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The Consistency of Topological Expansions in Field Theory: 'BRST Anomalies' in Strings and Yang-Mills

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Abstract

Many field theories of physical interest have configuration spaces consisting of disconnected components. Quantum mechanical amplitudes are then expressed as sums over these components. We use the Faddeev-Popov approach to write the terms in this topological expansion as moduli space integrals. A cut-off is needed when these integrals diverge. This introduces a dependence on the choice of parametrisation of configuration space which must be removed if the theory is to make physical sense. For theories that have a local symmetry this also leads to a breakdown in BRST invariance. We discuss in detail the cases of Bosonic Strings and Yang-Mills theory, showing how this arbitrariness may be removed by the use of a counter-term in the former case, and by compactification on S^4 in the latter.

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1. Introduction

Despite their very different physical interpretations there are many similarities between the mathematical structures of Yang-Mills theory and first quantised String Theories. Both have local symmetries which are crucial to their consistency. For Yang-Mills theory this is gauge invariance, for String Theory this is invariance under world-sheet reparametrisations. After gauge-fixing and the introduction of Faddeev-Popov [1] ghosts they are BRST invariant [2], and this may be made the basis of a quantisation procedure. Both theories are invariant classically under local scalings of the metric. In Yang-Mills theory this is useful in constructing classical solutions and the breaking of the invariance at a quantum level underlies the use of the renormalisation group. For critical strings the maintenance of invariance under Weyl transformations of the world-sheet metric is a crucial constraint on the quantisation of the system [3]. Also both theories have topologically non-trivial sectors so that any transition amplitude is an infinite sum over contributions from these sectors weighted with an appropriate coupling constant. For closed strings this is a sum over closed Riemann surfaces of increasing genus corresponding to loops of virtual strings [4]. The coupling constant is the string coupling, κ, to the power of minus the Euler characteristic which counts the number of three-string-interactions vertices. In the Euclidean formulation of Yang-Mills theory stereographically projected onto S^4 the configurations of the gauge-potential fall into distinct homotopy classes [5]. These are classified by the second Chern class or instanton number, and the coupling is essentially the theta angle. The coupling constant dependence of these two expansions appears to result from very different physical considerations but in both cases it may be ultimately traced back to unitarity. Furthermore the functional integral for a fixed topological sector may be reduced to a finite dimensional integral over a number of parameters or moduli. In String Theory these are the moduli of the Riemann surfaces, in Yang-Mills theory they are the instanton moduli. Typically these finite dimensional integrals diverge in some region of the moduli space, unless there are good reasons otherwise, for example supersymmetry in the case of strings. The one instanton sector contribution to the partition function of Yang-Mills theory for the gauge group $SU(N)$ reduces to an integral over the instanton position y^μ and scale ρ. When space-time is taken to be R^4 this integral is, in the semi-classical approximation at one loop [6],

$$Z_1^{YM} = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y\, d\rho}{\rho^5} \rho^{11N/3}. \tag{1.1}$$

This diverges for large scales. The integral over position is less troubling since it may reasonably be taken to yield the volume of space-time, in which case Z_1 is the integral of a constant density. Both divergences are infra-red, or large distance effects and for this reason are not considered as pathologies of the theory so much as defects of the approximation, which is only considered valid at short distances where asymptotic freedom holds sway. The one string loop partition function is an integral over the complex modulus τ [7]

$$Z_1^{string} = \int_F d^2\tau\,(\Im\tau)^{-2} C(\tau), \quad C(\tau) = 4(\frac{1}{2}\Im\tau)^{-12} e^{4\pi\Im\tau} |\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})|^{-48}. \qquad (1.2)$$

The fundamental domain is usually taken to be $F: \quad -\frac{1}{2} \leq \Re\tau \leq \frac{1}{2}, \quad \Im\tau > 0, \quad |\tau| \geq 1$. The integral diverges as $\Im\tau$ becomes large. This is again interpreted as an infra-red divergence, although by modular invariance it may be mapped to a small τ one if the integral is taken over a different, but equivalent, domain. These divergences are the subject of this paper.

Given an (ill-defined) integral over a domain M', $\int_{M'} d^n t\, f(t)$, that diverges in some region the simplest thing to do is to introduce a cut-off by restricting the range of values of the variables t. Suppose we do this by restricting just one of the variables, t^1 say. Then M' is replaced by a new domain M with a boundary ∂M on which t_1 takes its cut-off value. The well-defined integral $\int_M d^n t\, f(t)$ now depends strongly on the value of the cut-off and our choice of the parameter t^1. If we were to make a reparametrisation $t^A \to \tilde{t}^A = t^A + \epsilon^A(t)$ then the cut-off changes and consequently so does the value of the integral. The change in the integral can be expressed using Stokes' theorem as

$$\int_M d^n t\, \frac{\partial}{\partial t^A}\left(\epsilon^A(t)\, f(t)\right) = \int_{\partial M} d\Sigma_A\, \epsilon^A(t)\, f(t). \qquad (1.3)$$

If $\int d^n t\, f$ is the contribution to an amplitude from a particular topological sector then we have a regulated expression that depends on our choice of parametrisation if (1.3) is non-zero. This is not acceptable and a way must be found of removing this parametrisation dependence from the theory. Now the field configurations responsible for the divergences are those that degenerate to configurations belonging to a different topological sector. In the String Theory case we will see that the one string loop contribution to scattering amplitudes is proportional to the tree-level, or zero-loop, contribution as $\Im\tau \to \infty$, and in the case of Yang-Mills theory the one-instanton solution to the classical equations of motion approaches the (vanishing) zero-instanton solution as $\rho \to \infty$. Care must be exercised in making these statements precise because the degenerating configurations belong to the same topological class, characterised by the same invariants, as the non-degenerate configurations, but roughly speaking the field configurations corresponding to ∂M approximate to configurations belonging to a different topological class. This means that it may be possible to cancel (1.3) with a counter-term added to the contribution from a different topological sector, leading to a renormalisation of the topological expansion parameter. This 'topological renormalisation' can be implemented for the Bosonic String, although we will see that it is unnecessary for Yang-Mills theory. There is a number of other field theories that possess the same topological characteristics, for example Higgs Models and CP^n-models, so in the next section we will present a general formulation of the reduction of the path-integral for a topologically non-trivial sector to an integral over moduli and discuss its parametrisation dependence.

2. General Formulation

The general setting in which we are interested is a theory with fields ϕ and action $S[\phi]$ for which amplitudes are expressed as sums of functional integrals each of which is over a distinct homotopy class of configurations of ϕ. Suppose these classes, $\mathcal{C}_n$, are labelled by an integer n, then if we concentrate to begin with on the partition function Z

$$Z = \sum_n \kappa_n \, Z_n, \quad Z_n = \int_{\mathcal{C}_n} \mathcal{D}\phi \, e^{-S[\phi]} \tag{2.4}$$

