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A CONVERGING LAGRANGIAN FLOW IN THE SPACE OF ORIENTED LINES

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Abstract. Under mean radius of curvature flow, a closed convex surface in Euclidean space is known to expand exponentially to infinity. In the three-dimensional case we prove that the oriented normals to the flowing surface converge to the oriented normals of a round sphere whose centre is the Steiner point of the initial surface, which remains constant under the flow.

To prove this we show that the oriented normal lines, considered as a surface in the space of all oriented lines, evolve by a parabolic flow which preserves the Lagrangian condition. Moreover, this flow converges to a holomorphic Lagrangian section, which forms the set of oriented lines through a point.

The coordinates of the Steiner point are projections of the support function into the first non-zero eigenspace of the spherical Laplacian and are given by explicit integrals of initial surface data.

0. Introduction

Consider the evolution of a sphere $f : S^n \times [0, \infty) \to \mathbb{R}^{n+1}$ by *mean radius of curvature flow* (MRCF):

$$\frac{\partial f}{\partial t}^{\perp} = \sum_{j=1}^{n} r_j \, \mathbf{N}, \tag{1}$$

where $\mathbf{N}$ is the unit normal vector of $S_t = f_t(S^n) \subset \mathbb{R}^{n+1}$ and $r_1, r_2, \ldots, r_n$ are the radii of curvature of S_t.

As noted in [**1**], this flow, referred to there as the inverse harmonic mean curvature flow, is expanding and the support function r of the surface evolves by the linear strictly parabolic equation

$$\left(\frac{\partial}{\partial t} - \triangle_{S^n}\right) r = nr.$$

As a result, the support function for a closed convex surface increases exponentially, and the surface expands to infinity. Moreover, in [**7**] it is proven that by rescaling the flow about the origin, the surface converges to a round sphere with centre at 0.

In what follows we extract more information about this flow for $n = 2$ and prove the following theorem.

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MAIN THEOREM. *Under mean radius of curvature flow, the oriented normal lines to any convex surface with support function r_0 as initial data converge to those of the round sphere with the centre at Steiner point (x^1, x^2, x^3) of S_0:*

$$x^1 + ix^2 = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi} r_0(\theta, \phi)\sin^2\theta e^{i\phi}\, d\theta\, d\phi,$$
$$x^3 = \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi} r_0(\theta, \phi)\sin\theta\cos\theta\, d\theta\, d\phi, \qquad (2)$$

where (θ, ϕ) are standard spherical coordinates.

Thus, the Steiner point is invariant under the flow.

We prove this by computing the flow of the oriented normal lines to the surface as it evolves by MRCF. These normals form a surface in the space of all oriented lines.

In particular, recall that, given any smooth oriented convex surface S_t in $\mathbb{R}^3$, the set of oriented normal lines to S_t forms a surface Σ_t in the space $\mathbb{L}(\mathbb{R}^3)$ of all oriented lines of $\mathbb{R}^3$. This surface is of necessity Lagrangian with respect to the canonical symplectic structure on $\mathbb{L}(\mathbb{R}^3)$ and, since S_t is convex, Σ_t is a section of the bundle $\pi : \mathbb{L}(\mathbb{R}^3) = T\mathbb{S}^2 \to \mathbb{S}^2$ [**3, 4**].

Conversely, for any Lagrangian section Σ of $\mathbb{L}(\mathbb{R}^3)$, the line congruence determined by Σ is integrable; that is, there exists an embedded surface S in $\mathbb{R}^3$ orthogonal to the lines (see [**4**, Proposition 10]).

The evolution of the Lagrangian section induced by MRCF, in contrast to the flow in $\mathbb{R}^3$, converges without rescaling, and we prove the following theorem.

MAIN THEOREM (reformulated). *Under mean radius of curvature flow, any initial Lagrangian section converges smoothly to a quadratic holomorphic Lagrangian section of $\mathbb{L}(\mathbb{R}^3)$ determined by*

$$F = \tfrac{1}{2}[x^1 + ix^2 - 2x^3\xi - (x^1 - ix^2)\xi^2]$$

with coefficients given by the formulae (2).

A holomorphic Lagrangian section corresponds to those oriented lines that pass through a fixed point in $\mathbb{R}^3$, and so, while the convex surfaces in $\mathbb{R}^3$ run out to infinity under the flow, their normal lines converge to the normals of a round sphere (without rescaling the flow). Note that a surface can be both Lagrangian and holomorphic in this setting because the associated Kaehler metric is of neutral signature [**4**].

