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# Reconstructing the Vacuum Functional of Yang-Mills from its Large Distance Behaviour

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## Abstract

For fields that vary slowly on the scale of the lightest mass the logarithm of the vacuum functional can be expanded as a sum of local functionals. For Yang-Mills theory the leading term in the expansion dominates large distance effects and leads to an area law for the Wilson loop. However, this expansion cannot be expected to converge for fields that vary more rapidly. By studying the analyticity of the vacuum functional under scale transformations we show how to re-sum this series so as to reconstruct the vacuum functional for arbitrary fields.

# 1 Introduction

When the Schrödinger representation vacuum functional of a massive quantum field theory is evaluated for slowly varying fields its logarithm reduces to a sum of local integrals. For example, the vacuum functional of a free scalar field theory in  $D + 1$  dimensions with mass  $m$  is  $\Psi[\varphi] \equiv \langle \varphi | 0 \rangle = \exp -\frac{1}{2} \int d^D \mathbf{x} \varphi \sqrt{-\nabla^2 + m^2} \varphi$ , so that if the Fourier transform of  $\varphi$  vanishes for momenta with magnitude greater than the mass,  $m$ , the logarithm of  $\Psi$  can be expanded in the convergent series

$$\ln \Psi = - \int d^D \mathbf{x} \left( \frac{m}{2} \varphi^2 + \frac{1}{4m} (\nabla \varphi)^2 - \frac{1}{16m^3} (\nabla^2 \varphi)^2 + \dots \right). \quad (1)$$

The terms of this expansion are local in the sense that they involve the field and a finite number of its derivatives at the same spatial point. The same is true for an interacting theory in which the lightest particle has non-zero mass, because the logarithm of the vacuum functional is a sum of connected Feynman diagrams, and since massive propagators are exponentially damped at large distances these diagrams reduce to local integrals for slowly varying fields. In the case of 3+1 dimensional Yang-Mills theory the lightest glueball mass must depend non-perturbatively on the coupling so that the local expansion of the vacuum functional could not emerge from the usual semi-classical expansion of the functional integral. Nonetheless we would expect to obtain for slowly varying gauge-potential

$$W[\mathbf{A}] \equiv \ln \Psi[\mathbf{A}] = \int d^3 \mathbf{x} (a_1 \text{tr} \mathbf{B} \cdot \mathbf{B} / m + a_2 \text{tr} D \wedge \mathbf{B} \cdot D \wedge \mathbf{B} / m^3 + a_3 \text{tr} \mathbf{B} \cdot (\mathbf{B} \wedge \mathbf{B}) / m^3 + a_4 \text{tr} \mathbf{B} \cdot \mathbf{B} \mathbf{B} \cdot \mathbf{B} / m^5 + \dots) \quad (2)$$

where  $\mathbf{B}$  is the Yang-Mills magnetic field  $\nabla \wedge \mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ , and  $D$  is the gauge covariant derivative. (This is qualitatively different to the case of pure Abelian gauge theory where  $W = -\frac{1}{4\pi e^2} \int d^3 \mathbf{x} d^3 \mathbf{y} \mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{y}) / (\mathbf{x} - \mathbf{y})^2$  is conformally invariant and cannot be expanded as an integral of local quantities.) The unknown coefficients  $a_i$  are dimensionless constants. In principle these coefficients can be determined from the Schrödinger equation. In [1] we showed that the Schrödinger equation does not take its usual form because the process of removing the cut-off does not commute with expanding in terms of local quantities. However, by re-summing in terms of the cut-off an appropriate Schrödinger equation can be constructed, and we suggested a method of solution that does not rely on a semi-classical perturbation theory. Alternatively the first few terms can be computed in lattice gauge theory. Approximating the vacuum functional by the first term of (2) was originally suggested by Greensite [2], who used Monte Carlo techniques to compute its coefficient for the gauge group  $SU(2)$  in 2+1 dimensions [3] and, with Iwasaki, in 3+1 dimensions [4]. When these results are translated from lattice to continuum fields they give  $a_1/m = \frac{a}{8} \exp[\frac{3}{11} \pi^2 (\beta - \beta_0)]$ , [4], where  $a$  is the lattice spacing,  $\beta$  the conventional coupling that enters the Wilson action for the 3+1 dimensional theory, and  $\beta_0 \approx 1.74$ . The 2+1 dimensional result was extended by Arisue in [6], and the case of gauge group  $SU(3)$  considered by Chen *et al* in [7]. For  $SU(2)$  the 2+1 dimensional theory yields  $W_3[\mathbf{A}] = \int d^2 \mathbf{x} \left( \frac{b_1}{e^4} \text{tr} B^2 + \frac{b_2}{e^8} \text{tr} D_i B D_i B + \dots \right)$ , [6], where  $b_1 = 0.91 \pm 0.02$ ,  $b_2 = -0.19 \pm 0.05$  and  $e$  is the continuum coupling constant in 2+1 dimensions corresponding to the normalisation