Here κ_n is some function of the topological coupling constant. In practice we usually compute functional integrals by semi-classical expansions. Within each sector $\mathcal{C}_n$ there will be a solution, ϕ_0, to the classical equations of motion depending, in general, on a finite number of parameters $\{t^A\}$, so we set $\phi = \phi_0(t) + \bar{\phi}$. $\bar{\phi}$ is continuously deformable to zero, so we could expand the action in powers of $\bar{\phi}$ to obtain a Gaussian to leading order, the linear term vanishing by the equations of motion. However the derivatives of ϕ_0 with respect to the moduli will be zero-modes of the quadratic action, so that $\bar{\phi}$ should be made transverse to them by imposing some constraints. The integral over $\bar{\phi}$ can then be done yielding a function of the t. This must then be integrated over the moduli corresponding to the remaining degrees of freedom of ϕ, the zero-modes. We want to liberate ourselves from the restrictions of the semi-classical expansion so we will extract this dependence on the moduli using the Faddeeev-Popov trick without resorting to approximation. This is particularly useful because in the two cases in which we are primarily interested there is also a gauge-invariance that can be treated simultaneously using this method. So we will suppose that the action has an infinite number of invariances parametrised by group elements g i.e. if $\phi \rightarrow \phi^g$ then $S[\phi] \rightarrow S[\phi^g] = S[\phi]$. These symmetries will form a closed algebra, so for g close to the identity, $g = 1 + \omega$, and ω^a the components of ω in some basis then $\delta_\omega \phi = \omega^a \delta_a \phi$ and

$$[\delta_a, \delta_b]\phi = f_{ab}^c \delta_c \phi \tag{2.5}$$

We will introduce the moduli and gauge-fixing conditions simultaneously. Choose an infinite number of functions $F_j(\phi, t)$ such that the conditions $F_j(\phi^g, t) = 0$ for any $\phi \in \mathcal{C}_n$ have (locally) a unique solution for g and the moduli t^A. (In general there will be a Gribov ambiguity globally.) For example in the Yang-Mills case we could choose the conditions to consist of the background gauge condition as well as a finite number of constraints requiring $\bar{\phi}$ to be orthogonal to the zero-modes. With $\mathbf{A}_\mu$ denoting the gauge-potential, and $\mathcal{A}_\mu(t)$ the instanton solution to the classical equations of motion depending on moduli t, we would take on R^4

$$[\partial_\mu + \mathcal{A}_\mu(t), \mathbf{A}_\mu - \mathcal{A}_\mu(t)] = 0, \quad \int d^4x \, tr \left(\frac{\partial \mathcal{A}_\mu(t)}{\partial t^A} \left(\mathbf{A}_\mu - \mathcal{A}_\mu(t) \right) \right) = 0. \tag{2.6}$$

Following the usual Faddeev-Popov construction [1] we want to impose these constraints using delta-functions, so to this end we look for a functional $\Delta[\phi, t]$ such that

$$\int \mathcal{D}g\, dt\, \Delta[\phi, t] \prod_j \delta(F_j(\phi^g, t)) = 1. \tag{2.7}$$

Here $\mathcal{D}g$ is the Haar measure on the group of gauge tranformations. Invariance of this measure under multiplication of g by a group element implies that $\Delta[\phi^g, t] = \Delta[\phi, t]$. Suppose that for $\phi = \tilde{\phi}$ the constraints have a solution $g = \hat{g}$ and $t = \hat{t}$. If we expand about this solution $g = (1 + \omega)\hat{g}$ and $t = \hat{t} + \tilde{t}$ then, with ∂_A denoting $\partial/\partial t^A$

$$F_j(\tilde{\phi}^g, t) \simeq \left(\delta_\omega + \tilde{t}^A \partial_A\right) F_j(\phi, t)|_{\phi=\tilde{\phi}^{\hat{g}}, t=\hat{t}}. \tag{2.8}$$

so that

$$\int \mathcal{D}g\, dt \prod_j \delta\left(F_j(\tilde{\phi}^g, t)\right) = \int \mathcal{D}\omega\, d\tilde{t} \prod_j \delta\left((\delta_\omega + \tilde{t}^A \partial_A) F(\phi, t)|_{\phi=\tilde{\phi}^{\hat{g}}, t=\hat{t}}\right) \tag{2.9}$$

By the usual rules for integrating out delta-functions (2.9) becomes

$$Det^{-1}\left(\delta_a F_j(\phi, t), \quad \partial_A F_j(\phi, t)\right)|_{\phi=\tilde{\phi}^{\hat{g}}, t=\hat{t}}, \tag{2.10}$$

so that Δ is the determinant itself. This may be represented using Grassmann numbers. For each transformation parameter ω^a we have a ghost c^a, for each constraint F_j an anti-ghost b^j and for each modulus t^A a quasi-ghost τ^A, enabling the determinant to be represented as

$$\Delta = \int \mathcal{D}(b, c)\, d\tau\, exp\, -\left((c^a \delta_a + \tau^A \partial_A)(b^j\, F_j)\right). \tag{2.11}$$

Inserting the identity (2.7) into Z_n and changing the order of integration gives

$$Z_n = \int \mathcal{D}g\, dt \int \mathcal{D}\phi\, e^{-S[\phi]} \Delta[\phi, t] \prod_j \delta\left(F_j(\phi^g, t)\right) \tag{2.12}$$

If we take ϕ^g as a new integration variable then assuming that $\mathcal{D}\phi = \mathcal{D}\phi^g$ and using the invariance of Δ and $S[\phi]$ we obtain, on renaming ϕ^g as ϕ

$$Z_n = \int \mathcal{D}g\, dt \int \mathcal{D}\phi\, e^{-S[\phi]} \Delta[\phi, t] \prod_j \delta\left(F_j(\phi, t)\right) \equiv \left(\int \mathcal{D}g\right) \int dt\, z(t) \tag{2.13}$$

where the infinite volume of the gauge group is now explicitly factored out. With the representation (2.11) and writing the delta-functions as integrals over λ^j we finally arrive at the moduli space density $z(t)$ expressed as

$$z(t) = \int \mathcal{D}(\phi, b, c, \lambda)\, d\tau\, e^{-S_{FP}}$$

$$S_{FP} \equiv S[\phi] + i\lambda^j F_j(\phi, t) + (c^a \delta_a + \tau^A \partial_A) b^j\, F_j(\phi, t), \tag{2.14}$$

This can be written more economically using the BRST transformation [2]. In our case, due to the presence of the moduli, the BRST transformation will *not* be a symmetry of S_{FP}, although it is still useful. This transformation is parametrised by a Grassmann number, η say, and acts on the fields $\phi, b, c, \lambda, \tau$, but not on the moduli, and may be written as $\delta_\eta \phi = \eta \varsigma \phi$ etc., where the ς operation is given by

$$\varsigma\phi = c^a \delta_a \phi, \quad \varsigma c^a = \frac{1}{2} c^b c^c f^a_{bc}, \quad \varsigma b^j = i\lambda^j, \quad \varsigma\lambda^j = 0, \quad \varsigma\tau^A = 0. \tag{2.15}$$

ς is nilpotent by construction, i.e. $\varsigma^2 = 0$. With its use we can write the action as

$$S_{FP} = S[\phi] + (\varsigma + \tau^A \partial_A)(b^j\, F_j(\phi, t)). \tag{2.16}$$