The proof involves showing that, under MRCF, the Lagrangian sections flow by a linear strictly parabolic equation system – see Proposition 2. Then, utilizing spherical harmonics to solve the equation in terms of the initial spectral decomposition we study the asymptotic behaviour. The fundamental result for the parabolic equation that we use is contained in Proposition 4.

In the next section, we describe the geometric relationship between $\mathbb{R}^3$ and $\mathbb{L}(\mathbb{R}^3)$. Comparison of MRCF in the two spaces is carried out in Section 2, while in the final section we prove the Main Theorem.

1. The space of oriented lines

The space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines of Euclidean $\mathbb{R}^3$ can be identified with $T\mathbb{S}^2$, the total space of the tangent bundle to the 2-sphere. $T\mathbb{S}^2$ carries a neutral Kähler structure $(\mathbb{G}, \mathbb{J}, \Omega)$ which is invariant under the Euclidean group acting on oriented lines. In what follows, the terms holomorphic and Lagrangian refer to the complex structure $\mathbb{J}$ and symplectic structure Ω, respectively. The metric $\mathbb{G}$ is of neutral signature; hence planes can be both holomorphic and Lagrangian. Further details on the neutral Kähler structure can be found in [**3, 4**].

For local computations, let ξ be the standard complex coordinate on $\mathbb{S}^2$ coming from stereographic projection from the south pole so that $\xi = \tan(\theta/2)e^{i\phi}$ for the spherical polar coordinates $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Extend this to complex coordinates (ξ, η) on an open set of $T\mathbb{S}^2$ by identifying $X \in T_\xi\mathbb{S}^2$ with $(\xi, \eta) \in \mathbb{C}^2$ when

$$X = \eta\frac{\partial}{\partial\xi} + \bar{\eta}\frac{\partial}{\partial\bar{\xi}}.$$

Consider the set of oriented normal lines to a surface S. These form a Lagrangian surface $\Sigma \subset T\mathbb{S}^2$. As there are no flat points, the Gauss map of S is invertible and hence Σ is a Lagrangian section of the canonical bundle $\pi : T\mathbb{S}^2 \to \mathbb{S}^2$. In terms of local coordinates, the surface Σ is given by $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$ for some complex-valued function F.

The link between these holomorphic coordinates and flat coordinates (x^1, x^2, x^3) in $\mathbb{R}^3$ is provided by the map $\Phi : T\mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^3$:

$$x^1 + ix^2 = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1+\xi\bar{\xi})r}{(1+\xi\bar{\xi})^2}, \qquad x^3 = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1-\xi^2\bar{\xi}^2)r}{(1+\xi\bar{\xi})^2}, \tag{3}$$

which sends an oriented line (ξ, η) and a real number r to the point on the line in $\mathbb{R}^3$ that is an oriented distance r from the closest point on the line to the origin.

These equations can be recast as

$$\eta = \frac{1}{2}(x^1 + ix^2 - 2x^3\xi - (x^1 - ix^2)\xi^2), \quad r = \frac{(x^1 + ix^2)\bar{\xi} + (x^1 - ix^2)\xi + x^3(1 - \xi\bar{\xi})}{1 + \xi\bar{\xi}}.$$

The perpendicular distance χ of an oriented line (ξ, η) to the origin is found to be

$$\chi^2 = \frac{4\eta\bar{\eta}}{(1+\xi\bar{\xi})^2}. \tag{4}$$

Definition 1. The *support function* of a convex surface is the map $r : S \to \mathbb{R}$ which takes a point p to the signed distance between p and the point on the oriented normal line to S at p which lies closest to the origin. Alternatively, it is the signed perpendicular distance between the oriented tangent plane to S at p and the origin.

The relationship between the support function r of S and the complex-valued function F determining the Lagrangian section corresponding to S is

$$F = \tfrac{1}{2}(1+\xi\bar{\xi})^2\bar{\partial}r. \tag{5}$$

Define the complex slopes of F by

$$\bar{\partial}F = -\bar{\sigma}, \quad (1+\xi\bar{\xi})^2\partial\left(\frac{F}{(1+\xi\bar{\xi})^2}\right) = \rho + i\lambda. \tag{6}$$

A section is Lagrangian if and only if $\lambda = 0$ and this implies the existence of the real function r satisfying equation (5). In addition, the radii of curvature of the surface S are determined by

$$|\sigma|^2 = \tfrac{1}{4}(r_1 - r_2)^2, \quad (r+\rho)^2 = \tfrac{1}{4}(r_1 + r_2)^2.$$

Finally, translations in $\mathbb{R}^3$ act on our functions as follows. Suppose we consider the translation that takes the origin to $(x^1 + ix^2, x^3) = (\alpha, b)$. Then we have

$$\eta \mapsto \eta + \frac{1}{2}(\alpha - 2b\xi - \bar{\alpha}\xi^2), \quad r \mapsto r + \frac{\alpha\bar{\xi} + \bar{\alpha}\xi + b(1 - \xi\bar{\xi})}{1 + \xi\bar{\xi}},$$

while σ and $r + \rho$ are invariant under translations.