of the Lie algebra generators such that  $\text{tr } T^A T^B = -\delta^{AB}/2$ . (Note, that whilst the naive continuum limit of the lattice strong coupling expansion [5] would yield a similar expansion the coefficients would be wrong as they do not scale appropriately in the continuum limit.) To get some idea of the relevance of these results to large distance effects we can attempt to compute the string tension. We will simplify the calculation by retaining only the leading term. In the 2+1 dimensional theory this leads to an area law for the Wilson loop [8] because the leading term in  $W_3$  is the action for two-dimensional Yang-Mills which is free and confining. Applying the calculation given in [9] to the result of [6] gives the string tension as  $\sigma_3 = 3e^4/(16b_1) \approx 0.21 e^4$  which should be compared with the recent direct Monte Carlo estimate  $\sigma = (0.112 \pm 0.001) e^4$  [10]. Similarly the large Wilson loops in the 3+1 dimensional theory may be estimated using just the first term in (2) which reduces to a calculation in the three-dimensional theory, and thence to a calculation in the two-dimensional theory, so that the area law emerges from a kind of dimensional reduction [11]. Putting this together gives the string tension in the 3+1 dimensional theory as  $\sigma_4 \approx 0.21\tilde{e}^4$  with  $\tilde{e}^2 = m/(2a_1)$  so that  $\sigma_4 a^2 \approx 3.4 \exp[-\frac{6\pi^2}{11}(\beta - 1.74)]$ . For  $\beta = 2.85$  this yields  $\sigma_4 a^2 \approx 0.0086$  which compares with the accurate direct Monte Carlo measurement  $\sigma_4 a^2 = 0.00363(17)$  [12]. Some comments are in order. Firstly, the coefficients  $a_1, b_1, b_2$  are obtained by applying the naive continuum limit to a fit of the lattice vacuum functional obtained for slowly varying plaquette variables. This is justified since the transition from plaquette variables to continuum ones is itself a derivative expansion in continuum variables. If we had computed Wilson loops using the form of the vacuum functional originally found in terms of plaquette variables we would also have obtained an area law, but with a different string tension that would not scale appropriately in the continuum limit. This is because the integration over plaquette variables would include rapidly varying fields that were excluded when we made the transition to continuum variables, and for which the form of the vacuum functional used is inappropriate. The calculation of the string tension using just  $a_1$  and  $b_1$ , although numerically incorrect by about a factor of two, does at least scale properly. Presumably the numerical value would be improved were we to include the other terms in the expansion of  $W$ . In spite of this qualified success in describing large Wilson loops we can only expect the local expansion to converge for configurations that vary slowly on the scale of the mass of the lightest glueball so it would appear not to be relevant, for example, to the computation of the glueball spectrum as this involves heavier particles. The purpose of this letter is to show that in fact the vacuum functional for *arbitrary*  $\mathbf{A}$  can be reconstructed from this local expansion (2), given the coefficients  $a_i$ .