ς and $\tau^A \partial_A$ anti-commute because ∂_A acts only on the moduli whereas ς does not. Furthermore $\tau^A \partial_A$ is nilpotent since derivatives with respect to the moduli commute with each other, so the sum $\varsigma + \tau^A \partial_A$ is also nilpotent. Given that $S[\phi]$ is gauge invariant and independent of the moduli, which only enter S_{FP} via the constraints, it follows that

$$(\varsigma + \tau^A \partial_A)\, S_{FP} = (\varsigma + \tau^A \partial_A)\, S[\phi] + (\varsigma + \tau^A \partial_A)^2\, b^j\, F_j(\phi, t) = 0 \tag{2.17}$$

so that the gauge-fixed action is not BRST invariant, but rather $\varsigma S_{FP} = -\tau^A \partial_A S_{FP}$. If we denote the fields $\phi, b, c, \lambda, \tau$ collectively by Ψ then the Jacobian for the transformation $\Psi \to \Psi + \delta_\eta \Psi$ is one plus the super-trace of the derivative of $\delta_\eta \Psi$ with respect to Ψ. This super-trace is

$$\frac{\delta}{\delta\phi} \eta c^a \delta_a \phi - \frac{\partial}{\partial c^a} \left(\frac{1}{2} \eta c^b c^c f^a_{bc} \right) = \eta c^a \left(\frac{\delta}{\delta\phi} \delta_a \phi + f^b_{ba} \right) \tag{2.18}$$

so the volume element $\mathcal{D}\Psi$ is invariant if

$$\frac{\delta}{\delta\phi} \delta_a \phi + f^b_{ba} = 0. \tag{2.19}$$

The partition function is given by a sum of moduli-space integrals $\int dt\, z(t)$, with $z(t) = \int \mathcal{D}\Psi\, exp - S_{FP}$. If these integrals diverge then we cut off the region of integration to M introducing additional components to the boundary ∂M. We need to know if this procedure depends on the choice of arbitrary conditions F_j. If it does, then this is unsatisfactory and we have to find some way of repairing the damage. Suppose we make an arbitrary variation of the constraints $F_j \to F_j + \delta F_j$, then the change in the action is $\delta S_{FP} = (\varsigma + \tau^A \partial_A) b^j \delta F_j$ so

$$\delta e^{-S_{FP}} = -(\varsigma + \tau^A \partial_A)(b^j \delta F_j)\, e^{-S_{FP}} = -(\varsigma + \tau^A \partial_A) \left(b^j \delta F_j e^{-S_{FP}} \right). \tag{2.20}$$

Thus

$$\delta z(t) = -\int \mathcal{D}\Psi(\varsigma + \tau^A \partial_A)\left(b^j\, \delta F_j\, e^{-S_{FP}}\right). \tag{2.21}$$

Consider $\int \mathcal{D}\Psi\, b^j \delta F_j\, exp - S_{FP}$. This vanishes because it is Grassmann odd. If we change the integration variables $\Psi \to \Psi + \delta_\eta \Psi$ then the change in the integral is

$$\int \mathcal{D}\Psi\, \eta\varsigma(b^j \delta F_j\, e^{-S_{FP}}), \tag{2.22}$$

but the value of the integral does not change under a change of integration variable, so (2.22) vanishes for all η. Consequently

$$\delta z(t) = -\partial_A \int \mathcal{D}\Psi \tau^A b^j\, \delta F_j\, e^{-S_{FP}}. \tag{2.23}$$

Using Stokes' theorem the change in the density integrated over the cut-off moduli space is

$$\delta \int_M dt\, z(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi \tau^A b^j\, \delta F_j\, e^{-S_{FP}}. \tag{2.24}$$

If this is not zero then we have a problem because the partition function depends on our choice of constraints. On ∂M at least one of the moduli takes its cut-off value so the field configurations contributing to $z(t)|_{\partial M}$ approximate to those of a different topological sector, thus it may be possible to cancel (2.24) with a counter-term added into the contribution to the partition function from this other sector. The overall value of the partition function may then be made independent of the choice of constraints. The details of how this is done will vary from theory to theory. In section 3 we will see how the breakdown of BRST invariance in the bosonic string at one-string-loop can be corrected for by a tree-level counter-term.

More generally we should consider the contribution from $\mathcal{C}_n$ to the expectation value of some gauge-invariant operator $V(\phi)$,

$$\langle V \rangle_n \equiv \int dt \int \mathcal{D}\Psi\, V(\phi)\, e^{-S_{FP}}, \quad \delta_a V(\phi) = 0 \tag{2.25}$$

Since V is gauge-invariant and independent of the moduli which only enter via the constraints it is annihilated by $\varsigma + \tau^A \partial_A$. Repeating the above argument shows that the change in $\langle V \rangle_n$ resulting from a change in the constraints is

$$\delta \langle V \rangle_n = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi\, V(\Psi) \tau^A b^j\, \delta F_j\, e^{-S_{FP}}. \tag{2.26}$$

We need to ensure that any counter-term constructed to re-instate reparametrisation invariance in the sum over topological sectors for the partition function has the effect of making expectation values of gauge-invariant operators reparametrisation invariant too.

Formula (2.24) also describes the effect on the partition function of a transformation, $\phi \to \phi + \delta\phi$, that is an additional symmetry of $S[\phi]$ that commutes with the gauge invariance. For example Weyl invariance in Yang-Mills theory. Under this transformation the change in the Faddeev-Popov action is entirely due to the change in F_j, and assuming that $[\varsigma, \delta] = 0$ (2.24) gives the resulting change in the contribution of $\mathcal{C}_n$ to the partition function provided the Jacobian is unity.

The usual Ward identities that express the BRST symmetry [2] are modified by the above considerations. As a consequence the theory will not be BRST invariant unless the invariance under the choice of F_j is repaired. In the usual case the Ward identities state that the expectation value of an operator of the form $\varsigma V(\Psi)$ vanishes. This is crucial to the usual perturbative renormalisation of Yang-Mills theory [8], and to the decoupling of spurious states in String Theory, and so clearly is a feature of the theory that we do not want to spoil. However, we will see that the contribution of the topological sector $\mathcal{C}_n$ to this expectation value reduces to an integral over ∂M. Consider the contribution of $\mathcal{C}_n$ to $\int \mathcal{D}\Psi\, V(\Psi) e^{-S_{FP}}$, with V Grassmann odd. Applying the change of integration variables $\Psi \to \Psi + \delta_\eta \Psi$ on the assumption that (2.18) holds we conclude that $\int \mathcal{D}\Psi\, \varsigma(V(\Psi) exp{-}S_{FP})$ vanishes. Writing this out, and using the fact that $\varsigma + \tau^A \partial_A$ annihilates $exp - S_{FP}$ we obtain

$$\begin{aligned} 0 &= \int \mathcal{D}\Psi\, (\varsigma V(\Psi)) e^{-S_{FP}} - \int \mathcal{D}\Psi\, V(\Psi) \left(\varsigma e^{-S_{FP}}\right) \\ &= \int \mathcal{D}\Psi\, (\varsigma V(\Psi)) e^{-S_{FP}} + \int \mathcal{D}\Psi\, V(\Psi)\, \tau^A \partial_A\, e^{-S_{FP}}. \end{aligned} \tag{2.27}$$

The contribution of $\mathcal{C}_n$ to the expectation value of $\varsigma V(\Psi)$ is thus

$$\langle \varsigma V(\Psi) \rangle_n = \int_M dt \int \mathcal{D}\Psi\, (\varsigma V(\Psi)) e^{-S_{FP}} = - \int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi\, \tau^A\, V(\Psi)\, e^{-S_{FP}}. \tag{2.28}$$

If this does not vanish then we can think of it as a 'BRST anomaly'. It needs to be cancelled by a counter-term if we are to have all the usual nice properties guaranteed by BRST invariance.

