Q-branes

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Abstract

Non-topological solitons (Q-balls) are discussed in some stringy settings. Our main result is that the dielectric D-brane system of Myers admits non-abelian Q-ball solutions on their world-volume, in which N Dp-branes relax to the standard dielectric form outside the Q-ball, but assume a more diffuse configuration at its centre. We also consider how Q-balls behave in the bulk of extra-dimensional theories, or on wrapped branes. We demonstrate that they carry Kaluza-Klein charge and possess a corresponding Kaluza-Klein tower of states just as normal particles, and we discuss surface energy effects by finding exact Q-ball solutions in models with a specific logarithmic potential.

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1 Introduction and Background

One of the most interesting features of field theories with conserved charges is the possibility of non-topological solitons, in particular Q-balls [1, 2, 3, 4, 5, 6, 7]. Q-balls are localized field configurations that are stable simply because they are a more energy efficient way of holding charge than a collection of asymptotically free quanta. Such objects have been shown to occur very generally in field theory. Indeed the charge Q can be global (the simplest case) or local [3, 4] and the corresponding symmetry (which is spontaneously broken in their interior) can be either abelian or non-abelian.

For certain types of potential, the energy deficit or binding energy grows with charge, so that Q-balls can in principle be large macroscopic objects, and naturally there has been much interest in their cosmological implications and their impact on scenarios beyond the Standard Model [8, 9, 10]. For example, Q balls have been proposed as dark matter candidates [11, 12] in particular in gauge mediated SUSY breaking models [13, 14, 15, 16]. Experimental searches for Q-balls have also been proposed and carried out. For these, the possible electric charge a Q-ball may carry obviously plays a central role in determining their experimental signature. For instance, neutral Q-balls can be detected by Super-Kamiokande [13, 17, 18] by probing proton absorption. Conversely charged Q-balls could be seen directly in detectors such as MACRO [13, 19].

Given this interest, it is important to determine the ubiquity of Q-balls in scenarios of physics at the most fundamental scales. In this paper we study Q-lumps in various stringy settings, including configurations with extra dimensions, namely charged bulks, and wrapped branes. Our main result, in section 5, is the explicit construction of stable Q-ball solutions on systems of Dp-branes, in which the scalar fields in the solutions describe their displacements. It is well known that the Dp-branes can be spread over a 2-sphere by turning on a background field, forming so-called dielectric branes [20]. The global minimum of a dielectric brane has a non-commutative form, with the vacuum falling into an $N \times N$ irreducible representation of $SU(2)$. However, in this minimum there are additional non-abelian symmetries that can be broken by reducible representations of $SU(2)$. We show that the resulting charges support Q-balls, with the N Dp-branes relaxing to the standard dielectric form outside the Q-ball, but assuming a more complicated dielectric configuration at its centre, in which the 2-sphere itself is diffuse. Remarkably, even in the simplest case the dielectric brane potential has the correct coefficients for the Q-ball configuration to be energetically stable.

As well as presenting this construction we will, as a warm-up exercise, look at a number of additional issues that make Q-balls in extra dimensional setups a somewhat more complex problem than in 4 dimensional field theory. The first is that generally they will be wrapped on compact dimensions of various size. The extent of the Q-ball can therefore be limited, forcing the configurations to be anisotropic. The second issue, is that the Q-balls carry a Kaluza-Klein momentum in the extra dimensions which is quantized. Thus one expects to find a tower of Q-balls, corresponding to Kaluza-Klein excitations. In the limit of large compactification, one naturally expects the momentum to become continuous corresponding to the Q-balls moving freely in the extra-dimensions.