We will work with the functional integral representation

$$\Psi[\mathbf{A}] = \int \mathcal{D}A e^{-S[A] - S_b[A, \mathbf{A}]} \quad (3)$$

where  $S[A]$  is the Yang-Mills action gauge-fixed in the Weyl-gauge  $A_0 = 0$ . Space-time is Euclidean with co-ordinates  $(\mathbf{x}, t)$  and  $t \leq 0$ , so

$$S[A] = -\frac{1}{g^2} \int d^3\mathbf{x} dt \text{tr} \left( \dot{A}^2 + (\nabla \wedge A + A \wedge A)^2 \right) \quad (4)$$

The boundary term in the action is

$$S_b[A, \mathbf{A}] = -\frac{2}{g^2} \int d^3\mathbf{x} \operatorname{tr} \left( (\mathbf{A} - A) \cdot \dot{A} \right) |_{t=0} \quad (5)$$

The boundary value of  $A$  is to be freely integrated over, i.e. we will not impose a condition such as  $A(\mathbf{x}, 0) = 0$ . We will assume that at spatial infinity the source  $\mathbf{A}$  is a pure gauge  $\mathbf{A} \sim g(\hat{\mathbf{x}})^{-1} \nabla g(\hat{\mathbf{x}})$ .  $S_b$  is chosen so that  $\Psi[\mathbf{A}]$  is invariant under the gauge transformation  $\delta_\omega \mathbf{A} = \nabla \omega + [\mathbf{A}, \omega]$ , since the effect of varying the source  $\mathbf{A}$  may be compensated by gauge transforming  $A$ . As  $\omega$  cannot depend on time this is the residual gauge symmetry of  $S[A]$  that preserves the gauge condition. Functionally differentiating with respect to the source leads to an insertion of  $\dot{A}$ . Given the Wick rotation between Euclidean time,  $t$ , and Minkowskian time, this yields the Schrödinger representation of the canonical momentum, which in this case is the non-Abelian electric field,  $E = -ig^2 \delta / \delta \mathbf{A}$ .

Consider the effect of a scale transformation on the configuration  $\mathbf{A}$  given by

$$\mathbf{A}^s(\mathbf{x}) = \frac{1}{\sqrt{s}} \mathbf{A}\left(\frac{1}{\sqrt{s}} \mathbf{x}\right). \quad (6)$$

For small  $s$  the scaled field is trivial everywhere except the vicinity of the origin, since for  $\mathbf{x} = 0$  we have  $\mathbf{A}^s(0) = s^{-1/2} \mathbf{A}(0)$ , but for  $\mathbf{x} \neq 0$   $\mathbf{A}^s(\mathbf{x}) \approx s^{-1/2} g^{-1}(\hat{\mathbf{x}}) \nabla g(\hat{\mathbf{x}})$ . For large  $s$  the field varies slowly in space since now  $\mathbf{A}^s(\mathbf{x}) \approx s^{-1/2} \mathbf{A}(0)$ . By studying the analyticity of  $\Psi[\mathbf{A}^s]$  in  $s$  we will be able to express its value at  $s = 1$  in terms of its value for large  $s$ , where we can apply the local expansion (2), and its value for small  $s$ , which is reliably computed in the usual semi-classical perturbation theory and turns out to be negligible. More specifically, we will show that  $\Psi[\mathbf{A}^s]$  can be analytically continued to the complex plane with the negative real axis removed. This enables us to compute the contour integral

$$I(\lambda) = \frac{1}{2\pi i} \int_C \frac{ds}{s-1} e^{\lambda(s-1)} \Psi[\mathbf{A}^s] \quad (7)$$