3. Bosonic Strings

In Polyakov's formulation of String Theory [3] the partition function for closed strings is given by a sum of functional integrals over closed Riemann surfaces of increasing genus, h, weighted by a power of the coupling κ [4]

$$Z = \sum_{h=0}^{\infty} \kappa^{2-2h} \int_h \mathcal{D}g_{ab}\, \mathcal{D}X^\mu\, e^{-S[g_{ab}, X^\mu]}. \tag{3.1}$$

These integrals are rather formal because they require the factoring out of an infinite gauge group volume. The fields g_{ab} and X^μ are functions of the world-sheet coordinates ξ^a and describe an intrinsic metric and the coordinates of a surface embedded in D-dimensional space-time with metric $\eta_{\mu\nu}$, respectively. The action is [9]

$$S[g_{ab}, X^\mu] = \frac{1}{2}\int d^2\xi \sqrt{g} g^{ab} \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}. \tag{3.2}$$

The volume elements are constructed from inner products on variations of the fields

$$(\delta X, \delta X) = \int d^2\xi\, \delta X^\mu \delta X^\nu \eta_{\mu\nu}, \quad (\delta g, \delta g) = \int d^2\xi\, \delta g_{ab}\delta g_{cd} g^{ac} g^{bd} \tag{3.3}$$

The action is invariant under a local scaling of the metric, or Weyl transformation, which is parametrised by an infinitesimal function ρ, $\delta_\rho g_{ab} = \rho g_{ab}$. For a critical string (D=26) we can treat this as a gauge symmetry. The action is also invariant under an infinitesimal change of the world-sheet coordinates parametrised by the infinitesimal vector ζ^a, i.e. $\delta_\zeta g_{ab} = \nabla_a \zeta_b + \nabla_b \zeta_a$, $\quad \delta_\zeta X^\mu = \zeta^a \partial X^\mu / \partial \xi^a$. This is also a gauge symmetry. The conventional (i.e. Yang-Mills-like) Faddeev-Popov treatment of the tree-level contribution to (3.1) was given in [10]. Using the general approach of the previous section we can consider the contribution of an arbitrary Riemann surface. If we look for a variation of the g_{ab} that is orthogonal to the gauge variations $\delta_\rho g_{ab}$ and $\delta_\zeta g_{ab}$ then since the world-sheet is closed we require that the variation satisfy $g^{ab}\delta g_{ab} = 0$ and $\nabla^a \delta g_{ab} = 0$. On a Riemann surface with $h > 1$, $\quad h = 1$, and $h = 0$ handles there are respectively $6h - 6, 2$, and 0 independent real solutions to these equations, ψ^A_{ab}, and these zero-modes are in one to one correspondence with the moduli which parametrise those changes of g_{ab} that are not gauge-transformations. As a gauge-fixing condition it is conventional to take the background gauge-condition $g_{ab} - \hat{g}_{ab}(t) = 0$, where $\hat{g}_{ab}$ is an arbitrary metric depending on the m moduli, $\{t^A\}$.Ultimately we require that the partition function, as well as transition amplitudes, be independent of this reference metric.

Carrying out the procedure of the previous section we introduce a ghost, c, for each Weyl transformation, ρ, a ghost, c^a, for each reparametrisation ζ^a, anti-ghosts and Lagrange multipliers b^{ab}, λ^{ab} corresponding to the constraints, and quasi-ghosts τ^A for each modulus t^A. The Faddeev-Popov action is then

$$S_{FP}[\Psi] = S[g_{ab}, X^\mu] + i\int d^2\xi\, \lambda^{ab}(g_{ab} - \hat{g}_{ab})$$

$$+ \int d^2\xi (cb^{ab} g_{ab} + 2\nabla_a c_b b^{ab} - \tau^A b^{ab}\partial_A \hat{g}_{ab}). \tag{3.4}$$

The h-string-loop contribution to the partition function $z(t) = \int \mathcal{D}\Psi\, exp(-S_{FP})$ can be simplified by performing a number of integrations. Firstly we integrate over the Lagrange multipliers to re-obtain delta-functions for the constraints. These are used to perform the integral over g_{ab}. The integral over the Weyl ghost may be performed to produce another set of delta-functions, this time imposing the tracelessness of the anti-ghosts, so these are used to integrate over the trace of the anti-ghost. Finally, we integrate over the 'quasi-ghosts', τ, to obtain $6h - 6$ anti-ghost insertions. The result is [11]

$$z(t) = \int \mathcal{D}(X^\mu, b^{ab}, c^a) \left(\prod_{A=1}^{m} \int d^2\xi \, b^{ab} \partial_A \hat{g}_{ab} \right) e^{-S[\hat{g}_{ab}, X^\mu] - \int d^2\xi \, 2\nabla_a c_b \, b^{ab}}, \tag{3.5}$$

where $\hat{g}_{ab} b^{ab} = 0$. If we decompose the anti-ghost into a piece in the space of zero-modes $\{\psi^A\}$ and an orthogonal piece as $b_{ab} = b^A \psi_{A\,ab} + \hat{b}_{ab}$, then the b^A do not appear in the exponent but only in the insertions, so integrating over them generates the determinant of $(\psi_A, \partial_B g_{ab}) \equiv m_{AB}$. From (2.24) the effect of varying $\hat{g}_{ab}$ on the integral of $z(t)$ over the cut-off moduli space is

$$\delta \int_M dt \, z(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi \, \tau^A \left(\int d^2\xi \, b^{ab} \delta \hat{g}_{ab} \right) e^{-S_{FP}}. \tag{3.6}$$

Performing the same integrations as before we arrive at

$$\delta \int_M dt \, z(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}(X^\mu, b^{ab}, c^a) \, \tau^A \, (\int d^2\xi \, b^{ab} \delta \hat{g}_{ab})$$
$$\times \left(\prod_{B \neq A} \int d^2\xi \, b^{ab} \partial_B \hat{g}_{ab} \right) e^{-S[\hat{g}_{ab}, X^\mu] - \int d^2\xi \, 2\hat{\nabla}_a c_b \, b^{ab}}, \tag{3.7}$$

where $\hat{\nabla}$ is the connection constructed from $\hat{g}_{ab}$. Again, if we decompose the anti-ghost as $b_{ab} = b_A \psi^A_{ab} + \hat{b}_{ab}$, then the b_A can only appear in the insertions, so these produce a factor of

$$-(\int d^2\xi \, \psi_B^{ab} \, \delta \hat{g}_{ab}) \, m^{-1}_{AB} \, det \, m. \tag{3.8}$$

Now we can decompose an arbitrary variation of $\delta \hat{g}_{ab}$ as $\rho \hat{g}_{ab} + \nabla_a \zeta_b + \nabla_b \zeta_a + \delta t^A \partial_A \hat{g}_{ab}$. Substituting this into (3.8) just picks out $\delta t^A \, det \, m$, so (3.7) becomes

$$\delta \int_M dt \, z(t) = \int_{\partial M} d\Sigma_A \, \delta t^A \, z(t). \tag{3.9}$$

This simply states that the only changes in the background metric on which the partition function depends are those corresponding to changes in the cut-off modulus, compare (1.3). It is the divergences in the integration over this modulus that we need to cancel. Rather than considering just the partition function it is necessary to address the more stringent problem of regulating divergences in the Ward identities (2.28) for arbitrary operators $V(\Psi)$. In [12] we used a formulation of the BRST invariance that was rather special to string theory to show that moduli space divergences lead to a breakdown in the invariance, and we proposed that re-establishing this invariance should be a constraint on any attempt to control these divergences, for example by the use of counter-terms. With the more general approach to gauge-fixing of the present paper, which is as applicable to Strings as to Yang-Mills theory, we again obtain a breakdown in the BRST invariance. We will now study the one-loop effect, and go further than [12] by explicitly constructing a tree-level counter-term to re-instate the BRST invariance. The divergence we consider occurs as $\Im\tau \to \infty$ in (1.2), and is due to the tachyon. This is normally considered a pathology of the theory, and so is not usually addressed. There are other divergences in the theory, even when supersymmetry is included, and they have been treated in [22].

