

## Durham Research Online

---

### Deposited in DRO:

16 January 2015

### Version of attached file:

Accepted Version

### Peer-review status of attached file:

Peer-reviewed

### Citation for published item:

Belmonte, R. and Hof, van 't P. and Kaminski, M. and Paulusma, D. (2014) 'Forbidden induced subgraphs and the price of connectivity for feedback vertex set.', in 39th International Symposium, MFCS 2014, Budapest, Hungary, 26-29 August 2014 ; proceedings, Part II. Berlin, Heidelberg: Springer, pp. 57-68. Lecture notes in computer science. (8635).

### Further information on publisher's website:

[http://dx.doi.org/10.1007/978-3-662-44465-8\\_6](http://dx.doi.org/10.1007/978-3-662-44465-8_6)

### Publisher's copyright statement:

The final publication is available at Springer via [http://dx.doi.org/10.1007/978-3-662-44465-8\\_6](http://dx.doi.org/10.1007/978-3-662-44465-8_6)

### Additional information:

## Use policy

---

The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a [link](#) is made to the metadata record in DRO
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.

Please consult the [full DRO policy](#) for further details.

# Forbidden Induced Subgraphs and the Price of Connectivity for Feedback Vertex Set

Rémy Belmonte<sup>1</sup>, Pim van 't Hof<sup>2,\*</sup>, Marcin Kamiński<sup>3,\*\*</sup>, and  
Daniël Paulusma<sup>4,\*\*\*</sup>

<sup>1</sup> Dept. of Architecture and Architectural Engineering, Kyoto University, Japan  
`remybelmonte@gmail.com`

<sup>2</sup> Department of Informatics, University of Bergen, Norway  
`pim.vanthof@ii.uib.no`

<sup>3</sup> Institute of Computer Science, University of Warsaw, Poland  
`mjk@mimuw.edu.pl`

<sup>4</sup> School of Engineering and Computing Sciences, Durham University, UK  
`daniel.paulusma@durham.ac.uk`

**Abstract.** Let  $\text{fvs}(G)$  and  $\text{cfvs}(G)$  denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph  $G$ , respectively. For a graph class  $\mathcal{G}$ , the price of connectivity for feedback vertex set (poc-fvs) for  $\mathcal{G}$  is defined as the maximum ratio  $\text{cfvs}(G)/\text{fvs}(G)$  over all connected graphs  $G$  in  $\mathcal{G}$ . It is known that the poc-fvs for general graphs is unbounded. We study the poc-fvs for graph classes defined by a finite family  $\mathcal{H}$  of forbidden induced subgraphs. We characterize exactly those finite families  $\mathcal{H}$  for which the poc-fvs for  $\mathcal{H}$ -free graphs is bounded by a constant. Prior to our work, such a result was only known for the case where  $|\mathcal{H}| = 1$ .

## 1 Introduction

A *feedback vertex set* of a graph is a subset of its vertices whose removal yields an acyclic graph, and a feedback vertex set is connected if it induces a connected graph. We write  $\text{fvs}(G)$  and  $\text{cfvs}(G)$  to denote the cardinalities of a minimum feedback vertex set and a minimum connected feedback vertex set of a graph  $G$ , respectively. Let  $\mathcal{G}$  be a class of graphs. The *price of connectivity for feedback vertex set* (poc-fvs) for  $\mathcal{G}$  is defined to be the maximum ratio  $\text{cfvs}(G)/\text{fvs}(G)$  over all connected graphs  $G$  in  $\mathcal{G}$ . Graphs consisting of two disjoint cycles that are connected to each other by an arbitrarily long path show that the poc-fvs for general graphs is not upper bounded by a constant, and the same clearly holds for planar graphs. Interestingly, Grigoriev and Sitters [6] showed that the poc-fvs for planar graphs of minimum degree at least 3 is at most 11. Schweitzer

---

\* Supported by the Research Council of Norway (197548/F20).

\*\* Supported by Foundation for Polish Science (HOMING PLUS/2011-4/8) and National Science Center (SONATA 2012/07/D/ST6/02432).

\*\*\* Supported by EPSRC (EP/G043434/1) and Royal Society (JP100692).

and Schweitzer [7] later improved this upper bound from 11 to 5, and showed the upper bound of 5 to be tight.

In a previous paper [1], we studied the poc-fvs for graph classes characterized by a single forbidden induced subgraph. We proved that the poc-fvs for  $H$ -free graphs is bounded by a constant  $c_H$  if and only if  $H$  is a linear forest, i.e., a disjoint union of paths. In fact, we obtained a more refined tetrachotomy result that determines, for every graph  $H$ , which of the following cases holds: (i)  $\text{cfvs}(G) = \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ ; (ii) there exists a constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$ ; (iii) there exists a constant  $c_H$  such that  $\text{cfvs}(G) \leq c_H \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ ; (iv) there does not exist a constant  $c_H$  such that  $\text{cfvs}(G) \leq c_H \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ .

The concept of “price of connectivity”, introduced by Cardinal and Levy [4], has been studied for other parameters as well. One such parameter is the vertex cover number of a graph. Let  $\tau(G)$  and  $\tau_c(G)$  denote the cardinalities of a minimum vertex cover and a minimum connected vertex cover of a graph  $G$ , respectively. For a graph class  $\mathcal{G}$ , the price of connectivity for vertex cover for  $\mathcal{G}$  is defined as the worst-case ratio  $\tau_c(G)/\tau(G)$  over all connected graphs  $G$  in  $\mathcal{G}$ . It is known that for general graphs, the price of connectivity for vertex cover is upper bounded by 2, and this bound is sharp [2]. Cardinal and Levy [4] showed that for  $n$ -vertex graphs with average degree  $\epsilon n$ , this bound can be improved to  $2/(1 + \epsilon)$ . Camby et al. [2] provided forbidden induced subgraph characterizations of graph classes for which the price of connectivity for vertex cover is upper bounded by 1,  $4/3$ , and  $3/2$ , respectively.

