ -free graph with  $\delta(G) \geq 3$  and  $\Delta(G) \leq 4$  that properly contains a graph  $F_i$  for some  $1 \leq i \leq 4$ . Let  $G'$  be the graph obtained from  $G$  after applying Rule 6 with respect to  $F_i$ .

We first prove that  $G'$  is 3-colorable if and only if  $G$  is 3-colorable. First suppose that  $G'$  is 3-colorable. Consider a 3-coloring of  $G'$ . We color all vertices in  $V \setminus V_{F_i}$  by the same colors as in  $G'$ , the pole vertices of  $F_i$  are colored by the same color as the vertex obtained from them by the identification. It remains to observe that if  $i = 1$  or  $i = 2$ , then the neighbors of the two centre vertices are colored by one color, and if  $i = 3$  or  $i = 4$ , then the neighborhood of the unique centre vertex is colored by two colors. Hence, we can safely color the centre vertices of  $F_i$ . Now suppose that  $G$  is 3-colorable. Because in any 3-coloring of  $F_i$  the identified vertices are necessarily colored with the same color,  $G'$  is 3-colorable as well.

Now we show that  $G'$  is  $H_3$ -free. To obtain a contradiction, assume that  $G'$  has a subgraph  $H$  isomorphic to  $H_3$ . Let  $u$  be the vertex of degree 4 in  $H$ , and let  $v$  be the vertex of degree 3. Because  $G$  is  $H_3$ -free, at least one of  $u, v$  must be obtained by identifying pole vertices of  $F_i$ .

First suppose that  $u$  is not obtained by identifying pole vertices of  $F_i$ . Then  $v$  must be obtained by identifying pole vertices of  $F_i$ . Then, in  $G$ , we find that  $u$  is adjacent to a vertex  $v'$  that is a pole vertex of  $F_i$  and that corresponds to  $v$  in  $G'$  by the identification of pole vertices. Moreover, because  $u$  has degree 4

in  $G'$ , we find that  $u$  has three other neighbors  $z_1, z_2, z_3$  not equal to  $v'$  in  $G$  that are not identified with each other or with  $v'$  after applying Rule 6; one of them may still be a pole vertex in the case that  $i = 3$  or  $i = 4$ , but then such  $z_i$  is identified with some vertex of  $G$  not in  $\{v', z_1, z_2, z_3\} \setminus \{z_i\}$ . Also,  $z_1, z_2, z_3$  cannot be centre vertices of  $F_i$ , as centre vertices are removed by Rule 6.

Because  $u$  is in  $G'$  and Rule 6 removes centre vertices of  $F_i$ , we find that  $u$  is not a centre vertex of  $F_i$ . Because  $u$  is not a pole vertex of  $F_i$  either, this means that  $u \in V \setminus V_{F_i}$ . If  $i = 1$  or  $i = 2$ , then let  $w_1$  and  $w_2$  be the two centre vertices of  $F_i$ . Then the subgraph of  $G$  with vertices  $u, v', w_1, w_2, z_1, z_2, z_3$  and edges  $uv', uz_1, uz_2, uz_3, v'w_1, v'w_2$  is isomorphic to  $H_3$ . This is a contradiction. Hence,  $i = 3$  or  $i = 4$ .