We shall look at these issues by way of introduction to Q-balls in the following two sections, using a $U(1)$ model in 5 space time dimensions with one compact space dimension (*i.e.* corresponding to a Q-ball on a wrapped 4-brane). We first discuss, in section 2, the large volume limit of Q-balls for complex fields carrying both global charge, Q, and Kaluza-Klein momentum of a single compact extra dimension, P_5. The solutions are found to have rather natural $O(3)$ and $O(4)$ symmetric limits depending on the size of the extra dimension. We find that the momentum modes correspond to an infinite set of Kaluza-Klein excitations of the lowest lying Q-ball; the spectrum has a tower of P_5 momenta,

$$P_5 = Q\left(p + \frac{n}{R}\right) \qquad n \in \mathbb{Z}\,, \tag{1}$$

where Q is the global charge, p is the lowest mode and R is the compactification radius. The states with non-zero n can be thought of as Kaluza-Klein excitation of the lowest mode. If $p = 0$ the lowest mode is precisely the usual $D = 4$ Q-ball, albeit possibly constrained by compact extra dimensions, while $p \neq 0$ corresponds to giving this state additional mass by the Scherk-Schwarz mechanism [21].

In section 3 we discuss Q-balls in a special logarithmic potential that allows us to reduce the task of finding a Q-ball solution to the canonical one-dimensional problem, showing in detail a Q-ball going from $O(4)$ to $O(3)$ symmetric configurations, and demonstrating the energetic preference of the surface tension term for large radii and more symmetric configurations. Finally in section 4 we discuss the non-abelian Q-ball solutions that are shown generally to exist on dielectric branes.

2 Warm up; large Q-balls in small boxes

In order to see how Q-balls behave with finite dimensions, consider the large charge limit of a Q-ball in 5 space-time dimensions. In this limit one neglects the surface effects. (In the following section we discuss these using a particular logarithmic potential.)

The specific set-up is as follows. We shall take a single scalar field in $M_4 \times S^1$. The Minkowski dimensions we call x, and the dimension that is compactified on S^1 we call y, with y and $y + 2\pi R$ identified. Almost certainly the discussion will hold also for the untwisted sector of orbifolded extra dimensions, and as will become clear the qualitative behaviour would most likely be the same in non-flat compactifications.

The action can be written as

$$S = \int \mathrm{d}^3x\mathrm{d}y\mathrm{d}t \left(\partial_\alpha\phi^*\partial^\alpha\phi - U_5(\phi\phi^*)\right). \tag{2}$$

Reparameterization invariance leads to the following conserved charges;

$$\begin{aligned} E &= \int \mathrm{d}^3x\mathrm{d}y \left(\partial_t\phi^*\partial_t\phi + \partial_i\phi^*\partial_i\phi + \partial_y\phi^*\partial_y\phi + U_5(\phi\phi^*)\right) \\ P_i &= \int \mathrm{d}^3x\mathrm{d}y \left(\partial_t\phi^*\partial_i\phi + \partial_i\phi^*\partial_t\phi\right) \\ P_5 &= \int \mathrm{d}^3x\mathrm{d}y \left(\partial_t\phi^*\partial_y\phi + \partial_y\phi^*\partial_t\phi\right), \end{aligned} \tag{3}$$

where $i = 1, 2, 3$. In addition, assume invariance under a global $U(1)$ transformation, $\phi \to e^{i\alpha}\phi$, so that there is a conserved charge,

$$Q = \frac{1}{i}\int \mathrm{d}^3x\mathrm{d}y\ \phi^* \overset{\leftrightarrow}{\partial_t} \phi. \tag{4}$$

By assumption the origin is a global minimum of U_5 and the global $U(1)$ symmetry is unbroken there. As mentioned in the Introduction, in more general cases the transformation could be that of any compact group, and the Q-ball could be constructed from a local as well as a global symmetry. These extensions will be discussed in more detail later when we come to consider dielectric brane configurations.