The price of connectivity for dominating set (poc-ds) for a graph class  $\mathcal{G}$  is defined as the maximum ratio  $\gamma_c(G)/\gamma(G)$  over all connected graphs  $G$  in  $\mathcal{G}$ , where  $\gamma_c(G)$  and  $\gamma(G)$  denote the domination number and the connected domination number of  $G$ , respectively. It is easy to prove that the poc-ds for general graphs is upper bounded by 3 [5]. Motivated by the work of Zverovich [8], Camby and Schaudt [3] studied the poc-ds for  $(P_k, C_k)$ -free graphs for several values of  $k$ . Their results show that the poc-ds for  $(P_8, P_9)$ -free graphs is upper bounded by 2, while the general upper bound of 3 is asymptotically sharp for  $(P_9, C_9)$ -free graphs.

*Our contribution.* We continue the line of research on the price of connectivity for feedback vertex set we initiated in [1]. For a family of graphs  $\mathcal{H}$ , a graph  $G$  is called  $\mathcal{H}$ -free if  $G$  does not contain an induced subgraph isomorphic to any graph  $H \in \mathcal{H}$ . The vast majority of well-studied graph classes have forbidden induced subgraphs characterizations, and such characterizations can often be exploited when proving structural or algorithmic properties of these graph classes. In fact, for every hereditary graph class  $\mathcal{G}$ , that is, for every graph class  $\mathcal{G}$  that is closed under taking induced subgraphs, there exists a family  $\mathcal{H}$  of graphs such that  $\mathcal{G}$  is exactly the class of  $\mathcal{H}$ -free graphs. Notable examples of graph classes that can be characterized using a *finite* family of forbidden induced subgraphs include claw-free graphs, line graphs, proper interval graphs, split graphs and cographs.

Our main result establishes a dichotomy between the finite families  $\mathcal{H}$  for which the price of connectivity for feedback vertex set for  $\mathcal{H}$ -free graphs is upper bounded by a constant  $c_{\mathcal{H}}$  and the families  $\mathcal{H}$  for which such a constant  $c_{\mathcal{H}}$  does not exist. This can be seen as an extension of the case (iii) from [1] (mentioned above) from monogenic to finitely defined classes of graphs. In order to formally state our main result, we need to introduce some terminology.

For two graphs  $H_1$  and  $H_2$ , we write  $H_1 + H_2$  to denote the disjoint union of  $H_1$  and  $H_2$ . We write  $sH$  to denote the disjoint union of  $s$  copies of  $H$ . For any  $r \geq 3$ , we write  $C_r$  to denote the cycle on  $r$  vertices. For any three integers  $i, j, k$  with  $i, j \geq 3$  and  $k \geq 1$ , we define  $B_{i,j,k}$  to be the graph obtained from  $C_i + C_j$  by choosing a vertex  $x$  in  $C_i$  and a vertex  $y$  in  $C_j$ , and adding a path of length  $k$  between  $x$  and  $y$ .

It is clear that the price of connectivity for feedback vertex set for the class of all butterflies is not bounded by a constant, since  $\text{fvs}(B_{i,j,k}) = 2$  and  $\text{cfs}(B_{i,j,k}) = k + 1$  for every  $i, j \geq 3$  and  $k \geq 1$ . Roughly speaking, our main result states that the price of connectivity for feedback vertex set for the class of  $\mathcal{H}$ -free graphs is bounded by a constant  $c_{\mathcal{H}}$  if and only if the forbidden induced subgraphs in  $\mathcal{H}$  prevent arbitrarily large butterflies from appearing as induced subgraphs. To make this statement concrete, we need the following definition.

**Definition 1.** *Let  $i, j \geq 3$  be two integers, let  $\mathcal{H}$  be a family of graphs, and let  $N = 2 \cdot \max_{H \in \mathcal{H}} |V(H)| + 1$ . The family  $\mathcal{H}$  covers the pair  $(i, j)$  if  $\mathcal{H}$  contains an induced subgraph of  $B_{i,j,N}$ . A graph  $H$  covers the pair  $(i, j)$  if  $\{H\}$  covers  $(i, j)$ .*

The following theorem provides a sufficient and necessary condition for a finite family  $\mathcal{H}$  to have the property that the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant.

**Theorem 1.** *Let  $\mathcal{H}$  be a finite family of graphs. Then the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant  $c_{\mathcal{H}}$  if and only if  $\mathcal{H}$  covers the pair  $(i, j)$  for every  $i, j \geq 3$ .*

Section 2 is devoted to the proof of Theorem 1. In Section 3, we prove a sequence of lemmata that show exactly which graphs  $H$  cover which pairs  $(i, j)$ . In Section 4, we present some applications of the results in Sections 2 and 3. In particular, we describe a procedure that, given a positive integer  $k$ , yields an explicit description of all the minimal graph families  $\mathcal{H}$  with  $|\mathcal{H}| = k$  for which the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant. For  $k = 1$ , this immediately yields the aforementioned result from [1], stating that the poc-fvs for  $H$ -free graphs is upper bounded by a constant if and only if  $H$  is a linear forest (Corollary 1). We also demonstrate the procedure for the case  $k = 2$ , and obtain an explicit description of exactly those families  $\{H_1, H_2\}$  for which the poc-fvs for  $\{H_1, H_2\}$ -free graphs is upper bounded by a constant (Corollary 2). Section 5 contains some concluding remarks.