Let  $w$  be the unique centre vertex of  $F_i$  and assume that  $v' \in \{x_1, x_2\}$ . Let  $v''$  denote the other vertex of  $\{x_1, x_2\}$ . If none of the vertices  $z_1, z_2, z_3$  is in  $V_{F_i}$ , then the subgraph of  $G$  that has vertices  $u, v', v'', w, z_1, z_2, z_3$  and edges  $uv', uz_1, uz_2, uz_3, v'w, v'v''$  is isomorphic to  $H_3$ . This is a contradiction. Therefore, one of the vertices  $z_1, z_2, z_3$ , say  $z_1$ , is a pole vertex of  $F_i$ . Note that  $z_2$  and  $z_3$  are not in  $F_i$ , as we already deduced. We also deduced that  $z_1$  is not identified with  $v'$ . Suppose that  $z_1 \in \{y_1, y_2\}$ . Then again the subgraph of  $G$  that has vertices  $u, v', v'', w, z_1, z_2, z_3$  and edges  $uv', uz_1, uz_2, uz_3, v'w, v'v''$  is isomorphic to  $H_3$ , which is a contradiction. Hence,  $z_1 \in \{x_1, x_2\}$ . If  $z_1 = x_1$ , then  $v' = x_2$ . Then the subgraph of  $G$  with vertices  $u, v', w, y_2, z_1, z_2, z_3$  and edges  $wv', uz_1, uz_2, uz_3, z_1w, z_1y_2$  is isomorphic to  $H_3$ . If  $z_1 = x_2$ , then  $v' = x_1$ . Then the subgraph of  $G$  with vertices  $u, v', w, y_2, z_1, z_2, z_3$  and edges  $wv', uz_1, uz_2, uz_3, v'w, v'y_2$  is isomorphic to  $H_3$ . Both cases are not possible. We conclude that  $u$  must be obtained by identifying pole vertices, namely  $x_1$  and  $x_2$  if  $i = 1$ ,  $x_1, x_2, x_3$  if  $i = 2$ , and we may assume without loss of generality that  $u$  is obtained by identifying  $x_1$  and  $y_1$  if  $i = 3$  or  $i = 4$ .

First suppose that  $i = 1$ . Because  $\Delta(G) \leq 4$  and  $d_{G'}(u) = 4$ , each pole  $x_j$  must have two neighbors  $s_1^j$  and  $s_2^j$  in  $G$  that are not in  $F_1$  for  $j = 1, 2$ . Because  $G'$  contains  $H_3$ , one of the vertices  $s_1^1, s_2^1, s_1^2, s_2^2$ , say  $s_1^1$ , has two neighbors  $t_1$  and  $t_2$  in  $G$  that are not in  $V_{F_1} \cup \{s_1^1, s_2^1, s_1^2, s_2^2\}$ . Let  $w_1$  and  $w_2$  denote the two centre vertices of  $F_1$ . We find that the subgraph of  $G$  with vertices  $s_1^1, s_2^1, t_1, t_2, w_1, w_2, x_1$  and edges  $x_1s_1^1, x_1s_2^1, x_1w_1, x_1w_2, s_1^1t_1, s_1^1t_2$  is isomorphic to  $H_3$ . This is a contradiction.

Now suppose that  $i = 2$ . Because  $\Delta(G) \leq 4$  and  $d_{G'}(u) = 4$ , one pole, say  $x_1$ , has two neighbors  $s_1$  and  $s_2$  in  $G$  that are not in  $F_2$ . Let  $w_1$  and  $w_2$  denote the two centre vertices of  $F_2$ . We find that the subgraph of  $G$  with vertices  $s_1, s_2, w_1, w_2, x_1, x_2, x_3$  and edges  $x_1s_1, x_1s_2, x_1w_1, x_1w_2, w_1x_2, w_1x_3$  is isomorphic to  $H_3$ . This is a contradiction.

Finally suppose that  $i = 3$  or  $i = 4$ . Recall that we assume that  $u \in V_H$  was obtained by identifying  $x_1$  and  $y_1$ . Then, because  $d_{G'}(u) = 4$  and  $\Delta(G) \leq 4$ , we find that  $i = 3$  and that  $y_1$  has two neighbors  $s_1$  and  $s_2$  in  $G$  that are not in  $F_3$ . Let  $w$  denote the centre vertex of  $F_3$ . We find that the subgraph of  $G$  with vertices  $s_1, s_2, w, x_1, x_2, y_1, y_2$  and edges  $y_1s_1, y_1s_2, y_1y_2, y_1w, wx_1, wx_2$

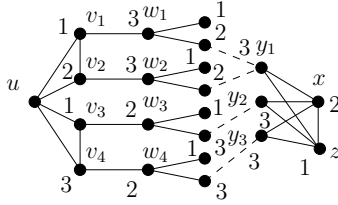
is isomorphic to  $H_3$ . This is a contradiction. We conclude that  $u$  cannot be obtained by identifying pole vertices. This completes the proof of Lemma 4.  $\square$

Before we can present our polynomial-time algorithm that solves COLORING for  $H_3$ -free graphs, we prove one final lemma.