2. Mean radius of curvature flow

Let us now consider the flow (1) for a strictly convex surface S_t in $\mathbb{R}^3$. Using coordinates $(x^1 + ix^2, x^3)$ on $\mathbb{R}^3$ and Gauss coordinates ξ on S_t, let $r_t : S^2 \to \mathbb{R}$ be the support function of S_t. Then, with $(r = r_t, \eta = \eta_t)$, differentiating equations (3) in time yields

$$\frac{\partial}{\partial t}(x^1 + ix^2) = \frac{2}{(1+\xi\bar{\xi})^2}\frac{\partial}{\partial t}\eta - \frac{2\xi^2}{(1+\xi\bar{\xi})^2}\frac{\partial}{\partial t}\bar{\eta} + \frac{2\xi}{1+\xi\bar{\xi}}\frac{\partial}{\partial t}r,$$
$$\frac{\partial}{\partial t}x^3 = -\frac{2\bar{\xi}}{(1+\xi\bar{\xi})^2}\frac{\partial}{\partial t}\eta - \frac{2\xi}{(1+\xi\bar{\xi})^2}\frac{\partial}{\partial t}\bar{\eta} + \frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}\frac{\partial}{\partial t}r,$$

and projecting this we obtain

$$\frac{\partial f}{\partial t}^{\perp} = \frac{\partial r}{\partial t}N = (r_1 + r_2)N.$$

Finally, from the relationship between section and support (5) we have

$$r_1 + r_2 = 2(r+\rho) = 2r + 2(1+\xi\bar{\xi})^2\partial\left(\frac{F}{(1+\xi\bar{\xi})^2}\right)$$
$$= 2r + (1+\xi\bar{\xi})^2\partial\bar{\partial}r = 2r + \triangle_{\mathbb{S}^2}r.$$

We have therefore proven the first part of Proposition 1.

PROPOSITION 1. *Under MRCF, the support function evolves by*

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)r = 2r, \tag{7}$$

while the perpendicular distance function of the normal lines evolves by

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)\chi^2 = 2\chi^2 - 4(\rho^2 + |\sigma|^2).$$

Proof. The first we have proven, and the second follows from a similar calculation. □

In the space of oriented lines, the set of oriented normal lines to S_t forms a Lagrangian section of $T\mathbb{S}^2 \to \mathbb{S}^2$ given locally by a complex function $F : \mathbb{C} \to \mathbb{C}$. We lift the flow to the space of oriented lines in Proposition 2.

PROPOSITION 2. *Under MRCF, the Lagrangian section F evolves in $T\mathbb{S}^2$ by the linear parabolic system*

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)F = -\frac{2\bar{\xi}}{1+\xi\bar{\xi}}\bar{\partial}F.$$

Proof. Differentiate the relationship (5) in time to obtain

$$\frac{\partial}{\partial t}F = \frac{1}{2}(1+\xi\bar{\xi})^2\bar{\partial}\frac{\partial}{\partial t}r = \triangle_{\mathbb{S}^2}F - \frac{2\bar{\xi}}{1+\xi\bar{\xi}}\bar{\partial}F. \qquad \square$$

Finally, we compute the flow of the derived quantities.

PROPOSITION 3. *Under MRCF, the slopes evolve by*

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)\rho = 2\rho, \qquad \left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)\lambda = -2\lambda,$$

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)\sigma = -2(1+2\xi\bar{\xi})\sigma + 2(1+\xi\bar{\xi})(\bar{\xi}\bar{\partial}\sigma - \xi\partial\sigma).$$

Proof. Differentiate the defining relationships (6) in time and use the previous proposition. □

Note that the flow equation for λ is such that, if $\lambda = 0$ initially, it remains so for all time. Since $\Omega|_\Sigma = \lambda d^2_{\mathbb{S}}A$, we say that the flow in $T\mathbb{S}^2$ is Lagrangian, since it preserves the Lagrangian condition.

In fact, even if the initial surface is not Lagrangian, we prove in Proposition 5 that under the flow it becomes Lagrangian.

3. Proof of the Main Theorem

Consider the flow

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)f = 2f, \tag{8}$$

for $f : S \times [0,\infty) \to \mathbb{R}$ with $f(\cdot, 0) = f_0(\cdot)$.