Since we seek a solution that is localized in the x coordinates the global minimum in energy must have $U(1)$ symmetry restored at large radius for any y. Hence it is convenient to separate out the time dependent $U(1)$ phase;

$$\phi(x,y,t) = \varphi(x,y,t)e^{i\theta(y,t)}, \tag{5}$$

where φ and θ are real. The equations of motion now give us two relations,

$$\begin{aligned} \varphi\partial^2\theta + 2\partial\theta\partial\varphi &= 0 \\ \partial^2\varphi - (\partial\theta)^2\varphi &= -\frac{1}{2}\frac{\partial U_5}{\partial\varphi}\,. \end{aligned} \tag{6}$$

By analogy with standard Q-balls we now choose θ to be linear, parameterizing it with α and ω;

$$\theta = \alpha(t + \omega y). \tag{7}$$

The equations of motion then require

$$\varphi = \varphi(y + \omega t) \tag{8}$$

and

$$(1-\omega^2)\partial_5^2\varphi + \partial_i^2\varphi = \frac{1}{2}\frac{\partial\hat{U}}{\partial\varphi} \tag{9}$$

where

$$\hat{U} = U_5 - (1-\omega^2)\alpha^2\varphi^2. \tag{10}$$

The Q-ball solution then corresponds to the usual problem of a real field rolling in the inverted potential $-\hat{U}$ where y and x_i replace time.

For completeness let us find the same result using the, perhaps more familiar, method which deduces the solution by minimising the energy of a generic field configuration whilst fixing the charges using Lagrange multipliers. That is we minimise the expression [1]

$$\begin{aligned} \varepsilon_{\omega,\omega'} &= E + \omega\left\{P_5 - \int \mathrm{d}^3x\mathrm{d}y\,\left(\partial_t\phi^*\partial_y\phi + \partial_y\phi^*\partial_t\phi\right)\right\} \\ &\quad +\omega'\left(Q - \frac{1}{i}\int \mathrm{d}^3x\mathrm{d}y\;\phi^*\overset{\leftrightarrow}{\partial_t}\phi\right) \end{aligned} \tag{11}$$

for a given ω and ω', and then minimise in ω, ω'. First completing the square in the kinetic terms gives

$$\begin{aligned} \varepsilon_{\omega,\omega'} &= \int \mathrm{d}^3x\mathrm{d}y\,\left(\left|\omega\partial_y\phi - \partial_t\phi + i\omega'\phi\right|^2 + (1-\omega^2)\left|\partial_y\phi - \frac{i\omega\omega'}{1-\omega^2}\phi\right|^2\right) \\ &\quad + \int \mathrm{d}^3x\mathrm{d}y\,\left(\partial_i\phi^*\partial_i\phi + \hat{U}_{\omega\omega'}(\phi\phi^*)\right) \qquad + \omega P_5 + \omega' Q\,, \end{aligned} \tag{12}$$

where

$$\hat{U}_{\omega\omega'}(\phi\phi^*) = U_5(\phi\phi^*) - \frac{\omega'^2}{(1-\omega^2)}\phi\phi^*. \tag{13}$$

Now θ only appears in the first integral as

$$\int \mathrm{d}^3x\mathrm{d}y \left(\left| i\varphi\left(\omega\partial_y\theta - \partial_t\theta + \omega'\right)\right) + \omega\partial_y\varphi - \partial_t\varphi \right|^2 + (1-\omega^2)\left| |\partial_y\varphi - i\varphi\left(\partial_y\theta - \frac{\omega\omega'}{1-\omega^2}\right) \right|^2 \right). \tag{14}$$

The energy is minimised where the imaginary contributions vanish, which independently determines θ,

$$\theta(y,t) = \frac{\omega'}{1-\omega^2}(\omega y + t), \tag{15}$$

and in addition,

$$\varphi = \varphi(y + \omega t). \tag{16}$$

As one might have expected, the solutions with non-zero P_5 are going to be "lumps" travelling in the y direction with speed ω. The energy is now

$$\begin{aligned} \varepsilon_{\omega,\omega'} &= \int \mathrm{d}^3x\mathrm{d}y \left((\partial_y\varphi)^2 + (\partial_i\varphi)^2 + \hat{U}_{\omega\omega'}(\varphi^2) \right) \\ &\qquad + \omega P_5 + \omega' Q, \end{aligned} \tag{17}$$

and clearly extremizing this gives the same equation of motion as before if we identify $\alpha = \omega'/(1-\omega^2)$. The physical interpretation is that the boost factor (squared) $1/(1-\omega^2)$ is a result of both Lorentz contraction and time dilation in the phase factor, that will ultimately feed into the charge Q.