**Lemma 5.** *Let  $G$  be an  $H_3$ -free graph with  $\delta(G) \geq 3$  and  $\Delta(G) \leq 4$  that does not properly contain any of the graphs  $F_1, \dots, F_4$ . Then  $G$  is 3-colorable if and only if  $G$  is  $K_4$ -free.*

*Proof.* Let  $G = (V, E)$  be an  $H_3$ -free graph with  $\delta(G) \geq 3$  and  $\Delta(G) \leq 4$  that does not properly contain any of the graphs  $F_1, \dots, F_4$ . First suppose that  $G$  is 3-colorable. This immediately implies that  $G$  is  $K_4$ -free.

Now suppose that  $G$  is  $K_4$ -free. If  $\Delta(G) \leq 3$ , then Brooks' Theorem (cf. [6]) tells us that  $G$  is 3-colorable unless  $G = K_4$ , which is not the case. Hence, we may assume that  $G$  contains at least one vertex of degree 4. To obtain a contradiction, assume that  $G$  is a minimal counter-example, that is,  $\chi(G) \geq 4$  and the graph obtained from  $G - v$  by removing vertices of degree at most 2 as long as possible is 3-colorable for all  $v \in V$ ; note that this graph may be empty.



**Fig. 7.** The structure of the graph  $G$ . We note that neighbors of  $w_1, \dots, w_4$  not equal to  $v_1, \dots, v_4$  may not be distinct.

Let  $u$  be a vertex of degree 4 in  $G$ , and let  $N_G(u) = \{v_1, v_2, v_3, v_4\}$ . We first show the following four claims.

- (a)  $G[N_G(u)]$  is  $C_3$ -free;
- (b)  $G[N_G(u)]$  contains no vertex of degree 3;
- (c)  $G[N_G(u)]$  is not isomorphic to  $P_4$ ;
- (d)  $G[N_G(u)]$  is not isomorphic to  $C_4$ .

Claims (a)–(d) can be seen as follows. If  $G[N_G(u)]$  contains  $C_3$  as a subgraph, then  $G[N_G(u)]$ , and consequently,  $G$  contains  $K_4$  as a subgraph of  $G$ . This proves (a). If  $G[N_G(u)]$  contains a vertex of degree 3, then  $G$  properly contains  $F_2$ , as  $G[N_G(u)]$  is  $C_3$ -free due to (a). This proves (b). If  $G[N_G(u)]$  is isomorphic to  $P_4$ , then  $G$  properly contains  $F_3$ . This proves (c). If  $G[N_G(u)]$  is isomorphic to  $C_4$ , then  $G$  properly contains  $F_4$ . This proves (d).

Because  $G$  is  $H_3$ -free, each  $v_j$  has at most one neighbor in  $V \setminus N_G[u]$ . Because  $\delta(G) \geq 3$ , this means that  $G[N_G(u)]$  contains no isolated vertices. Then, by claims (a)–(d), we find that  $G[N_G(u)]$  contains exactly two edges. Moreover,  $d_G(v_j) = 3$  for  $j \in \{1, \dots, 4\}$  as  $\delta(G) \geq 3$ .

We assume without loss of generality that  $v_1v_2$  and  $v_3v_4$  are edges in  $G$ . Let  $w_j$  be the neighbor of  $v_j$  in  $V \setminus N_G[u]$  for  $j = 1, \dots, 4$ . We note that  $w_1 \neq w_2$  and  $w_3 \neq w_4$ , as otherwise  $G$  properly contains  $F_1$ .