in two ways. We take  $C$  to be a key-hole shaped contour running just under the negative real axis up to  $s = 1 - s_0$ , around the circle of radius  $s_0$  centred on  $s = 1$  and then back to  $s = -\infty$  running just above the negative real axis. If we take  $s_0$  to be large then we can compute the integral using the local expansion for  $W$ . Each term in this expansion scales so we can express it in terms of  $\mathbf{A}$  rather than  $\mathbf{A}^s$ ,

$$\begin{aligned} W[\mathbf{A}^s] = & \int d^3\mathbf{x} (a_1 s^{-1/2} \operatorname{tr} \mathbf{B} \cdot \mathbf{B} / m + a_2 s^{-3/2} \operatorname{tr} D \wedge \mathbf{B} \cdot D \wedge \mathbf{B} / m^3 + \\ & a_3 s^{-3/2} \operatorname{tr} \mathbf{B} \cdot (\mathbf{B} \wedge \mathbf{B}) / m^3 + a_4 s^{-5/2} \operatorname{tr} \mathbf{B} \cdot \mathbf{B} \mathbf{B} \cdot \mathbf{B} / m^5 + \dots) \end{aligned} \quad (8)$$

This yields an expansion of  $\Psi[\mathbf{A}^s]$  in inverse powers of  $s - 1$ , with coefficients that depend on the original configuration,  $\Psi[\mathbf{A}^s] \sim \sum (s - 1)^{-n} \psi_n[\mathbf{A}]$ , enabling us to compute  $I(\lambda)$  as

$$I(\lambda) = \sum_n \frac{\lambda^n \psi_n[\mathbf{A}]}{\Gamma(n+1)} \quad (9)$$

We can also evaluate the integral by collapsing the contour  $C$  until it breaks into two disconnected pieces, a small circle centred on  $s = 1$  and a contour that just surrounds

the negative real axis. The integral over the circle gives  $\Psi[\mathbf{A}]$ . By taking  $\lambda$  to be real, positive and very large the contribution from the negative real axis will be exponentially suppressed, (provided it is not singular, which we check in perturbation theory), so we obtain for large  $\lambda$

$$\Psi[\mathbf{A}] \approx \sum_n \frac{\lambda^n \psi_n[\mathbf{A}]}{\Gamma(n+1)} \quad (10)$$

which provides a re-summation of the local expansion. Note that only terms of order up to  $s^{-n}$  in (8) will contribute to  $\psi_n$ . We might expect to obtain an approximation by truncating (10) at some order in  $\lambda$ .

As an illustration we will show how the vacuum functional of a free massive field theory may be reconstructed from its local expansion. First scale the field by setting  $\varphi^s(\mathbf{x}) = s^{-(D-1)/2} \varphi(\mathbf{x}/\sqrt{s})$ . The vacuum functional for the scaled field is then  $\exp W[\varphi^s] = \exp -\frac{1}{2} \int d^D \mathbf{x} \varphi \sqrt{-\nabla^2 + sm^2} \varphi$  which can be continued to an analytic function in the complex  $s$ -plane with the negative real axis removed, and is finite at the origin. For large  $s$  we can expand this, or alternatively (1) evaluated for  $\varphi^s$ , in inverse powers of  $s - 1$  obtaining the local series

$$W[\varphi^s] = -\frac{m}{2} \sum_0^\infty \frac{\Gamma(3/2)}{\Gamma(n+1)\Gamma(3/2-n)} (s-1)^{1/2-n} \int d^D \mathbf{x} \varphi \left(1 - \frac{\nabla^2}{m^2}\right)^n \varphi \quad (11)$$

which yields an expansion of  $\Psi$  in inverse powers of  $s - 1$ . The term quadratic in  $\varphi$  is just  $W[\varphi^s]$  itself, so if we concentrate on this term rather than the whole of  $\Psi$  we should consider

$$\frac{1}{2\pi i} \int_C \frac{ds}{s-1} e^{\lambda(s-1)} W[\varphi^s] = \frac{m}{4\sqrt{\pi}} \sum_0^\infty \frac{(-)^n \lambda^{n-1/2}}{n!(n-1/2)} \int d^D \mathbf{x} \varphi \left(1 - \frac{\nabla^2}{m^2}\right)^n \varphi \quad (12)$$