The operators of interest are integrals over the world-sheet of vertex operators, i.e. $\int d^2\xi\,\sqrt{g}\,V$. For $h = 0, 1$ there are ghost zero-modes due to the existence of conformal Killing vectors, these are vectors which generate reparametrisations of the metric equivalent to Weyl transformations. The integral over the ghosts will vanish unless there are insertions to saturate these zero-modes. For $h = 0$ there are six real conformal Killing vectors, for $h = 1$ there are 2. For $h = 0$ it is convenient to replace three of the vertex operator integrals by insertions of the form $\epsilon_{ab}c^a c^b V$, and for $h = 1$ just one replacement is necessary. This also corrects for the over-counting of the gauge degrees of freedom. If we associate one power of the coupling with each vertex operator, then tree-level scattering amplitudes, which have no moduli, are given by [11]

$$\begin{aligned}\mathcal{A}(\{V_i\})_0 &= \kappa^{(n-6)} \int \mathcal{D}\Psi \left(\prod_{i=1}^{3} \epsilon_{ab}\, c^a\, c^b\, V_i\right) \left(\prod_{j=4}^{n} \int d^2\xi\, \sqrt{g}\, V_j\right) e^{-S_{FP}} \\ &= \kappa^{(n-6)} \int \mathcal{D}(X^\mu, b^{ab}, c^a)\, e^{-S[\hat{g}_{ab}, X^\mu] - \int d^2\xi\, 2\hat{\nabla}_a c_b\, b^{ab}} \\ &\quad \times \left(\prod_{i=1}^{3} \epsilon_{ab}\, c^a\, c^b\, V_i\right) \left(\prod_{j=4}^{n} \int d^2\xi\, \sqrt{\hat{g}_{ab}}\, V_j\right) \end{aligned} \tag{3.10}$$

whereas one-string-loop amplitudes, which have two real moduli corresponding to the complex modulus of (1.2), are given by

$$\begin{aligned}\mathcal{A}(\{V_i\})_1 &= \kappa^{n} \int_F dt \int \mathcal{D}\Psi \epsilon_{ab}\, c^a\, c^b\, V_1 \left(\prod_{j=2}^{n} \int d^2\xi\, \sqrt{g}\, V_j\right) e^{-S_{FP}} \\ &= \kappa^{n} \int_F dt \int \mathcal{D}(X^\mu, b^{ab}, c^a)\, e^{-S[\hat{g}_{ab}, X^\mu] - \int d^2\xi\, 2\hat{\nabla}_a c_b\, b^{ab}} \\ &\quad \times \left(\prod_{A=1}^{2} \int d^2\xi\, b^{ab} \partial_A \hat{g}_{ab}\right) \left(\hat{\epsilon}_{ab}\, c^a\, c^b\, V_1\right) \left(\prod_{j=2}^{n} \int d^2\xi\, \sqrt{\hat{g}_{ab}} V_j\right) \end{aligned} \tag{3.11}$$

$\hat{\epsilon}$ is the anti-symmetric tensor constructed using $\hat{g}_{ab}$.

Consider now the Ward identities (2.28) for one-string-loop. As an illustration we take $V(\Psi)$ to have the form $V = \int d^2\xi b^{rs}\partial_r X^\mu \partial_s X^\nu l_{\mu\nu} e^{ik_0\cdot X} (\prod_{i=1}^{n-1} \int d^2\xi\, V_i) \epsilon_{ab} c^a c^b V_n$. This contains the anti-ghost, since otherwise the expectation value of ςV will vanish, as we will soon see. On higher genus surfaces V would need the anti-ghost anyway, in order that ςV have ghost number zero, on the torus we need it to have ghost number two in order to saturate the ghost zero-modes due to the conformal Killing vectors. We will take

$k_0^2 = -8$ and $\eta^{\mu\nu} l_{\mu\nu} = k_0^\mu l_{\mu\nu} = 0$, in which case $b^{rs}\partial_r X^\mu \partial_s X^\nu l_{\mu\nu} e^{ik_0\cdot X}$ is the vertex operator of a level two state, the BRST transform of which is a spurious state representing a gauge degree of freedom at level two. We will take the V_i to be tachyon vertex operators, $V_i = e^{ik_i\cdot X}, \quad k_i^2 = 8$. Putting this in (2.28) we obtain the Ward identity

$$\langle \varsigma V(\Psi)\rangle_1 = \int_{\partial F} d\Sigma_A \int \mathcal{D}\Psi\, \tau^A\, V(\Psi)\, e^{-S_{FP}} =$$

$$\int_{\partial M} d\Sigma_A \int \mathcal{D}(X^\mu, b^{ab}, c^a)\, e^{-S[\hat{g}_{ab}, X^\mu] - \int d^2\xi\, 2\hat{\nabla}_a c_b\, b^{ab}} \left(\int d^2\xi\, b^{rs}\partial_r X^\mu \partial_s X^\nu l_{\mu\nu} e^{ik_0\cdot X}\right)$$

$$\times \left(\epsilon_{ab}\, c^a\, c^b\, V_n\right) \left(\prod_{j=1}^{n-1} \int d^2\xi\, \sqrt{\hat{g}} V_i\right) \epsilon_{AB} \int d^2\xi\, b^{ab} \partial_B \hat{g}_{ab} \tag{3.12}$$

Were this to vanish it would express the decoupling from a scattering process of the level two gauge degree of freedom represented by the spurious state. To compute the derivatives $\partial_A \hat{g}_{ab}$ we take coordinates independent of the moduli, so we fix $\hat{g}_{ab}$ by taking $ds^2 = |d\xi^1 + \tau d\xi^2|^2/\Im\tau$ and $0 \le \xi^a \le 1$, with opposite sides of the unit square identified. The cut-off boundary ∂F is defined by taking $\Im\tau = T >> 1$. Decomposing the anti-ghost as before, $b_{ab} = b^A \psi_{A\,ab} + \hat{b}_{ab}$, we see that the integral over the b^A gives zero unless, as we stated, V contains b^{rs}. In fact the integral over the b^A yields a factor of $(det\, m)\,(m^{-1})_{AB}\psi_B^{rs}$. So that we can rewrite (3.12) as

$$\int_{\partial F} d\Sigma_A \int \mathcal{D}\Psi\, e^{-S_{FP}} \left(\int d^2\xi\, (m^{-1})_{AB} \psi_B^{rs} \partial_r X^\mu \partial_s X^\nu l_{\mu\nu} e^{ik_0\cdot X}\right)$$

$$\times \left(\epsilon_{ab}\, c^a\, c^b\, V_n\right) \left(\prod_{j=1}^{n-1} \int d^2\xi\, \sqrt{\hat{g}}\, V_i\right). \tag{3.13}$$