Because  $G$  is a minimal counterexample, we find that the graph obtained from  $G - u$  by removing vertices of degree at most 2 as long as possible is 3-colorable. Hence,  $G - u$  is 3-colorable. Let  $c$  be an arbitrary 3-coloring of  $G - u$ . We show that the following two claims are valid for  $c$  up to a permutation of the colors 1, 2, 3.

- (1)  $c(v_1) = c(v_3) = 1$ ,  $c(v_2) = 2$  and  $c(v_4) = 3$ ;
- (2)  $c(w_1) = c(w_2) = 3$  and  $c(w_3) = c(w_4) = 2$ ;

Claims (1) and (2) can be seen as follows. If  $c$  uses at most two different colors on  $v_1, \dots, v_4$ , then we can extend  $c$  to a 3-coloring of  $G$ , which is not possible as  $\chi(G) \geq 4$ . Hence,  $c$  uses three different colors on  $v_1, \dots, v_4$ . Then we may assume without loss of generality that  $c(v_1) = c(v_3) = 1$ ,  $c(v_2) = 2$  and  $c(v_4) = 3$ . This proves (1). We now prove (2). In order to obtain a contradiction, assume that  $c(w_1) \neq c(w_2)$ . Because  $c(v_2) = 2$ , we find that  $c(w_2) = 1$  or  $c(w_2) = 3$ . If  $c(w_2) = 1$ , then we change the color of  $v_2$  into 3, contradicting (1). Hence,  $c(w_2) = 3$ . Then, as  $c(v_1) = 1$ , we obtain  $c(w_1) = 2$ . However, we can now change the colors of  $v_1$  and  $v_2$  into 3 and 1, respectively, again contradicting (1). We conclude that  $c(w_1) = c(w_2)$ . Hence,  $c(w_1) = c(w_2) = 3$ . By the same arguments, we find that  $c(w_3) = c(w_4)$ . Hence,  $c(w_3) = c(w_4) = 2$ . This proves (2).

The facts that  $w_1 \neq w_2$  and  $w_3 \neq w_4$  together with Claim (2) imply that  $w_1, w_2, w_3, w_4$  are four distinct vertices. We observe that  $d_G(w_j) = 3$  for  $j = 1, \dots, 4$ , as otherwise  $H_3$  is a subgraph of  $G$ . See Fig. 7 for an illustration. In this figure we also indicate that  $w_1, w_2$  have neighbors colored with colors 1 and 2, and that  $w_3, w_4$  have neighbors colored with colors 1 and 3, as otherwise we could recolor  $w_1, \dots, w_4$  such that  $c(w_1) \neq c(w_2)$  or  $c(w_3) \neq c(w_4)$ , and hence we would contradict Claim (2). We may also assume without loss of generality that  $c$  is chosen in such a way that the set of vertices with color 1 is maximal, that is, each vertex with color 2 or 3 has a neighbor with color 1.

Consider the subgraph  $Q$  of  $G - u$  induced by the vertices colored with colors 2 and 3. We claim that the vertices  $w_1$  and  $v_2$  are in the same connected component of  $Q$ . To show this, suppose that there is a connected component  $Q'$  of  $Q$  that contains  $w_1$  but not  $v_2$ . Then we recolor all vertices of  $Q'$  colored 2 with color 3 and all vertices of  $Q'$  colored 3 with color 2. We obtain a 3-coloring of  $G - u$  such that  $w_1$  and  $v_2$  are colored by distinct colors, contradicting Claim (2). Using the same arguments, we conclude that  $w_3$  and  $v_4$  are in the same connected component of  $Q$ . Now we show that all the vertices  $w_1, v_2, w_3, v_4$  are in the same connected component of  $Q$ . Suppose that there is a connected component  $Q'$  of  $Q$  that contains  $w_1, v_2$  but not  $w_3, v_4$ . Then we recolor all vertices of  $Q'$  colored 2

with color 3 and all vertices colored 3 with color 2. We obtain a 3-coloring of  $G - u$  such that  $w_1, w_2, w_3, w_4$  are colored with the same color, contradicting Claim (2).