The integrals will exist for all  $n$  provided  $\varphi$  has a momentum cut-off,  $k_0$  say, but the integral will converge for all  $\lambda$  and  $k_0$  because of the  $n!$  in the denominator. This is in contrast to our original expansion (1) which only converges for  $k_0 < m$ . The series (12) results from expanding in powers of  $\lambda$  the exponentials in

$$-\frac{m}{2\sqrt{\pi}} \int d^D \mathbf{x} \varphi \left( \frac{1}{\sqrt{\lambda}} e^{-\lambda(1-\nabla^2/m^2)} + \int_0^\lambda d\lambda \frac{1}{\sqrt{\lambda}} e^{-\lambda(1-\nabla^2/m^2)} \left(1 - \frac{\nabla^2}{m^2}\right) \right) \varphi, \quad (13)$$

which is just

$$-\frac{1}{2} \int d^D \mathbf{x} \varphi \sqrt{-\nabla^2 + m^2} \varphi + \frac{m}{4\sqrt{\pi}} \int d^D \mathbf{x} \varphi \left( \int_\lambda^\infty d\lambda \frac{1}{\sqrt{\lambda}^3} e^{-\lambda(1-\nabla^2/m^2)} \right) \varphi \quad (14)$$

We see that as  $\lambda \rightarrow \infty$  the series tends to  $W[\varphi]$ , as it should. Furthermore, if we keep  $\lambda$  large, but finite, the error in approximating  $W[\varphi]$  by the series (12), as given by the last integral in (14), is exponentially suppressed. For a finite value of  $\lambda$  we can also truncate the alternating series (12) at order  $\lambda^n$  with an error smaller than the first neglected term, so that by taking  $n$  sufficiently large in comparison to  $\lambda$  and  $k_0$  this error can be made small. In conclusion, we can represent  $W[\varphi]$  for cut-off, but otherwise arbitrary,  $\varphi$ , by the *local* series (12) truncated at order  $\lambda^n$  where  $\lambda$  and  $n$  are both large and chosen to give acceptable error.

## 2 Analyticity of the Vacuum Functional

To construct an analytic continuation of  $\Psi[\mathbf{A}^s]$  we adopt a similar approach to that used in [1], complicated by having to work with three spatial dimensions. We consider scaling each dimension separately, so we set  $\mathbf{A}^s = (s_1 s_2 s_3)^{-1/6} \mathbf{A}(x^1/\sqrt{s_1}, x^2/\sqrt{s_2}, x^3/\sqrt{s_3})$  and show that  $\Psi[\mathbf{A}^s]$  is analytic in  $s_1, s_2, s_3$  separately. Firstly we interchange the names of the Euclidean time,  $t$ , and one of the spatial co-ordinates,  $x^1$  say, in the functional integral (3) which we now interpret as the Euclidean time-ordered vacuum expectation value

$$\Psi[\mathbf{A}^s] = T \langle 0_r | \exp \left( \frac{2}{g^2} \int dx^2 dx^3 dt \text{tr} (\mathbf{A}^s \cdot A') \right)_{x^1=0} | 0_r \rangle \quad (15)$$

where  $|0_r\rangle$  is the vacuum for the Yang-Mills Hamiltonian defined on the space  $x^1 \leq 0$  in the axial gauge  $A_1 = 0$  with a boundary term in the action  $\frac{1}{g^2} \int dx^2 dx^3 dt \text{tr} (A \cdot A')$ , where the  $\prime$  denotes differentiation with respect to  $x^1$ . Expanding the exponential gives

$$\begin{aligned} \Psi[\mathbf{A}^s] = & \sum_n \int_{-\infty}^{\infty} dt_n \int_{-\infty}^{t_{n-1}} dt_{n-1} \dots \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 \prod_{i=1}^n \left( -\frac{1}{g^2} \int dx_i^2 dx_i^3 \mathbf{A}_{R_i}^s(t_i, x_i^2, x_i^3) \right) \\ & \langle 0_r | A'_{R_n}(0, x_n^2, x_n^3) e^{(t_{n-1}-t_n)H} \dots A'_{R_2}(0, x_2^2, x_3^3) e^{(t_2-t_1)H} A'_{R_1}(0, x_1^2, x_1^3) | 0_r \rangle. \end{aligned} \quad (16)$$