Definition 2. Define the *spherical area* $A_{\mathbb{S}^2}(f)$ of f by

$$A_{\mathbb{S}^2}(f) = \int_{\mathbb{S}^2} f\, dA.$$

PROPOSITION 4. *The above flow converges if and only if the $A_{\mathbb{S}^2}(f_0) = 0$.*

For $A_{\mathbb{S}^2}(f_0) = 0$, it converges smoothly to an eigenfunction for the spherical Laplacian with eigenvalue 2.

For $A_{\mathbb{S}^2}(f_0) \neq 0$, there exist a positive constant C_1 and a constant C_2 depending only on f_0 such that

$$|f| \geq C_1 e^{2t} + C_2.$$

Proof. The flow (8) is linear and strictly parabolic, and therefore by standard theory [**5**], given any initial function, there exists a smooth solution for all time. Let f_t be the solution of the flow for some initial f_0.

Integrating the flow equation over the 2-sphere we obtain

$$\frac{\partial}{\partial t}A_{\mathbb{S}^2}(f)=A_{\mathbb{S}^2}(f).$$

Thus, if $A_{\mathbb{S}^2}(f_0)=0$, then $A_{\mathbb{S}^2}(f)=0$ for all time, while $A_{\mathbb{S}^2}(f_0)\neq 0$ implies exponential growth in time for the spherical area.

For fixed time t, decompose $f_t:S\to\mathbb{R}$ in terms of spherical harmonics $Y_l^m:S\to\mathbb{R}$ [**6**]:

$$f_t=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}Y_l^m,$$

where B_{lm} are complex and satisfy $\overline{B_{lm}}=(-1)^lB_{l-m}$ for $m\neq 0$ and $\overline{B_{l0}}=B_{l0}$.

Since the flow is linear, we obtain a flow on the projection of f onto the spectrum of the Laplacian:

$$\frac{\partial B_{lm}}{\partial t}=[2-l(l+1)]B_{lm},$$

which integrates to yield

$$f_t=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\overset{\circ}{B}_{lm}\,e^{[2-l(l+1)]t}Y_l^m.$$

For convenience, we have denoted B_{lm} at $t=0$ by $\overset{\circ}{B}_{lm}$.

In more detail, the spectral decomposition is given by

$$\begin{aligned}f_t=&\overset{\circ}{B}_{00}e^{2t}Y_0^0+\overset{\circ}{B}_{1-1}\,Y_1^{-1}+\overset{\circ}{B}_{10}\,Y_1^0+\overset{\circ}{B}_{11}\,Y_1^1+\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\overset{\circ}{B}_{lm}e^{(2-l(l+1))t}Y_l^m\\
=&\frac{1}{2}\sqrt{\frac{1}{\pi}}\,\overset{\circ}{B}_{00}e^{2t}+\sqrt{\frac{3}{2\pi}}\,\overset{\circ}{B}_{1-1}\,\frac{\bar{\xi}}{1+\xi\bar{\xi}}+\frac{1}{2}\sqrt{\frac{1}{\pi}}\,\overset{\circ}{B}_{10}\,\frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}-\sqrt{\frac{3}{2\pi}}\,\overset{\circ}{B}_{11}\,\frac{\xi}{1+\xi\bar{\xi}}\\
&+\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\overset{\circ}{B}_{lm}e^{(2-l(l+1))t}Y_l^m\\
=&\frac{1}{2}\sqrt{\frac{1}{\pi}}A_{\mathbb{S}^2}(f_0)e^{2t}-\frac{\alpha\bar{\xi}+\bar{\alpha}\xi+b(1-\xi\bar{\xi})}{1+\xi\bar{\xi}}+\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\overset{\circ}{B}_{lm}e^{(2-l(l+1))t}Y_l^m,\end{aligned}$$

where we note that, by the orthogonality properties of the spherical harmonics,

$$A_{\mathbb{S}^2}(f_0)=\int_{\mathbb{S}^2}f_0\,dA=\overset{\circ}{B}_{00},$$

and we have introduced $\alpha\in\mathbb{C}$, $b\in\mathbb{R}$:

$$\alpha=-\sqrt{\frac{3}{2\pi}}\,\overset{\circ}{B}_{11},\qquad b=\frac{1}{2}\sqrt{\frac{1}{\pi}}\,\overset{\circ}{B}_{10}.$$