We observe that  $d_Q(w_1) = d_Q(v_2) = d_Q(w_3) = d_Q(v_4) = 1$ . Then, because  $w_1, v_2, w_3, v_4$  belong to the same connected component of  $Q$ , we find that  $Q$  contains a vertex  $x$  with  $d_Q(x) \geq 3$ .

Let  $y_1, \dots, y_r$  denote the neighbors of  $x$  in  $Q$  for some  $r \geq 3$ . Because  $y_1, \dots, y_r$  are colored with the same color, they are pairwise non-adjacent. Because  $\Delta(G) \leq 4$ , we find that  $r \leq 4$ . First suppose that  $r = 4$ . Because  $d_G(y_1) \geq 3$  as  $\delta(G) \geq 3$  and  $y_1, \dots, y_4$  are pairwise non-adjacent,  $y_1$  has at least two neighbors in  $V \setminus N_G[x]$ . However, then  $G$  contains  $H_3$  as a subgraph. This is a contradiction. Now suppose that  $r = 3$ . Recall that the set of vertices with color 1 is maximal. Hence  $x$  is adjacent to a vertex  $z$  with color 1. Because  $G$  is  $H_3$ -free and  $d_G(y_i) \geq 3$  for  $i = 1, 2, 3$ , we find that  $z$  is adjacent to  $y_1, y_2, y_3$ . However, since  $\Delta(G) \leq 4$ , this means that  $G[N_G[z]]$  is isomorphic to  $F_2$ . Consequently,  $G$  properly contains  $F_2$ . This contradiction completes the proof of Lemma 5.  $\square$

We are now ready to prove that COLORING can be solved in polynomial time for  $H_3$ -free graphs. Let  $G$  be an  $H_3$ -free graph on  $n$  vertices, and let  $k \geq 1$  be an integer.

**Case 1.**  $k \leq 2$ .

Then COLORING can be solved in polynomial time even for general graphs.

**Case 2.**  $k \geq 3$ .

By Lemma 1, we may assume without loss of generality that  $\delta(G) \geq 3$  and that  $G$  contains no universal vertices. By Lemma 3 we find that  $\Delta(G) \leq 4$ . Because  $G$  has no universal vertices,  $G \neq K_5$ . Then applying Brooks' Theorem (cf. [6]) yields that  $G$  is 4-colorable.

**Case 2a.**  $k \geq 4$ .

Then  $(G, k)$  is a yes-answer.

**Case 2b.**  $k = 3$ .

We apply Rule 6 exhaustively. This takes polynomial time, because each application of Rule 6 takes linear time and reduces the size of  $G$ . In order to maintain the properties of having minimum degree at least 3 and containing no universal vertices, we first apply Rules 1–5 exhaustively before another application of Rule 6. Afterward, by Lemmas 1 and 4, we have found in polynomial time a (possibly empty) set  $\mathcal{G}$  of at most  $n$  graphs, such that  $G$  is 3-colorable if and only if each graph in  $\mathcal{G}$  is 3-colorable. Moreover, each  $G' \in \mathcal{G}$  is  $H_3$ -free, has minimum degree at least 3, contains no universal vertices, and in addition, does not properly contain any of the graphs  $F_1, \dots, F_4$ . Then, by Lemma 3, each  $G' \in \mathcal{G}$  has  $\Delta(G') \leq 4$ . As a consequence, we may apply Lemma 5. This lemma tells us that a graph  $G' \in \mathcal{G}$  is 3-colorable if and only if it does not contain  $K_4$  as a subgraph. As we can check the latter condition in polynomial time and  $|\mathcal{G}| \leq n$ , that is, we have at most  $n$  graphs to check, also the last step of our algorithm runs in polynomial time.



### 3.3 The Cases $H = H_4$ and $H = H_5$

For these cases we replace Rule 4 by a new rule. Let  $G = (V, E)$  be a graph and  $k$  be an integer.

**Rule 4\***. If  $k \geq 3$  and  $V \setminus N_G[u]$  is an independent set for some  $u \in V$ , take  $(G[N_G(u)], k - 1)$ .

The next lemma shows that Rule 4 is correct.