We end this section by defining some additional terminology that will be used throughout the paper. For any  $k, p, q \geq 1$ , let  $P_k$  denote the path on  $k$  vertices,

and let  $T_k^{p,q}$  denote the graph obtained from  $P_k + P_p + P_q$  by making a new vertex adjacent to one end-vertex of each path. For any  $k \geq 0$  and  $r \geq 3$ , let  $D_k^r$  denote the graph obtained from  $P_k + C_r$  by adding an edge between a vertex of the cycle and an end-vertex of the path; in particular,  $D_0^r$  is isomorphic to  $C_r$ .

## 2 Proof of Theorem 1

In this section, we prove the dichotomy result given in Theorem 1. We will make use of the following simple observation.

**Observation 1** *Let  $i, j, k, \ell$  be integers such that  $i, j \geq 3$  and  $\ell \geq k \geq 1$ . A graph on at most  $k$  vertices is an induced subgraph of  $B_{i,j,k}$  if and only if it is an induced subgraph of  $B_{i,j,\ell}$ .*

*Proof (of Theorem 1).* First suppose there exists a pair  $(i, j)$  with  $i, j \geq 3$  such that  $\mathcal{H}$  does not cover  $(i, j)$ . For contradiction, suppose there exists a constant  $c_{\mathcal{H}}$  as in the statement of the theorem. By Definition 1,  $\mathcal{H}$  does not contain an induced subgraph of  $B_{i,j,N}$ , and hence  $B_{i,j,N}$  is  $\mathcal{H}$ -free. As a result of Observation 1,  $B_{i,j,k}$  is  $\mathcal{H}$ -free for every  $k \geq N$ . In particular, the graph  $B_{i,j,N+2c_{\mathcal{H}}}$  is  $\mathcal{H}$ -free. Note that  $\text{fvs}(B_{i,j,N+2c_{\mathcal{H}}}) = 2$  and  $\text{cfvs}(B_{i,j,N+2c_{\mathcal{H}}}) = N + 2c_{\mathcal{H}} + 1$ . This implies that  $\text{cfvs}(B_{i,j,N+2c_{\mathcal{H}}}) > c_{\mathcal{H}} \cdot \text{fvs}(B_{i,j,N+2c_{\mathcal{H}}})$ , yielding the desired contradiction.

For the converse direction, suppose  $\mathcal{H}$  covers the pair  $(i, j)$  for every  $i, j \geq 3$ . Let  $G$  be a connected  $\mathcal{H}$ -free graph. Observe that  $\text{cfvs}(G) = \text{fvs}(G)$  if  $G$  is a cycle or a tree, so we assume that  $G$  is neither a cycle nor a tree. Let  $F$  be a minimum feedback vertex set of  $G$ , and without loss of generality assume that each vertex in  $F$  lies on a cycle and has degree at least 3 in  $G$ . Below, we will prove that the distance in  $G$  between any two vertices of  $F$  is at most  $5N$ . To see why this suffices to prove the theorem, observe that we can transform  $F$  into a connected feedback vertex set of  $G$  of size at most  $5N \cdot |F| = 5N \cdot \text{fvs}(G)$  by choosing an arbitrary vertex  $x \in F$  and adding, for each  $y \in F \setminus \{x\}$ , all the internal vertices of a shortest path between  $x$  and  $y$ .

Let  $x, y \in F$ , and let  $P$  be a shortest path from  $x$  to  $y$ . For contradiction, suppose  $P$  has length at least  $5N + 1$ . Recall that by the definition of  $F$ , there exist cycles  $C_x$  and  $C_y$  that contain  $x$  and  $y$ , respectively; assume, without loss of generality, that  $C_x$  and  $C_y$  are induced cycles in  $G$ . Let  $X = \{v \in V(C_x) \mid d_{G[V(C_x)]}(v, x) \leq N\}$ . Note that  $X$  induces the cycle  $C_x$  in case  $|V(C_x)| \leq 2N$ , and  $X$  induces a path of length at most  $2N$  otherwise. We also define  $Y = \{v \in V(C_y) \mid d_{G[V(C_y)]}(v, y) \leq N\}$ . We partition the vertex set of  $P$  into three sets:  $L = \{v \in V(P) \mid d_G(v, x) \leq 2N + 1\}$ ,  $M = \{v \in V(P) \mid d_G(v, x) \geq 2N + 2 \text{ and } d_G(v, y) \geq 2N + 2\}$ , and  $R = \{v \in V(P) \mid d_G(v, y) \leq 2N + 1\}$ . For any two distinct vertices  $u$  and  $v$  on the path  $P$ , we say that  $u$  is to the left of  $v$  (and, equivalently,  $v$  is to the right of  $u$ ) if the subpath of  $P$  from  $x$  to  $u$  does not contain  $v$ .

*Claim 1.*  $G[X \cup L]$  contains a graph in  $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$  as an induced subgraph.

We prove Claim 1 as follows. Let  $x'$  be the vertex of  $P$  closest to  $y$  that has a neighbor  $x_1 \in X \setminus \{x\}$ ; possibly  $x' = x$ . Let  $P'$  be the subpath of  $P$  from  $x$  to  $x'$ . By the definition of  $X$ , the distance between  $x_1$  and  $x$  is at most  $N$ , implying that  $d_G(x, x') \leq N + 1$ . Since  $P$  is a shortest path from  $x$  to  $y$ , we find that the length of  $P'$  is at most  $N + 1$ . Let  $x''$  be the unique vertex of  $P$  such that  $x''$  is to the right of  $x'$  and  $d_G(x'', x') = N$ , and let  $P''$  be the subpath of  $P$  from  $x'$  to  $x''$ . Since  $|L| = 2N + 2$ , path  $P'$  has length at most  $N + 1$ , and path  $P''$  has length  $N$ , it follows that  $V(P'') \subseteq L$ . Observe that  $x'$  is the only vertex of  $P''$  that has a neighbor in  $X \setminus \{x\}$ .