We can now proceed to the large and small (in a sense to be defined shortly) limits of compactification radius, R. In the large R limit the φ configuration that minimises ε is approximately the same as that in the decompactified space. The variation of φ proceeds as for tunneling in $d = 4$ Euclidean space dimensions in the potential $\hat{U}_{\omega\omega'}$. In this limit the symmetry of the problem dictates that, for the stationary ($\omega = 0$) Q-ball, we have a fully $O(4)$ symmetric solution and so the minimum is the action $S_4[\varphi]$ of the bounce solution. Since φ is a function of $y+\omega t$, the factor $(1-\omega^2)$ includes a Lorentz contraction which squashes the solution in the y direction. Clearly for the $O(4)$ limit to apply, R should be much greater than the radius of this solution (which we shall call r_4).

In the opposite limit, where $R < r_4$, the $(\partial_y\varphi)^2$ term makes any significant variation of φ in the interval $y \in [0, 2\pi R]$ very costly in energy. In this limit we can therefore take $\varphi(x,y) = \varphi(x)$ and write

$$\begin{aligned} \varepsilon_{\omega,\omega'} &= 2\pi R \int \mathrm{d}^3x \left((\partial_i\varphi)^2 + \hat{U}_{\omega\omega'}(\varphi^2) \right) \\ &\qquad + \omega P_5 + \omega' Q. \end{aligned} \tag{18}$$

Now the variation of φ proceeds as for tunneling in $d = 3$ Euclidean dimensions in the potential $\hat{U}_{\omega\omega'}$, and the minimum energy is the action $S_3[\varphi]$ of the relevant bounce solution.