**Lemma 6.** *Let  $k \geq 2$  be an integer, and let  $u$  be a vertex of a graph  $G = (V, E)$  such that  $V \setminus N_G[u]$  is an independent set. Then  $G$  is  $k$ -colorable if and only if  $G[N_G(u)]$  is  $(k - 1)$ -colorable.*

*Proof.* First suppose that  $G$  is  $k$ -colorable. Let  $c$  be a  $k$ -coloring of  $G$ . Then the vertices of  $N_G(u)$  are colored with at most  $k - 1$  colors, which are different from  $c(u)$ . Hence,  $G[N_G(u)]$  is  $(k - 1)$ -colorable. Now suppose that  $G[N_G(u)]$  is  $(k - 1)$ -colorable. Then we extend this coloring to a  $k$ -coloring of  $G$  by coloring  $V \setminus N_G(u)$  with a new color.  $\square$

We will also need the following lemma.

**Lemma 7.** *Let  $G = (V, E)$  be a 2-connected graph with  $\delta(G) \geq 3$  such that  $V \setminus N_G[u]$  contains at least two adjacent vertices for all  $u \in V$ . If  $G$  is  $H_4$ -free or  $H_5$ -free, then  $\Delta(G) \leq 3$ .*

*Proof.* Let  $G = (V, E)$  be a 2-connected graph with  $\delta(G) \geq 3$  such that  $V \setminus N_G[u]$  contains at least two adjacent vertices for all  $u \in V$ . Assume that  $G$  has a vertex  $u$  with  $d_G(u) \geq 4$ . We will show that  $G$  contains a subgraph isomorphic to  $H_4$  and a subgraph isomorphic to  $H_5$ .

By our assumption,  $V \setminus N_G[u]$  contains two adjacent vertices  $v$  and  $w$ . We choose  $v$  and  $w$  so that at least one of them, say  $v$ , is adjacent to a vertex  $z_1 \in N_G(u)$ . Because  $d_G(u) \geq 4$ , we find that  $N_G(u)$  contains at least three other vertices, which we denote by  $z_2, z_3$  and  $z_4$ . Then the subgraph of  $G$  with vertices  $u, v, w, z_1, z_2, z_3, z_4$  and edges  $uz_1, uz_2, uz_3, uz_4, z_1v, vw$  is isomorphic to  $H_4$ . Because  $G$  is 2-connected,  $G$  contains a path  $P$  from  $w$  to  $u$  that neither uses  $v$  nor  $z_1$ . Let  $v'$  be the vertex of  $P$  that is in  $V \setminus N_G[u]$  and that is adjacent to a neighbor of  $u$ , say to  $z_2$ . Then the subgraph of  $G$  with vertices  $u, v, v', z_1, z_2, z_3, z_4$  and edges  $uz_1, uz_2, uz_3, uz_4, z_1v, z_2v'$  is isomorphic to  $H_5$ .  $\square$

We are now ready to prove that COLORING can be solved in polynomial time for  $H_4$ -free graphs and for  $H_5$ -free graphs. Let  $G = (V, E)$  be a graph, and let  $k \geq 1$  be an integer. If  $k \leq 2$ , then COLORING can be solved in polynomial time even for general graphs. Now suppose that  $k \geq 3$ . Lemma 6 shows that Rule 4\* is correct. Moreover, an application of Rule 4\* takes linear time and reduces the number of vertices of  $G$  by at least one. Hence, we can replace Rule 4 by Rule 4\* in Lemma 1. Due to this, we may assume without loss of generality that  $G$  is 2-connected and has  $\delta(G) \geq 3$ , and moreover, that  $V \setminus N_G[u]$  contains at least

two adjacent vertices for all  $u \in V$ . Then Lemma 7 tells us that  $\Delta(G) \leq 3$ . By using Brooks' Theorem (cf. [6]) we find that  $G$  is 3-colorable, unless  $G = K_4$ . Hence,  $(G, k)$  is a yes-answer when  $k \geq 4$ , whereas  $(G, k)$  is a yes-answer when  $k = 3$  if and only if  $G \neq K_4$ .