Suppose  $x = x'$ . Then  $X \cap V(P) = \{x\}$ , and hence  $G[X \cup V(P'')]$  is isomorphic to either  $D_N^{|V(C_x)|}$  or  $T_N^{N,N}$ , implying that the claim holds in this case. From now on, we assume that  $x' \neq x$ . We distinguish two cases, depending on how many neighbors  $x'$  has in  $X$ .

If  $x'$  has at least two neighbors in  $X$ , then  $x'$  has two neighbors  $x_1, x_2$  in  $X$  such that there is a path in  $X$  from  $x_1$  to  $x_2$  whose internal vertices are not adjacent to  $x'$ . This path, together with the edges  $x_1x'$  and  $x_2x'$ , forms an induced cycle  $C$  in  $G$ . Then  $G[V(C) \cup V(P'')]$  is isomorphic to  $D_N^{|V(C)|}$ , so the claim holds.

Now suppose  $x'$  has exactly one neighbor  $x_1 \in X$ . If  $X$  induces a cycle in  $G$ , then the cycle  $G[X]$ , the path  $P''$ , and the edge  $x'x_1$  together form a graph that is isomorphic to  $D_N^{|X|}$ , so the claim holds. Suppose  $X$  induces a path in  $G$ ; recall that this path has exactly  $2N + 1$  vertices, and  $x$  is the middle vertex of this path. If  $x_1 = x$ , then  $G[X \cup V(P'')]$  is isomorphic to  $T_N^{N,N}$ . Suppose  $x_1 \neq x$ . Let  $P_X$  be the unique path in  $G[X]$  from  $x_1$  to  $x$ . Then the graph  $G[V(P_X) \cup V(P'')]$  contains an induced cycle  $C$  such that  $x'$  lies on  $C$ , and the graph  $G[V(C) \cup V(P'')]$  is isomorphic to  $D_N^{|V(C)|}$ . This completes the proof of Claim 1.

Let  $G_x$  be an induced subgraph of  $G[X \cup L]$  that is isomorphic to a graph in  $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$  and that is constructed from the cycle  $C_x$  in the way described in the proof of Claim 1. In particular, let  $x''$  be the vertex of  $G_x$  that is closest to  $y$  in  $G$ . Recall that  $x''$  is a vertex of  $P$  and has degree 1 in  $G_x$ . It is clear from the construction of  $G_x$  that every vertex in  $G_x$  has distance at most  $2N + 1$  to  $x$ . By symmetry, we can define an induced subgraph  $G_y$  of  $G[Y \cup R]$  and a vertex  $y''$  in  $G_y$  in an analogous way, that is,  $G_y$  is isomorphic to a graph in  $\{D_N^i \mid i \geq 3\} \cup \{T_N^{N,N}\}$ , and  $y''$  is the vertex of  $G_y$  that is closest to  $x$  in  $G$ .

Let  $P^*$  be the subpath of  $P$  from  $x''$  to  $y''$ . The fact that  $P$  is a shortest path from  $x$  to  $y$  implies that  $x''$  and  $y''$  are the only two vertices of  $G_x$  and  $G_y$  that are adjacent to internal vertices of  $P^*$ . Moreover, there are no edges between  $G_x$  and  $G_y$ , as otherwise there would be a path from  $x$  to  $y$  of length at most  $4N + 2$ , contradicting the fact that  $P$  is a shortest path from  $x$  to  $y$ . Let  $G^*$  denote the induced subgraph of  $G$  obtained from  $G_x + G_y$  by connecting  $x''$  and  $y''$  using the path  $P^*$ . We distinguish four cases, and obtain a contradiction in each case. We will repeatedly use the fact that in each case,  $G^*$  can be obtained from a “large” butterfly by deleting at most two vertices.

*Case 1.*  $G_x$  is isomorphic to  $D_N^i$  and  $G_y$  is isomorphic to  $D_N^j$  for some  $i, j \geq 3$ .

In this case,  $G^*$  is isomorphic to  $B_{i,j,k}$  for some  $k \geq 2N$ . Since  $\mathcal{H}$  covers the pair  $(i, j)$ , there exists a graph  $H \in \mathcal{H}$  such that  $H$  is an induced subgraph of  $B_{i,j,N}$  by Definition 1. Due to Observation 1,  $H$  is also an induced subgraph of  $G^*$  and hence also of  $G$ . This contradicts the assumption that  $G$  is  $\mathcal{H}$ -free.

*Case 2.*  $G_x$  is isomorphic to  $D_N^i$  for some  $i \geq 3$  and  $G_y$  is isomorphic to  $T_N^{N,N}$ .

Since  $\mathcal{H}$  covers the pair  $(i, 2N)$ , there exists a graph  $H \in \mathcal{H}$  such that  $H$  is an induced subgraph of  $B_{i,2N,N}$ . Since  $|V(H)| \leq N$ , the graph  $H$  contains at most one cycle, and this cycle, if it exists, is of length  $i$ . Hence it is clear that  $H$  is also an induced subgraph of  $G^*$ . This contradicts the assumption that  $G$  and thus  $G^*$  is  $\mathcal{H}$ -free.

*Case 3.*  $G_x$  is isomorphic to  $T_N^{N,N}$  and  $G_y$  is isomorphic to  $D_N^i$  for some  $i \geq 3$ .