The functional integral is given by a standard calculation [7]. It is usual to express the result using different world-sheet coordinates to the above. Take complex coordinates $z = \xi^1 + \tau\xi^2$ with the domain $0 \le \Im z \le \Im\tau, \quad -\frac{1}{2} \le \Re z \le \frac{1}{2}, \quad z_n = \tau$. The vertex operators in (3.13) are all Weyl invariant by virtue of the anomalous behaviour of $e^{ik\cdot X}$ [3,13]. This is true also for the level two operator because if we set $\psi_{A\,rs} = \sqrt{g}\tilde{\psi}_{A\,rs}$ then $\tilde{\psi}_{A\,rs}$ is Weyl invariant. m_{AB} is also Weyl invariant. So, in terms of these coordinates we can write (3.13) as

$$\int_{\partial F} d\Sigma_A \int \mathcal{D}\Psi\, e^{-S_{FP}} \int dz\, d\bar{z}\, (m^{-1})_{AB} \left(\psi_{B\,zz} \bar{\partial} X^\mu \bar{\partial} X^\nu l_{\mu\nu} + \psi_{B\,\bar{z}\bar{z}} \partial X^\mu \partial X^\nu l_{\mu\nu}\right) e^{ik_0\cdot X}$$

$$\times \left(\, c^z\, c^{\bar{z}}\, V_n\right) \left(\prod_{j=1}^{n-1} \int dz\, d\bar{z}\, V_i\right), \tag{3.14}$$

where $\partial = \partial/\partial z, \bar{\partial} = \partial/\partial\bar{z}$. This may be evaluated as $(\Im\tau)^{-2}\, C(\tau)\, F(\tau)$ with

$$F(\tau) = \pi^n \Im\tau \int \left(\prod_{i=0}^{n-1} dz_i \bar{z}_i\right) \left(\prod_{i<j} (\chi(z_i - z_j))^{k_i \cdot k_j}\right) P(\{z_i\})$$

$$P = (m^{-1})_{AB} \tilde{\psi}_{B\,\bar{z}\bar{z}} \sum_{ij} k_i^\mu l_{\mu\nu} k_j^\nu (\partial\, ln\, \chi(z_0 - z_i))(\partial\, ln\, \chi(z_0 - z_j))$$

$$+(m^{-1})_{AB} \tilde{\psi}_{B\,zz} \sum_{ij} k_i^\mu l_{\mu\nu} k_j^\nu (\partial\, ln\, \chi(z_0 - z_i))^\dagger (\partial\, ln\, \chi(z_0 - z_j))^\dagger,$$

$$\chi(z) = 2\pi\, sin\, \pi z \prod_1^\infty \frac{1 - 2e^{2n\pi i\tau} cos\, 2\pi z + e^{4n\pi i\tau}}{(1 - e^{2n\pi i\tau})^2}, \tag{3.15}$$

On ∂F $\Im\tau = T$ and we can approximate $\chi(z) \simeq 2\pi z$. We now change the integration variables to $\xi_i = exp - 2\pi i(z_i - iT/2)$ with integration ranges $exp - T/2 < |\xi| < exp\, T/2$. Computing the Jacobian, and taking account of the mass-shell conditions, and the conditions on the polarisation $l_{\mu\nu}$ we obtain the leading order contribution to (3.13)

$$2\left(\frac{T}{2}\right)^{-13} e^{4\pi T} \pi^n \int \left(\prod_1^{n-1} d\xi_i d\bar{\xi}_i\right) \left(\prod_{n>i>j} |\xi_i - \xi_j|^{k_i \cdot k_j/2}\right)$$

$$\times (m^{-1})_{AB} (\tilde{\psi}_{B\,\bar{\xi}\bar{\xi}} \sum_{ij} k_i^\mu l_{\mu\nu} k_j^\nu (\xi_0 - \xi_i)^{-1} (\xi_0 - \xi_j)^{-1}$$

$$+\tilde{\psi}_{B\,\xi\xi} \sum_{ij} k_i^\mu l_{\mu\nu} k_j^\nu \left((\xi_0 - \xi_i)^{-1} (\xi_0 - \xi_j)^{-1}\right)^\dagger). \tag{3.16}$$

The integral is almost a tree-level amplitude, $\mathcal{A}_0$, for world-sheet metric $ds^2 = d\xi d\bar{\xi}$, and ξ having the whole complex plane as its domain, and vertex operators $V_0 = (\tilde{\psi}_{B\,\bar{\xi}\bar{\xi}} \partial X^\mu \partial X^\nu + \tilde{\psi}_{B\,\xi\xi} \bar{\partial} X^\mu \bar{\partial} X^\nu) l_{\mu\nu} e^{ik_0 \cdot X}$ and $V_i = e^{ik_i \cdot X}$. Apart from the fact that the position of only the n-th tachyon vertex has been fixed and the integration range of the ξ_I is not the whole complex plane. So as $T \to \infty$ the integral approaches the tree-level amplitude multiplied by a divergent function of T, which we will call $v(T)$ to take account of the gauge symmetry generated by the conformal Killing vectors. Thus

$$\langle \varsigma V(\Psi) \rangle_1 \simeq \left(\int_{\partial F} d\Sigma_A\, (m^{-1})_{AB}\, 2\, (T/2)^{-13}\, e^{4\pi T} v(T)\right) \mathcal{A}(\{V_i\})_0 \tag{3.17}$$

This may be compared to the tree-level Ward-identity. The corresponding operator is $V = \int d^2\xi b^{rs} \partial_r X^\mu \partial_s X^\nu e^{ik_0 \cdot X} (\prod_{i=1}^{n-4} \int d^2\xi \sqrt{g}\, V_i) \prod_{j=n-3}^n \epsilon_{ab} c^a c^b V_j$. Because there are no moduli the Ward identity is just

$$\langle \varsigma V \rangle_0 = \int \mathcal{D}\Psi\, \varsigma V(\Psi)\, e^{-S_{FP}} = 0 \tag{3.18}$$

However, since (3.17) factorises into a product of a tree-level amplitude and an integral over ∂F we can hope to cancel the right-hand-side of (3.12) by adding a counter-term to the tree-level contribution. To this end consider the tree-level Ward identitiy for the product of V and a counter-term $\Lambda(\Psi)$. Taking into account the Grassmann character of ς and V we have $\langle\varsigma(V\Lambda)\rangle_0 = \langle(\varsigma V\Lambda - V\varsigma\Lambda)\rangle_0 = 0$. If we take $\Lambda = \int d^2\xi\sqrt{g}\, g^{rs}u_{rs}$ with u a world-sheet tensor independent of Ψ, then $\varsigma\Lambda = \int d^2\xi\,\sqrt{g}(\nabla^r c^s + \nabla^s c^r - g^{rs}g_{ab}\nabla^a c^b)u_{rs}$, so

$$\langle V\varsigma\Lambda\rangle_0 = \kappa^{(n-6)}\int \mathcal{D}(X^\mu, b^{ab}, c^a)\, e^{-S[\hat{g}_{ab},X^\mu]-\int d^2\xi\, 2\hat{\nabla}_a c_b\, b^{ab}}$$
$$\times\left(\int d^2\xi\, b^{rs}\partial_r X^\mu \partial_s X^\nu\, l_{\mu\nu}\, e^{ik_0\cdot X}\right)\left(\int d^2\xi\,\sqrt{\hat{g}}(\hat{\nabla}^r c^s + \hat{\nabla}^s c^r - \hat{g}^{rs}\hat{g}_{ab}\hat{\nabla}^a c^b)u_{rs}\right)$$
$$\times\left(\prod_{i=n-3}^{n}\hat{\epsilon}_{ab}\, c^a\, c^b\, V_i\right)\left(\prod_{j=1}^{n-4}\int d^2\xi\,\sqrt{\hat{g}_{ab}}\, V_j\right) \tag{3.20}$$

The contraction of b^{rs} and the ghosts in $\varsigma\Lambda$ results in u_{rs}, and the remainder of the expression is just the tree-level amplitude $\mathcal{A}(\{V_i\})_0$. If we take

$$u_{rs} = \int_{\partial F} d\Sigma_A\,(m^{-1})_{AB}\,\tilde{\psi}_{B\,rs} 2\,(T/2)^{-13}\, e^{4\pi T} v(T), \tag{3.21}$$

then $\langle\varsigma V\rangle_1 = \langle(\varsigma V)\Lambda\rangle_0$. The full amplitude for n vertex operators is a sum weighted by powers of the coupling, $\sum_{h=0}^{\infty}\kappa^{(6h-6+n)}\langle V\rangle_h$. We will now modify the tree-level contribution by including Λ as a counter-term, so we define a new sum to one-string-loop order

$$\langle\langle V\rangle\rangle = \kappa^{(n-6)}\langle V\,(1-\kappa^6\Lambda)\rangle_0 + \kappa^n\langle V\rangle_1. \tag{3.22}$$

This is constructed so that $\langle\langle\varsigma V\rangle\rangle = 0$, which means that the new sum is independent of how we parametrise the functional integration over Ψ. Note that the counter-term is equivalent to a modification of the action $S_{FP} \to S_{FP} - \kappa^6\Lambda$.