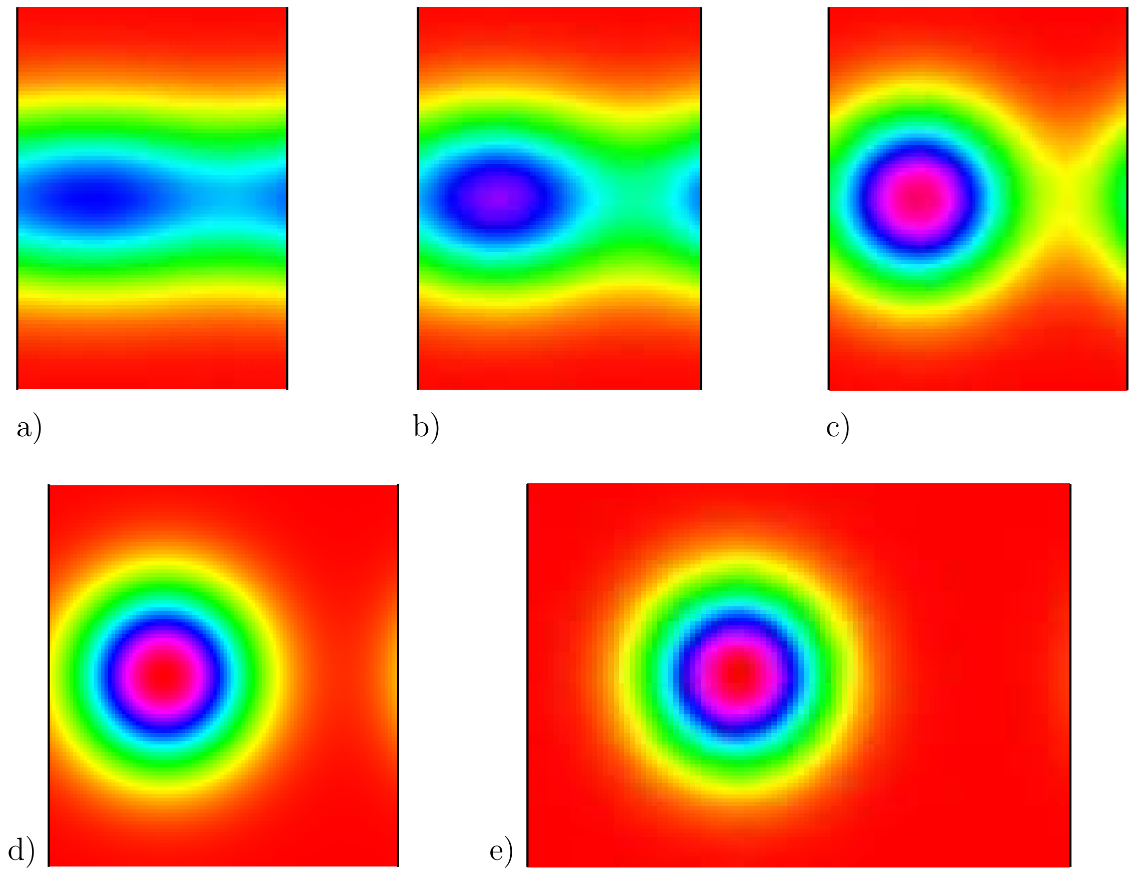


Figure 1: *A Q-ball condensing as the bulk radius increases. Here the centre of the Q-ball is offset by 1/2 a bulk radius. Figure 1e is close to the solution for an infinite radius. Figure 1a is slightly larger than the radius corresponding to the "natural" frequency of oscillation in the upturned potential. For a transverse radius smaller than a certain critical value, the only solution is trivial in the compact direction (i.e. constant in y). The slight squashing in the compactified direction is Lorentz contraction due to the non-zero P_5.*

An example of the two limits, which we will be discussing in detail in the following sections, is shown in Figure 1.

Consider now the large volume solution, where the field is approximately constant, φ_0, inside a 4-volume V_4 (whose form depends on whether we are considering the large or small R limit). In this case we find

$$\begin{aligned} Q &= \frac{2\omega'}{1-\omega^2} V_4 \varphi_0^2 \\ P_5 &= \frac{2\omega\omega'^2}{(1-\omega^2)^2} V_4 \varphi_0^2 \end{aligned} \tag{19}$$

and

$$E = V_4 \tilde{U} + \frac{1}{4}\frac{Q^2}{V_4 \varphi_0^2}\,, \tag{20}$$

where

$$\tilde{U} = U_5(\varphi_0^2) + \frac{P_5^2}{Q^2}\varphi_0^2. \tag{21}$$

Note that $P_5/Q = \omega'\omega/(1-\omega^2)$ so that eq.(21) should be compared with

$$\phi(x,y,t) = \varphi(x,y)\exp\left(i\frac{P_5}{Q}(y+\frac{t}{\omega})\right). \tag{22}$$

Hence $\tilde{U}$ is the potential after applying the Scherk-Schwarz mechanism to the field ϕ [21]. One can now minimise the energy with respect to V_4 to find

$$E = Q\sqrt{\frac{\tilde{U}}{\varphi_0^2}}, \tag{23}$$

where φ_0 is the field value that minimises E. Since the volume is proportional to Q, the point where R becomes relatively small is determined by Q:

$$Q \approx 4\pi R^4\sqrt{\tilde{U}\varphi_0^2}. \tag{24}$$

Note that the potential $\hat{U}_{\omega\omega'}$ is the same for the large and small R solutions (from now on we will drop the $\omega\omega'$ suffix), and actually the energy is independent of R.

We can see this in an interesting scaling limit of the thin wall approximation, given by

$$\omega' \ll 1 \qquad ; \qquad \omega^2 = 1 - \mathcal{O}(\omega'). \tag{25}$$

Without the intervention of P_5 small ω' would always imply small Q-balls, but because of the simultaneous second limit we have

$$Q \sim P_5 \sim V_4\varphi_0^2, \tag{26}$$

and at the same time $\hat{U} = U - \mathcal{O}(\omega')$, thereby maintaining the thin-wall requirement of $\hat{U}(\varphi_0) \approx 0$. We conclude that in this limit the potential at φ_0 only needs to be shifted down by a parametrically small amount in order to develop a Q-ball solution, which nevertheless has large Q. In more physical terms, the squared boost factor $1/(1-\omega^2)$ is able to keep Q and P_5 large even though ω' is small. In this limit the energy is given by