By symmetry, we obtain a contradiction in the same way as in Case 2.

*Case 4.* Both  $G_x$  and  $G_y$  are isomorphic to  $T_N^{N,N}$ .

Since  $\mathcal{H}$  covers the pair  $(2N, 2N)$ , there exists a graph  $H \in \mathcal{H}$  such that  $H$  is an induced subgraph of  $B_{2N,2N,N}$ . This graph  $H$  has at most  $N$  vertices, which implies that  $H$  has no cycle. But then  $H$  is an induced subgraph of  $G^*$ , again yielding the desired contradiction. This completes the proof of Theorem 1.  $\square$

### 3 Which Graphs $H$ Cover Which Pairs $(i, j)$ ?

Recall that by Definition 1, a graph  $H$  covers a pair  $(i, j)$  if and only if  $H$  is an induced subgraph of  $B_{i,j,N}$ , where  $N = 2 \cdot |V(H)| + 1$ . In particular, if a graph  $H$  is not an induced subgraph of a butterfly, then  $H$  does not cover any pair  $(i, j)$ . However, it is important to note that some induced subgraphs of  $B_{i,j,N}$  cover more pairs than others. For example, as we will see in Lemma 6, a linear forest covers all pairs  $(i, j)$  with  $i, j \geq 3$ , but this is not the case for any induced subgraph of  $B_{i,j,N}$  that is not a linear forest.

In this section, we will prove exactly which pairs  $(i, j)$  are covered by which graphs  $H$ . For convenience, we first describe all the possible induced subgraphs of  $B_{i,j,N}$  in the following observation.

**Observation 2** *Let  $H$  be a graph, let  $N = 2 \cdot |V(H)| + 1$ , and let  $i, j \geq 3$  be two integers. Then  $H$  is an induced subgraph of  $B_{i,j,N}$  if and only if  $H$  is isomorphic to the disjoint union of a linear forest (possibly on zero vertices) and at most one of the following graphs:*

- (i)  $D_\ell^i$  for some  $\ell \geq 0$ ;
- (ii)  $D_\ell^j$  for some  $\ell \geq 0$ ;
- (iii)  $D_\ell^i + D_{\ell'}^j$  for some  $\ell, \ell' \geq 0$ ;
- (iv)  $T_k^{p,q}$  for some  $k, p, q \geq 1$  such that  $p + q + 2 \leq \max\{i, j\}$ ;
- (v)  $T_k^{p,q} + T_{k'}^{p',q'}$  for some  $k, p, q, k', p', q' \geq 1$  such that  $p + q + 2 \leq i$  and  $p' + q' + 2 \leq j$ ;

- (vi)  $D_\ell^i + T_k^{p,q}$  for some  $\ell \geq 0$  and  $k, p, q \geq 1$  such that  $p + q + 2 \leq j$ ;
- (vii)  $D_\ell^j + T_k^{p,q}$  for some  $\ell \geq 0$  and  $k, p, q \geq 1$  such that  $p + q + 2 \leq i$ .

The lemmata below show, for each of the induced subgraphs described in Observation 2, exactly which pairs  $(i, j)$  they cover. In the statement of each of the lemmata, we refer to a table in which the set of covered pairs is depicted. This will be helpful in the applications presented in Section 4. The rather straightforward proofs of Lemmata 2–5 have been omitted due to page restrictions.

**Lemma 1.** *Let  $H$  be a graph, let  $p \geq 3$ , and let  $\mathcal{X}$  be the set consisting of the pairs  $(i, j)$  with  $i, j \geq 3$  and  $p \in \{i, j\}$ ; see the left table in Figure 1 for an illustration of the pairs in  $\mathcal{X}$ .*

- (i) *If  $H$  is an induced subgraph of  $D_k^p$  for some  $k \geq 0$ , then  $H$  covers all the pairs in  $\mathcal{X}$ .*
- (ii) *If  $D_k^p$  is an induced subgraph of  $H$  for some  $k \geq 0$ , then  $H$  covers only pairs in  $\mathcal{X}$ .*

*Proof.* Let  $N = 2 \cdot |V(H)| + 1$ . Suppose  $H$  is an induced subgraph of  $D_k^p$  for some  $k \geq 0$ . Then  $H$  is also an induced subgraph of  $B_{i,j,N}$  for every  $i, j \geq 3$  such that  $p \in \{i, j\}$ . Hence, by Definition 1,  $H$  covers the pairs  $(p, j)$  and  $(i, p)$  for every  $i, j \geq 3$ .

Now suppose  $D_k^p$  is an induced subgraph of  $H$  for some  $k \geq 0$ . Then  $H$  contains a cycle of length  $p$ . Hence it is clear that if  $H$  is an induced subgraph of a butterfly  $B_{i,j,N}$ , then we must have  $p \in \{i, j\}$ . This shows that  $H$  only covers pairs that belong to  $\mathcal{X}$ .  $\square$

	$j$	3	...	...	$p$	...	...
$i$							
3					✓		
⋮					✓		
⋮					✓		
$p$		✓	✓	✓	✓	✓	✓
⋮					✓		
⋮					✓		

	$j$	3	...	$p$	...	$q$	...
$i$							
3							
⋮							
$p$						✓	
⋮							
$q$				✓			
⋮							
⋮							

**Fig. 1.** The ticked cells represent the pairs  $(i, j)$  covered by  $H$  when  $H$  is isomorphic to  $D_k^p$  for some  $k \geq 0$  (left table) and when  $H$  is isomorphic to  $D_k^p + D_k^q$  for some  $k \geq 0$  (right table).