4. Yang-Mills

We now turn to the problem of moduli space divergences in the expansion of Yang-Mills theory as a sum over instanton sectors, labelled by instanton number, n. The partition function in a Euclidean space-time with metric $g_{\mu\nu}$ is formally [4]

$$Z = \sum_{n=-\infty}^{\infty} e^{-in\theta}\int_n \mathcal{D}A\, e^{-S_{YM}},$$
$$S_{YM} = -\frac{1}{4g^2}\int d^4x\sqrt{g}g^{\mu\rho}g^{\nu\sigma}\, tr\,(\mathbf{F}_{\mu\nu}\mathbf{F}_{\rho\sigma}). \tag{4.1}$$

The field-strength is $\mathbf{F}_{\mu\nu} = [\partial_\mu + \mathbf{A}_\mu, \partial_\nu + \mathbf{A}_\nu]$ and the gauge potential is an element of a Lie algebra which we take to be su(N), $\mathbf{A} = A^a T_a$, $\quad T_a^\dagger = -T_a$, $\quad [T_a, T_b] = f_{ab}^c T_c$, and

$tr\,(T_aT_b) = -\delta_{ab}$. The theta-angle plays the role of a coupling for the expansion. The action is invariant under local scalings of the metric, and so if we take $g_{\mu\nu} = \delta_{\mu\nu}\Omega^2(x)$ then it is independent of Ω, consequently the classical equations of motion are conformally covariant, which means that if we apply a conformal transformation to one solution we obtain another. However, the need to introduce a mass-scale in the quantisation of the theory breaks this invariance and there is a Weyl anomaly. That is, the regulated volume element $\mathcal{D}A$ does not share the invariance [14]. In the one-instanton sector the action is minimised by the classical solution [15]

$$A^a_\mu = \eta^a_{\mu\nu}\,\frac{2(x-y)^\mu}{(x-y)^2+\rho^2} \equiv \mathcal{A}_1(x;y,\rho) \tag{4.2}$$

$(x-y)^2 = (x^\mu - y^\mu)(x^\mu - y^\mu)$ and the $\eta^a_{\mu\nu}$, $a = 1,2,3$ form a basis for self-dual tensors. The parameters y^μ and ρ are a complete set of moduli for this sector. With the constraints (2.6) these yield 't Hooft's one-loop result (1.1) for the one-instanton sector partition function when $\Omega = 1$. As stated earlier this diverges for large y^μ and ρ. This implies that BRST invariance is broken as the Ward identities are replaced by (2.28). This infra-red divergence is usually thought to be an artefact of the approximation, which due to asymptotic freedom is only really valid at short distances and scales. However, it was pointed out in [16] that when space-time is compactified to S^4, as it should be to control the infra-red behaviour of the one-loop calculation, the large y^μ, ρ behaviour is reliably computable and not given by (1.1), but rather, leads to a convergent integral. There is then no problem with BRST invariance. We will outline this argument and then discuss the problem of divergences in the two-loop sector which, in contrast to those in the one-loop sector, include apparent short distance divergences.

For a spherical space-time of radius a the function $\Omega = 2/(1 + x^2/a^2)$, and the one-instanton partition function may be obtained by integrating the Weyl anomaly for (1.1). However, it is not necessary to do this to discover the limiting behaviour of the integrand. For small y^μ and ρ the integrand is approximately the same as in the case of R^4 because at short distances R^4 and S^4 look the same. Now there is an invariance of the metric corresponding to inversion through the centre of the sphere followed by a parity transformation to reinstate the original orientation. In terms of coordinates $x^\mu \to \tilde{x}^\mu = a^2\, m^\mu_\nu\, x^\nu/x^2$, $m = diag\,(1,-1,-1,-1)$ and $(1+\tilde{x}^2/a^2)^{-2}d\tilde{x}^2 = (1+x^2/a^2)^{-2}dx^2$. This is a conformal transformation so its effect on the instanton solution is equivalent to a change of the moduli, i.e.

$$\mathcal{A}_\mu(\tilde{x};\rho,y)\,d\tilde{x}^\mu = \mathcal{A}_\mu(x;\tilde{\rho},\tilde{y})\,dx^\mu, \quad \tilde{\rho} = \frac{a^2\rho}{y^2+\rho^2}, \quad \tilde{y}^\mu = \frac{a^2 m^\mu_\nu y^\nu}{y^2+\rho^2}. \tag{4.3}$$

The partition function is reparametrisation invariant under changes in the space-time coordinates and so is the same whether we use x^μ or $\tilde{x}^\mu$. Consequently for small $\tilde{y}, \tilde{\rho}$ where the semi-classical approximation is reliable

$$Z^{YM}_1 = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4\tilde{y}\,d\tilde{\rho}}{\tilde{\rho}^5}\,\tilde{\rho}^{11N/3}. \tag{4.4}$$

Furthermore the measure $d^4y\,d\rho/\rho^5$ is invariant under the action of conformal transformations on the moduli, so $d^4y\,d\rho/\rho^5 = d^4\tilde{y}\,d\tilde{\rho}/\tilde{\rho}^5$, and

$$Z_1^{YM} = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y\,d\rho}{\rho^5}\tilde{\rho}^{11N/3} = e^{-\frac{8\pi^2}{g^2(\mu)}} \int \frac{d^4y\,d\rho}{\rho^5}\left(\frac{a^2\rho}{y^2+\rho^2}\right)^{11N/3}, \tag{4.5}$$

which is valid for small $\tilde{\rho}, \tilde{y}^\mu$ which means *large* ρ, y^μ. Hence the moduli space integration converges in the one-instanton sector.

The two-instanton contribution to the partition function has divergences that appear to be short distance effects. The Yang-Mills action is minimised by the configuration [17]

$$\mathcal{A}_2(x; y_0, y_1, y_2, \lambda_1, \lambda_2) = \frac{1}{2}\eta^a_{\mu\nu}\frac{\partial}{\partial x^\nu} ln\,\phi, \quad \phi = \sum_{i=0}^{2}\frac{\lambda_i^2}{(x-y_i)^2}, \quad \lambda_0 = 1. \tag{4.6}$$

$\lambda_1, \lambda_2, y_0^\mu, y_1^\mu, y_2^\mu$ provide one too many parameters to coordinatise the two instanton moduli space, so there is a gauge symmetry amongst them. When the space-time is taken to be R^4 the partition function is [18]

$$Z_2^{YM} = e^{-\frac{16\pi^2}{g^2(\mu)}}\left(\frac{4\pi}{g^2(\mu)}\right)^8 e^{-2\alpha(1)}\frac{4\pi}{3}\int \frac{d\lambda_1}{\lambda_1}\frac{d\lambda_2}{\lambda_2}\,d^4y_0\,d^4y_1\,d^4y_2\,W^4\,\frac{N_A^{4/3}}{N_S^{1/3}}\sqrt{\Gamma},$$

$$W = z^3\lambda_1\lambda_2, \quad z^2 = \frac{1}{1+\lambda_1^2+\lambda_2^2},$$

$$N_A = z^2(\lambda_2^2(y_0-y_1)^2 + \lambda_0^2(y_1-y_2)^2 + \lambda_1^2(y_2-y_0)^2),$$

$$\begin{aligned} N_S &= W^2(y_0-y_1)^2(y_1-y_2)^2(y_2-y_0)^2, \\ \Gamma &= \Gamma((y_0-y_1)^2,\,(y_1-y_2)^2,\,(y_2-y_0)^2), \\ \Gamma(a,b,c) &= 2(ab+bc+ca) - a^2 - b^2 - c^2. \end{aligned} \tag{4.7}$$