If $A_{\mathbb{S}^2}(f_0)\neq 0$, then $|f_t|$ exponentially blows up as $t\to\infty$, while for $A_{\mathbb{S}^2}(f_0)=0$, f_t converges to an eigenfunction of the spherical Laplacian with an eigenvalue equal to 2:

$$f_t\to-\frac{\alpha\bar{\xi}+\bar{\alpha}\xi+b(1-\xi\bar{\xi})}{1+\xi\bar{\xi}}\quad\text{as } t\to\infty,$$

as claimed. □

Proof of Main Theorem. We have seen that under MRCF the support flows by equation (7). By Proposition 4, we need to determine the behaviour of the area $A_{\mathbb{S}^2}(r)$. Note that this integral is invariant under translation:

$$A_{\mathbb{S}^2}(r) = A_{\mathbb{S}^2}\left(r - \frac{\alpha\bar{\xi} + \bar{\alpha}\xi + b(1 - \xi\bar{\xi})}{1 + \xi\bar{\xi}}\right).$$

If we move the origin to the interior of S_0 (which is assumed to be convex), we can make $r > 0$ and conclude that $A_{\mathbb{S}^2}(r) > 0$. Thus, by Proposition 4, the support function blows up exponentially. More particularly, the spectral decomposition is

$$r = r_{00}e^{2t} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} r_{lm}e^{[2-l(l+1)]t}Y_l^m,$$

where $r_{00} > 0$. Clearly, r_{00} is the radius of the limit of the rescaled flow for $\tilde{r} = re^{-2t}$ [**7**] – it is the projection of the support function into the 0-eigenspace of the spherical Laplacian:

$$r_{00} = \frac{1}{2}\sqrt{\frac{1}{\pi}}\int_0^{2\pi}\int_0^{\pi} r_0 \sin\theta\, d\theta\, d\phi.$$

Here, we have introduced standard spherical polar coordinates $\xi = \tan(\theta/2)e^{i\phi}$.

Now consider the flow in $T\mathbb{S}^2$. By the evolution equation for ρ in Proposition 3 we require $A_{\mathbb{S}^2}(\rho)$. In this case, however, we have

$$A_{\mathbb{S}^2}(\rho) = \frac{1}{2}\int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} r\, dA = 0,$$

and by the proof of Proposition 4,

$$\rho = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \rho_{lm}e^{(2-l(l+1))t}Y_l^m,$$

where the constants ρ_{lm} are determined by the initial surface. Let

$$\alpha = -\sqrt{\frac{3}{2\pi}}\rho_{11}, \quad b = \frac{1}{2}\sqrt{\frac{1}{\pi}}\rho_{10}.$$

We claim that the oriented normal lines to the flowing surface converge to the set of oriented lines passing through $(x^1 + ix^2, x^3) = (\alpha, b)$. Using the power series expansion for r corresponding to the one for f_t in the proof of Proposition 4, we have

$$F = \tfrac{1}{2}(1 + \xi\bar{\xi})^2\bar{\partial}r = -\tfrac{1}{2}(\alpha - 2b\xi - \bar{\alpha}\xi^2) + R(t, \xi, \bar{\xi}),$$

where $R(t, \xi, \bar{\xi}) \to 0$ uniformly in any $C^k(\mathbb{S}^2)$ as $t \to \infty$. This completes the proof of the Main Theorem since the first term represents a global Lagrangian holomorphic section and we have

$$
\begin{aligned}
x^1 + ix^2 = \alpha = -\sqrt{\frac{3}{2\pi}}\rho_{11} &= -\sqrt{\frac{3}{2\pi}}\int_{\mathbb{S}^2} Y_1^1 \rho\, dA \\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\int_{\mathbb{S}^2} Y_1^1 \triangle_{\mathbb{S}^2} r\, dA \\
&= \sqrt{\frac{3}{2\pi}}\int_{\mathbb{S}^2} Y_1^1 r\, dA \\
&= \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi} r \sin^2\theta e^{i\phi}\, d\theta\, d\phi,
\end{aligned}
$$

and similarly for x^3.

The point (α, b) is the Steiner point of the initial surface S_0 [**2, 8**], which remains fixed under the flow. □

In fact, one could drop the Lagrangian condition on the initial global section.

PROPOSITION 5. *The flow converges to a Lagrangian holomorphic section even if the initial section is not Lagrangian.*

Proof. Recall that the section is Lagrangian if and only if $\lambda = 0$. The flow equation for λ given in Proposition 3 can be written as

$$\left(\frac{\partial}{\partial t} - \triangle_{\mathbb{S}^2}\right)\lambda e^{2t} = 0,$$

which by the maximum principle implies that there exists a constant C depending only on λ_0 such that

$$|\lambda| \leq Ce^{-2t},$$

so that $\lambda \to 0$ as $t \to \infty$.

Now the proof follows that of the Lagrangian case.

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