**Lemma 2.** Let  $H$  be a graph, let  $p, q \geq 3$ , and let  $\mathcal{X} = \{(p, q), (q, p)\}$ ; see the right table in Figure 1 for an illustration of the pairs in  $\mathcal{X}$ .

- (i) If  $H$  is an induced subgraph of  $D_k^p + D_k^q$  for some  $k \geq 0$ , then  $H$  covers all the pairs in  $\mathcal{X}$ .
- (ii) If  $D_k^p + D_k^q$  is an induced subgraph of  $H$  for some  $k \geq 0$ , then  $H$  covers only pairs in  $\mathcal{X}$ .

$i \backslash j$	3	...	...	$p+q+2$	...	...
3				✓	✓	✓
⋮				✓	✓	✓
⋮				✓	✓	✓
$p+q+2$	✓	✓	✓	✓	✓	✓
⋮	✓	✓	✓	✓	✓	✓
⋮	✓	✓	✓	✓	✓	✓

$i \backslash j$	3	...	$p+q+2$	...	$p'+q'+2$	...
3						
⋮						
$p+q+2$					✓	✓
⋮					✓	✓
$p'+q'+2$			✓	✓	✓	✓
⋮			✓	✓	✓	✓

**Fig. 2.** The ticked cells represent the pairs  $(i, j)$  covered by  $H$  when  $H$  is isomorphic to  $T_r^{p,q}$  for some  $r \geq 1$  (left table) and when  $H$  is isomorphic to  $T_r^{p,q} + T_r^{p',q'}$  for some  $r \geq 1$  (right table).

**Lemma 3.** Let  $H$  be a graph, let  $p, q \geq 1$ , and let  $\mathcal{X}$  be the set consisting of the pairs  $(i, j)$  with  $i, j \geq 3$  and  $\max\{i, j\} \geq p + q + 2$ ; see the left table in Figure 2 for an illustration of the pairs in  $\mathcal{X}$ .

- (i) If  $H$  is an induced subgraph of  $T_r^{p,q}$  for some  $r \geq 1$ , then  $H$  covers all the pairs in  $\mathcal{X}$ .
- (ii) If  $T_r^{p,q}$  is an induced subgraph of  $H$  for some  $r \geq 1$ , then  $H$  covers only pairs in  $\mathcal{X}$ .

**Lemma 4.** Let  $H$  be a graph, let  $p, q, p', q' \geq 1$  be such that  $p + q \leq p' + q'$ , and let  $\mathcal{X}$  consist of all the pairs  $(i, j)$  with  $\min\{i, j\} \geq p + q + 2$  and  $\max\{i, j\} \geq p' + q' + 2$ ; see the right table in Figure 2 for an illustration of the pairs in  $\mathcal{X}$ .

- (i) If  $H$  is an induced subgraph of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r \geq 1$ , then  $H$  covers all the pairs in  $\mathcal{X}$ .
- (ii) If  $T_r^{p,q} + T_r^{p',q'}$  is an induced subgraph of  $H$  for some  $r \geq 1$ , then  $H$  covers only pairs in  $\mathcal{X}$ .

	$j$	3	...	$p$	...	$p'+q'+2$	...
$i$							
3							
...							
$p$						✓	✓
...							
$p'+q'+2$				✓			
...				✓			

	$j$	3	...	$p'+q'+2$	...	$p$	...
$i$							
3							
...							
$p'+q'+2$						✓	
...						✓	
$p$				✓	✓	✓	✓
...						✓	

**Fig. 3.** The ticked cells represent the pairs  $(i, j)$  covered by  $H$  when  $H$  is isomorphic to  $D_k^p + T_r^{p', q'}$  for some  $k \geq 0$  and  $r \geq 1$  in the case where  $p < p' + q' + 2$  (left table) and in the case where  $p > p' + q' + 2$  (right table).

	$j$	3	...	$p$	...	...
$i$						
3						
...						
$p$				✓	✓	✓
...				✓		
...				✓		

**Fig. 4.** The ticked cells represent the pairs  $(i, j)$  covered by  $H$  when  $H$  is isomorphic to  $D_k^p + T_r^{p', q'}$  for some  $k \geq 0$  and  $r \geq 1$  in the case where  $p = p' + q' + 2$ .

**Lemma 5.** Let  $H$  be a graph, let  $p \geq 3$  and  $p', q' \geq 1$ , and let  $\mathcal{X}$  be the set consisting of the pairs  $(i, j)$  with either  $i = p$  and  $j \geq p' + q' + 2$  or  $i \geq p' + q' + 2$  and  $j = p$ ; see the left and right tables in Figure 3 and the table in Figure 4 for an illustration of the pairs in  $\mathcal{X}$  in the cases where  $p < p' + q' + 2$ ,  $p > p' + q' + 2$ , and  $p = p' + q' + 2$ , respectively.

- (i) If  $H$  is an induced subgraph of  $D_k^p + T_r^{p', q'}$  for some  $k \geq 0$  and  $r \geq 1$ , then  $H$  covers all the pairs in  $\mathcal{X}$ .
- (ii) If  $D_k^p + T_r^{p', q'}$  is an induced subgraph of  $H$  for some  $k \geq 0$  and  $r \geq 1$ , then  $H$  covers only pairs in  $\mathcal{X}$ .

**Lemma 6.** *A graph  $H$  covers every pair  $(i, j)$  with  $i, j \geq 3$  if and only if  $H$  is a linear forest.*

*Proof.* If  $H$  is a linear forest, then  $H$  is an induced subgraph of a path on  $2 \cdot |V(H)|$  vertices. Hence  $H$  is also an induced subgraph of  $B_{i,j,2|V(H)|+1}$  for every  $i, j \geq 3$ . By Definition 1,  $H$  covers every pair  $(i, j)$  with  $i, j \geq 3$ .