Here  $R_i$  stands for both Lie algebra and spatial indices. The time integrals may be done after Fourier transforming the sources. To do this we define the  $s_1$ -independent Fourier mode

$$a(k, x_2, x_3) \equiv (s_2 s_3)^{-1/6} \int dt e^{-ikt} \mathbf{A}(t, x^2/\sqrt{s_2}, x^3/\sqrt{s_3}), \quad (17)$$

so that

$$\mathbf{A}^s(t, x^2, x^3) = \frac{1}{2\pi} \int dk e^{ikt/\sqrt{s}} s^{-1/6} a(k, x^2, x^3). \quad (18)$$

Substituting this into (16) gives

$$\begin{aligned} \Psi[\mathbf{A}^s] = & \sum_n \prod_{i=1}^n \left( -\frac{1}{\pi g^2} \int dk_i dx_i^2 dx_i^3 s_1^{1/3} a_{R_i}(k_i, x_i^2, x_i^3) \right) \delta \left( \sum k_i \right) \\ & \langle 0_r | A'_{R_n}(0, x_n^2, x_n^3) \frac{1}{\sqrt{s_1} H - i \sum_1^{n-1} k_j} \dots \frac{1}{\sqrt{s_1} H - i k_1} A'_{R_1}(0, x_1^2, x_1^3) | 0_r \rangle. \end{aligned} \quad (19)$$

This makes explicit the  $s_1$ -dependence of  $\Psi[\mathbf{A}^s]$ . Although  $s_1$  was originally real and positive we may use this expression to define an analytic continuation to complex values, yielding a function that is analytic away from the zeroes of the denominators in (19). Since the eigenvalues of the Hermitian Hamiltonian are real, the singularities lie on the negative real  $s_1$ -axis. Similarly we may show that  $\Psi[\mathbf{A}^s]$  continues to an analytic function in  $s_2$  and  $s_3$  and, by setting  $s_1 = s_2 = s_3 = s$ , that  $\Psi[\mathbf{A}^s]$  continues to an analytic function in  $s$  on the complex plane with the negative real axis removed.

We have seen that for small values of  $s$  the configuration  $\mathbf{A}^s$  is non-trivial only over a short distance about the origin. Since asymptotic freedom under-writes semi-classical

perturbation theory at short distances we can reliably use this to calculate  $\Psi[\mathbf{A}^s]$  for small  $s$ . We will assume that Symanzik's work on the vacuum functional of  $\varphi^4$  theory [13] can be generalised to the present case, and further, that gauge invariance implies that the source  $\mathbf{A}$  needs no renormalisation [14]. Thus we assume that the cut-off necessary to define (3) may be removed leaving a finite vacuum functional that depends on the source  $\mathbf{A}$ , an arbitrary mass-scale  $\mu$  and a renormalised coupling,  $g(\mu)$ . (This assumption is supported by the scaling behaviour observed in lattice estimates of  $\Psi$ , [3], [4], [6]). Now if in computing  $\Psi[\mathbf{A}^s]$  we were to choose a new unit of length so as to undo the scaling of  $\mathbf{A}$ , and at the same time we scale  $\mu$  appropriately, then nothing would change so we must have that  $\Psi[\mathbf{A}^s]$  computed using  $\mu$  and  $g(\mu)$  is equal to  $\Psi[\mathbf{A}, ]$  computed using  $\mu/\sqrt{s}$  and  $g(\mu/\sqrt{s})$ . As  $s$  decreases so does the coupling, since in perturbation theory  $g^{-2}(\mu/\sqrt{s}) = g^{-2}(\mu) - 11N(\ln s)/(48\pi)$  for gauge group  $SU(N)$ , so for small enough  $s$  it is sufficient to take the tree-level approximation