α is a function tabulated in [6]. The $N_A^{4/3}/N_S^{1/3}$ contribution is approximate [19], but is accurate near configurations which degenerate into those of lower instanton number, which is the region of interest to us. These are the configurations for which $(y_i - y_j)^2 \to 0$. The integrand itself does not diverge too badly near these configurations, for example as $y_2 \to y_1$ it behaves as $((y_1-y_2)^2)^{-1/3}$. However it then depends on y_0 and y_1 only through the difference $y_0 - y_1$, so that if the integrals over y_1 and y_0 are written as integrals over the difference $y_0 - y_1$ and the average $(y_0+y_1)/2$ then this last integration gives an infinite volume factor. Thus the partition function appears to diverge as the small distance $(y_1-y_2)^2$ goes to zero. The obvious question to consider is whether the partition function for S^4 has this divergence. We will see that, as in the one instanton sector, compactifying the theory on S^4 removes these moduli space divergences.

The metric on S^4, $g_{\mu\nu} = (2/(1+x^2/a^2))^2\delta_{\mu\nu} = \Omega^2\delta_{\mu\nu}$ differs from that on R^4, $g_{\mu\nu} = \delta_{\mu\nu}$ by a local scaling. So, to construct the partition function on S^4 we study the effect of a Weyl transformation on the metric. The classical action, S_{YM}, is Weyl invariant, but the regulated volume element $\mathcal{D}A$ is not, so there is a Weyl anomaly. The change in the partition function resulting from a symmetry of the classical action that commutes with the gauge invariance, and is respected by the volume element, was given by (2.26). This must now be modified to take account of the anomaly. Thus, under the transformation $\delta_\phi g_{\mu\nu} = \phi g_{\mu\nu}$ the volume element changes, so

$$\delta_\phi \int_M dt\, z(t) = \int_M dt \int \mathcal{D}\Psi \left(\int d^4x\, \phi W(x) \right) e^{-S_{FP}} - \int_{\partial M} d\Sigma_A \int \mathcal{D}\Psi \tau^A b^j \left(\delta_\phi F_j \right) e^{-S_{FP}}. \tag{4.8}$$

If we work to one-loop the integral over the cut-off boundary does not contribute with the choice of F_j (2.6) which are linear in the 'quantum correction,' $\mathbf{A} - \mathcal{A}$, and so contribute at higher order in the expansion in powers of Planck's constant. If $z_a(t)$ and $\mathcal{D}_a\Psi$ denote the partition function moduli space density and volume element for a sphere of radius a, then taking $\phi = \delta a \frac{d}{da} ln\, \Omega^2$ gives, to one-loop

$$\frac{d}{da} \int_M dt\, z_a(t) = \int_M dt \int \mathcal{D}_a\Psi \left(\int d^4x\, \frac{d\, ln\, \Omega^2}{da} W \right) e^{-S_{FP}}. \tag{4.9}$$

Integrating with respect to a from ∞ to a we obtain

$$\int_M dt\, z_a(t) = \int_M dt \int \mathcal{D}_\infty \Psi\, exp \left(-S_{FP} + \int d^4x\, ln\, (\Omega^2/4)\, W \right). \tag{4.10}$$

On general grounds [20] the anomaly density W is a sum of the Lagrangian and the Euler density for the sphere [21]. The latter makes a contribution independent of the moduli which we will ignore. To one loop we evaluate W for the background field $\mathcal{A}$, so to this order

$$\int d^4x\, ln\, (\Omega^2)\, W = \frac{\beta(g^2)}{4g^4} \int d^4x\ ln(\Omega^2)\, tr\, (F_{\mu\nu} F_{\mu\nu}). \tag{4.11}$$

Where the beta-function is

$$\beta = \mu \frac{\partial (g^2(\mu))}{\partial \mu} = -\frac{g^4 11 N}{24\pi^2}. \tag{4.11}$$

For the instanton solution (4.6) we have [17]

$$tr\, F_{\mu\nu} F_{\mu\nu} = 2 \partial^2 \partial^2\, ln\, \sigma,$$

$$\sigma \equiv \left(\lambda_0^2 (x-y_1)^2 (x-y_2)^2 + \lambda_1^2 (x-y_2)^2 (x-y_0)^2 + \lambda_2^2 (x-y_0)^2 (x-y_1)^2 \right), \tag{4.12}$$

where ∂^2 is the flat four-dimensional Laplacian. When $y_2 = y_1$ the function σ has a factor of $(x-y_1)^2$, but $\partial^2 \partial^2\, ln\, (x-y_1)^2 = -16\pi^2 \delta(x-y_1)$, so that $exp \int d^4x\, ln(\Omega^2/4)\, W$ depends on y_1 via the factor $(1+y_1^2/a^2)^{-22N/3}$. The presence of this in the partition function density $z_a(t)$ makes the integral over $(y_0+y_1)/2$ converge. The same arguments may be applied to the divergences of the partition function on R^4 as the scales $\lambda_i \to 0$. Thus the moduli space divergences are indeed removed by compactifying the theory on S^4.

5. Conclusions

The configuration spaces of many field theories consist of disconnected pieces. Quantum mechanical amplitudes are then sums over these components weighted by a function of a coupling constant. For each component the amplitudes can be reduced to finite dimensional integrals over moduli. Within the semi-classical approach the moduli appear as the parameters that label physically distinct solutions to the equations of motion. In String Theory however, they label surfaces that are not equivalent under reparametrisations and local scalings of the metric. We used the Faddeev-Popov trick to make the dependence on the moduli explicit without resorting to approximation. This involves choosing coordinates on the configuration space by imposing constraints. In general the moduli space integrals will diverge. If we regulate them with cut-offs the integrals will depend strongly on the choice of cut-off surfaces in moduli space. As a result amplitudes are not independent of the constraints we use to parametrise configuration space. This dependence is expressed by (2.26) and is the origin of the 'BRST anomaly' (2.28). Physics must be independent of such arbitrary choices, and so a viable quantum theory must be capable of being cured of this disease. The two theories we have considered in detail, Bosonic Strings and Yang-Mills, cope with this in different ways. In String Theory the one-string-loop divergence is due to a toroidal world-sheet degenerating by becoming infinitely long. Scattering amplitudes computed for such a surface are given by the corresponding amplitudes for a spherical world-sheet multiplied by a divergent factor. Thus it is possible to cancel the one-loop 'BRST anomaly' by a tree-level counter-term (3.22). It is common that moduli space divergences are associated with configurations in a given component degenerating to those of another component. This gives the possibility of 'topological renormalisation' in which the pathologies of the topological expansion over the disconnected components of configuration space can be cured by counter-terms relating different components. In Yang-Mills theory the story is quite different. The divergences associated with the integration over instanton moduli in the one instanton sector disappear when the infra-red divergences of the theory are properly regulated by compactifying onto S^4 [16]. We have shown that the same thing happens to the divergences of the two-instanton sector. Although we have worked with theories that have a local invariance that requires gauge fixing, our considerations are not restricted to such theories.

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