For the reverse direction, suppose  $H$  covers every pair  $(i, j)$  with  $i, j \geq 3$ . For contradiction, suppose  $H$  is not a linear forest. Then, as a result of Definition 1 and Observation 2, either  $H$  contains  $T_r^{p,q}$  as an induced subgraph for some  $p, q, r \geq 1$ , or  $H$  contains  $D_k^p$  as an induced subgraph for some  $p \geq 3$  and  $k \geq 0$ . In the first case, it follows from Lemma 3(ii) that  $H$  does not cover the pair  $(3, 3)$ . In the second case, it follows from Lemma 1(ii) that  $H$  does not cover any pair  $(i, j)$  with  $r \notin \{i, j\}$ . In both cases, we obtain the desired contradiction.  $\square$

## 4 Applications of Our Results

In this section, we show how we can apply Theorem 1 and the lemmata from Section 3 in order to obtain some concrete characterizations. Let us first remark that the following result, previously obtained in [1], immediately follows from Theorem 1 and Lemma 6.

**Corollary 1 ([1]).** *Let  $H$  be a graph. Then the poc-fvs for  $H$ -free graphs is upper bounded by a constant  $c_H$  if and only if  $H$  is a linear forest.*

Obtaining similar characterizations for finite families  $\mathcal{H}$  with  $|\mathcal{H}| \geq 2$  is more involved, but can be done using the procedure we informally describe below. We then illustrate the procedure in Corollary 2 below for the case where  $|\mathcal{H}| = 2$ .

Let  $k \geq 2$ . Suppose we want to characterize the families of graphs  $\mathcal{H}$  with  $|\mathcal{H}| = k$  for which the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant. It follows from Theorem 1 and Lemma 6 that the poc-fvs for  $\mathcal{H}$ -free graphs is bounded whenever  $\mathcal{H}$  contains a linear forest. What about families  $\mathcal{H}$  that do not contain a linear forest?

Consider the infinite table containing all the pairs  $(i, j)$  with  $i, j \geq 3$ . From Lemmata 3–5 and Figures 1–4, we can observe two important things. First, the only graphs  $H$  that cover the pair  $(3, 3)$  are induced subgraphs of  $2D_\ell^3$  for some  $\ell \geq 0$ . Second, the only graphs  $H$  that cover infinitely many rows and columns of this table are induced subgraphs of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r, p, q, p', q' \geq 1$ . Hence, any finite family  $\mathcal{H}$  that covers all pairs  $(i, j)$  must contain at least one graph of both types. Formally, we have the following observation (observe that every linear forest is an induced subgraph of  $2D_\ell^3$  for some  $\ell \geq 0$  and of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r, p, q, p', q' \geq 1$ ):

**Observation 3** *Let  $\mathcal{H}$  be a finite family of graphs such that  $|\mathcal{H}| \geq 2$ . If the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant  $c_{\mathcal{H}}$ , then  $\mathcal{H}$  contains an induced subgraph of  $2D_\ell^3$  for some  $\ell \geq 0$  and an induced subgraph of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r, p, q, p', q' \geq 1$ .*

Suppose  $\mathcal{H}$  is a family of  $k$  graphs such that the poc-fvs for  $\mathcal{H}$ -free graphs is bounded by a constant. By Observation 3,  $\mathcal{H}$  contains a graph  $H_1$  that is an induced subgraph of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r, p, q, p', q' \geq 1$ .

If  $H_1$  is also an induced subgraph of  $T_r^{p,q}$  for some  $r, p, q \geq 1$ , or if  $\mathcal{H}$  contains another graph that is of this form, then Lemma 3 and Figure 2 show that there are only finitely many pairs  $(i, j)$  that are not covered by  $H_1$ . These cells need to be covered by the remaining graphs in  $\mathcal{H}$ . Using Lemmata 3–5, we can determine exactly which combination of graphs covers exactly those remaining pairs.

Suppose  $\mathcal{H}$  does not contain induced subgraph of  $T_r^{p,q}$  for any  $r, p, q \geq 1$ . Then, by Lemma 4, there are finitely many rows and columns in which no pair is covered by  $H_1$ . In particular, since  $p, q, p', q' \geq 1$ , the pairs  $(i, 3)$  and  $(3, j)$  are not covered for any  $i, j \geq 3$ . From the lemmata in Section 3 and the corresponding tables, it is clear that the only graphs  $H$  that cover infinitely many pairs of this type are induced subgraphs of  $T_r^{p,q}$  for some  $r, p, q \geq 1$  or of  $D_{r'}^3 + T_r^{p,q}$  for some  $r' \geq 0$  and  $p, q \geq 1$ . Hence,  $\mathcal{H}$  must contain a graph  $H_2$  that is isomorphic to such an induced subgraph. Similarly, if the pairs  $(i, 4)$  and  $(4, j)$  are not covered for any  $i, j \geq 3$ , then  $\mathcal{H}$  must contain an induced subgraph of  $T_r^{p,q}$  for some  $r, p, q \geq 1$  or of  $D_{r'}^4 + T_r^{p,q}$  for some  $r' \geq 0$  and  $p, q \geq 1$ , etcetera. Once all rows and columns contain only finitely many pairs that are not covered yet, we can determine all possible combinations of graphs that cover those last pairs.

To illustrate the above procedure, we now give an explicit description of exactly those families  $\{H_1, H_2\}$  for which the poc-fvs for  $\{H_1, H_2\}$ -free graphs is upper bounded by a constant.