## 4 Conclusions

We classified the complexity of COLORING restricted to strongly  $H$ -free graphs for all graphs  $H$  up to seven vertices. We also identified an infinite number of polynomial-time solvable and NP-complete cases. The only open cases left are when  $H$  is a forest on at least eight vertices that does not satisfy the conditions of Theorem 2 (for instance, we may assume that each connected component of  $H$  has at most one vertex of degree 4). However, the exact borderline between tractability and hardness is not clear. Even determining the computational complexity of COLORING restricted to strongly  $H$ -free graphs for some graphs  $H$  on eight vertices, such as the 8-vertex trees that contain the graph  $H_3$ , seems to be a difficult task.

As our current proof techniques are rather diverse, a more unifying approach may be required in order to complete the computational complexity classification of COLORING for strongly  $H$ -free graphs. Also the fact that COLORING (and even the more general problem PRECOLORING EXTENSION [4]) is polynomial-time solvable for graphs of maximum degree at most 3 makes the problem harder to classify for strongly  $H$ -free graphs than some other decision problems that are NP-complete for graphs of maximum degree at most 3. To illustrate this, we consider the INDEPENDENT SET problem, which is the problem of deciding whether a graph has an independent set of at least  $k$  vertices for some given integer  $k$ . It is well known that INDEPENDENT SET is already NP-complete for graphs of maximum degree at most 3 [7]. This allows us to use a well-known and simple edge-replacing gadget in order to prove that INDEPENDENT SET is NP-complete on strongly  $H$ -free graphs for almost all graphs  $H$ .

**Proposition 1.** *Let  $H$  be a graph. Then INDEPENDENT SET is polynomial-time solvable for strongly  $H$ -free graph if  $H$  is a forest with  $\Delta(H) \leq 3$ , each connected component of which contains at most one vertex of degree 3. In all other cases, INDEPENDENT SET is NP-complete for strongly  $H$ -free graphs.*

*Proof.* First suppose that  $H$  is a forest with  $\Delta(H) \leq 3$ , each connected component of which contains at most one vertex of degree 3. We apply exactly the same arguments as we used in the proof of Theorem 3(a) in order to show that INDEPENDENT SET is polynomial-time solvable on strongly  $H$ -free graphs.

Now suppose that  $H$  contains at least one connected component that contains either a vertex of degree at least 4 or two vertices of degree 3 or a cycle. Recall that INDEPENDENT SET is NP-complete on graphs of maximum degree at most 3 [7]. Hence, INDEPENDENT SET is NP-complete on strongly  $H$ -free graphs if  $H$  contains a vertex of degree at least 4. Due to this, we are left with the case

when  $H$  is a graph with  $\Delta(H) \leq 3$  that contains either two vertices of degree 3 or a cycle.

If INDEPENDENT SET is NP-complete for a graph class  $\mathcal{G}$ , then it remains NP-complete on the graph class obtained by subdividing each edge of each graph of  $\mathcal{G}$  exactly twice (the subdivision of an edge  $uv$  in a graph replaces  $uv$  by two new edges  $uw$  and  $wv$  for some new vertex  $w$ ). Hence, INDEPENDENT SET is NP-complete on graphs of maximum degree at most 3 that have girth at least  $g$  for any fixed  $g \geq 3$  (the girth of a graph is the length of a shortest induced cycle in the graph) such that any two vertices of degree 3 are of distance at least  $h$  for any fixed  $h \geq 1$ . As a consequence, INDEPENDENT SET is NP-complete for strongly  $H$ -free graphs.  $\square$

We note that, just as the complexity classification of  $k$ -COLORING (see Section 1.2), also the complexity classification of INDEPENDENT SET is wide open when  $H$  is forbidden as an induced subgraph, and that so far only partial results have obtained; very recently, Lokshtanov, Vatshelle, and Villanger [14] solved a long-standing open problem by showing that INDEPENDENT SET is polynomial-time solvable on  $P_3$ -free graphs.

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