$$\Psi[\mathbf{A}] \approx e^{-S[A_{cl}] - S_b[A_{cl}, \mathbf{A}]} \quad (20)$$

where  $A_{cl}$  satisfies the Euler-Lagrange equation  $\delta(S[A] - S_b[A, \mathbf{A}]) = 0$  under variations of  $A$  that are arbitrary at the boundary  $t = 0$ . Now

$$\begin{aligned} & \delta \left( \frac{1}{2} \int d^3\mathbf{x} dt tr \left( \dot{A}^2 + (\nabla \wedge A + A \wedge A)^2 \right) + \int d^3\mathbf{x} tr \left( (\mathbf{A} - A) \cdot \dot{A} \right) \Big|_{t=0} \right) = \\ & \int d^3\mathbf{x} dt tr \left( (\dot{A} \delta A) - \delta A \left( \ddot{A} + \nabla \wedge B + A \wedge B + B \wedge A \right) \right) \\ & + \int d^3\mathbf{x} tr \left( (\mathbf{A} - A) \cdot \delta \dot{A} - \delta A \cdot \dot{A} \right) \Big|_{t=0} \end{aligned} \quad (21)$$

where  $B$  is the non-Abelian magnetic field constructed from  $A$ . The first and last terms cancel using Stokes' theorem so we require that  $A_{cl}$  satisfy the Euclidean Yang-Mills equations  $\ddot{A} + \nabla \wedge B + A \wedge B + B \wedge A = 0$  with boundary condition  $A|_{t=0} = \mathbf{A}$ . The boundary integral  $S_b[A_{cl}, \mathbf{A}] = 0$ , so we are left with an expression for the small- $s$  dependence of  $\Psi[\mathbf{A}^s]$

$$\Psi[\mathbf{A}^s] \approx s^{-\frac{11N}{48\pi}} \int d^3\mathbf{x} dt tr \left( \dot{A}_{cl}^2 + B_{cl}^2 \right) \equiv s^\alpha \quad (22)$$

which, being a positive power of  $s$ , goes to zero with  $s$ . This gives a contribution to  $I(\lambda)$  from the cut along the negative real axis in the vicinity of the origin of

$$\frac{1}{2\pi i} \int \frac{ds}{s-1} e^{\lambda(s-1)} s^\alpha = \frac{\sin(\pi\alpha)}{\pi} e^{-\lambda} \int_0^\infty \frac{dx}{1+x} e^{-\lambda x} x^\alpha \quad (23)$$

Since  $1/(1+x) < 1/x$  for positive  $x$ , the last integral is less than  $\lambda^{-\alpha} \Gamma(\alpha)$  and we obtain the bound

$$\left| \frac{1}{2\pi i} \int \frac{ds}{s-1} e^{\lambda(s-1)} s^\alpha \right| < \frac{|\sin(\pi\alpha)| \Gamma(\alpha)}{\pi \lambda^\alpha e^\lambda} = \frac{1}{|\Gamma(1-\alpha)| \lambda^\alpha e^\lambda}, \quad (24)$$

which is negligible for sufficiently large  $\lambda$ . So we conclude that, for large  $\lambda$  we can ignore this semi-classical contribution to  $I(\lambda)$  and reconstruct the vacuum functional as  $\Psi[\mathbf{A}] \approx \sum_n \frac{\lambda^n \psi_n[\mathbf{A}]}{\Gamma(n+1)}$ , where the  $\psi_n$  are *local* functionals of the field computable from a knowledge of the vacuum functional evaluated for fields that vary slowly on the scale of the lightest glueball mass.



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