**Corollary 2.** *Let  $H_1$  and  $H_2$  be two graphs, and let  $\mathcal{H} = \{H_1, H_2\}$ . Then the poc-fvs for  $\mathcal{H}$ -free graphs is upper bounded by a constant  $c_{\mathcal{H}}$  if and only if there exist integers  $\ell \geq 0$  and  $r \geq 1$  such that one of the following conditions holds:*

- $H_1$  or  $H_2$  is a linear forest;
- $H_1$  and  $H_2$  are induced subgraphs of  $D_{\ell}^3$  and  $2T_r^{1,1}$ , respectively;
- $H_1$  and  $H_2$  are induced subgraphs of  $2D_{\ell}^3$  and  $T_r^{1,1}$ , respectively.

*Proof.* First suppose that the price of connectivity for feedback vertex set for  $\mathcal{H}$ -free graphs is bounded by some constant  $c_{\mathcal{H}}$ , and suppose that neither  $H_1$  nor  $H_2$  is a linear forest. Due to Observation 3, we may without loss of generality assume that  $H_1$  is an induced subgraph of  $2D_{\ell}^3$  for some  $\ell \geq 0$  and  $H_2$  is an induced subgraph of  $T_r^{p,q} + T_r^{p',q'}$  for some  $r, p, q, p', q' \geq 1$ . From Lemmata 1 and 2 and the assumption that  $H_1$  is not a linear forest, it follows that  $H_1$  does not cover the pair  $(4, 4)$ . Hence  $H_2$  must cover this pair. This, together with Lemma 4, implies that  $p = q = p' = q' = 1$ , i.e.,  $H_2$  is an induced subgraph of  $2T_r^{1,1}$  for some  $r \geq 1$ .

If  $H_1$  is an induced subgraph of  $D_{\ell'}^3$  for some  $\ell' \geq 0$ , then the second condition holds and we are done. Suppose this is not the case. Then  $H_1$  covers only the pair  $(3, 3)$  due to Lemma 2. This means that all the pairs  $(i, j)$  with  $i, j \geq 3$  and  $3 \in \{i, j\}$ , apart from  $(3, 3)$ , must be covered by  $H_2$ . From Lemma 3 and 4 it is clear that this only holds if  $H_2$  is an induced subgraph of  $T_{r'}^{1,1}$  for some  $r' \geq 1$ . Hence the third condition holds.

The converse direction follows by combining Theorem 1 with Lemma 6, Lemmata 1 and 4, and Lemmata 2 and 3, respectively.  $\square$

## 5 Conclusion

Recall that in [1], we proved for every graph  $H$  which of the following cases holds: (i)  $\text{cfvs}(G) = \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ ; (ii) there exists a constant  $c_H$  such that  $\text{cfvs}(G) \leq \text{fvs}(G) + c_H$  for every connected  $H$ -free graph  $G$ ; (iii) there exists a constant  $c_H$  such that  $\text{cfvs}(G) \leq c_H \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ ; (iv) there does not exist a constant  $c_H$  such that  $\text{cfvs}(G) \leq c_H \cdot \text{fvs}(G)$  for every connected  $H$ -free graph  $G$ . Theorem 1 extends the case of (iii) to all finite families  $\mathcal{H}$ . A natural question to ask is to characterize all finite families  $\mathcal{H}$  for (i) and (ii) as well.

Another natural question to ask is whether Theorem 1 can be extended to families  $\mathcal{H}$  which are not finite, i.e., to all hereditary classes of graphs. Definition 1 and Theorem 1 show that for any finite family  $\mathcal{H}$ , the poc-fvs for  $\mathcal{H}$ -free graphs is bounded essentially when the graphs in this class do not contain arbitrarily large induced butterflies. The following example shows that when  $\mathcal{H}$  is infinite, it is no longer only butterflies that can cause the poc-fvs to be unbounded. Let  $G$  be a graph obtained from  $K_3$  by first duplicating every edge once, and then subdividing every edge arbitrarily many times. Let  $\mathcal{G}$  be the class of all graphs that can be constructed this way. In order to make  $\mathcal{G}$  hereditary, we take its closure under the induced subgraph relation. Let  $\mathcal{G}'$  be the resulting graph class. Observe that graphs in this class have arbitrarily large minimum connected feedback vertex sets, while  $\text{fvs}(G) \leq 2$  for every graph  $G \in \mathcal{G}'$ . Hence, the poc-fvs for  $\mathcal{G}'$  is not bounded. However, no graph in this family contains a butterfly as an induced subgraph.

## References

1. Belmonte, R., van 't Hof, P., Kamiński, M., Paulusma, D.: The price of connectivity for feedback vertex set. In: Eurocomb 2013, CRM Series, vol. 16, pp. 123–128 (2013)
2. Camby, E., Cardinal, J., Fiorini, S., Schaudt, O.: The price of connectivity for vertex cover. Manuscript, arXiv:1303.2478 (2013)
3. Camby, E. and Schaudt, O.: A note on connected dominating set in graphs without long paths and cycles. Manuscript, arXiv:1303.2868 (2013)
4. Cardinal, J. and Levy, E.: Connected vertex covers in dense graphs. Theor. Comput. Sci. 411(26-28):2581–2590 (2010)
5. Duchet, P. and Meyniel, H.: On Hadwiger's number and the stability number. Ann. Discrete Math. 13:71–74 (1982)
6. Grigoriev, A. and Sitters, R.: Connected feedback vertex set in planar graphs. In: WG 2009, LNCS vol. 5911, pp. 143–153, Springer (2010)
7. Schweitzer, P. and Schweitzer, P.: Connecting face hitting sets in planar graphs. Inf. Process. Lett. 111(1):11–15 (2010)
8. Zverovich, I.E.: Perfect connected-dominant graphs. Discuss. Math. Graph Theory 23:159–162 (2003)