$$E = \frac{P_5^2}{Q^2}V\varphi_0^2 + \frac{1}{4}\frac{Q^2}{V\varphi_0^2} + \mathcal{O}(\omega'), \tag{27}$$

which is minimised when

$$V_{\min} = \frac{Q^2}{2P_5\varphi_0^2} + \mathcal{O}(\omega'). \tag{28}$$

We conclude that there is an energetically optimal volume for the Q-ball to occupy given by the parameters on the right of this equation, but the Q-ball can achieve this minimal volume for *any* radius of compact dimension R because there is no surface tension term in the energy. Note that substituting back in we find that the energy of this configuration is $E = P_5 + \mathcal{O}(\omega')$; i.e. the 4-dimensional rest-mass is made up almost entirely of P_5 in this limit, while the 5-dimensional rest-mass is negligible in this limit.

Returning to the generic case, we still need to show stability of the Q-ball with respect to decay into a collection of Kaluza-Klein modes. Decay is allowed into Q Kaluza-Klein modes $i = 1..Q$, with total momentum $\sum_i P_{5i} = P_5$, and total rest mass

$$E_{KK-modes} = \sum_i^Q \sqrt{\frac{1}{2}\mu^2 + P_{5i}^2}, \tag{29}$$

where $\mu^2 = \partial^2 U/\partial\varphi^2$. A simple geometric argument shows that this expression is minimised when the P_5 momentum is equally distributed amongst the Kaluza-Klein modes, $P_{5i} = P_5/Q$. Stability therefore requires

$$E_{KK-modes} = Q\sqrt{\frac{1}{2}\mu^2 + \frac{P_5^2}{Q^2}} > E = Q\sqrt{\frac{\tilde{U}}{\varphi_0^2}} \tag{30}$$

or

$$\mu^2 > \frac{2U}{\varphi_0^2}. \tag{31}$$

This is *precisely* the condition for a Q-ball to exist in the 4 dimensional theory. Hence the Q-balls with additional integer global Q-charge can be simply understood as the Kaluza-Klein ladder of the lowest lying Q-ball[1]. This can be trivially seen from Eqs.(21,23) which gives

$$E^2(P_5) = E^2(0) + P_5^2 \tag{32}$$

so that in the thin wall limit the Kaluza-Klein momentum P_5 can be boosted away to leave the rest-mass of the Q-ball in a non-compact volume: the large Q-ball is blind to the compactness of the extra dimension. The momentum P_5 is naturally decomposed into Kaluza-Klein modes as

$$P_5 = Q\left(p + \frac{n}{R}\right) \tag{33}$$

where n is an integer parametrising the Kaluza-Klein tower, whilst the non-integer p represents the Scherk-Schwarz phase with

$$\phi(x, y + 2\pi R) = e^{ip}\phi(x, y). \tag{34}$$

The interpretation of the phase p is that it is the non-integer momentum per unit charge.

Similar solutions can be found for a global unitary symmetry as we shall later see for Q-balls on dielectric branes. In the more general cases we have to replace the phase by a time dependent unitary rotation, but the rest of the analysis will go through unchanged.

2.1 Generalization to d compact dimensions

The treatment above can be straightforwardly extended to multi-dimensional compact flat spaces. Consider a toroidal compactification on an untilted torus with d compact radii R_a where $a = 1\cdots d$. Then eq.(11) becomes

$$\begin{aligned}\varepsilon_{\boldsymbol{\omega},\omega'} &= E + \boldsymbol{\omega}\cdot\left\{\boldsymbol{P} - \int \mathrm{d}^3x\mathrm{d}^d y\,\left(\partial_t\phi^*\nabla_d\phi + \nabla_d\phi^*\partial_t\phi\right)\right\} \\ &\quad + \omega'\left(Q - \frac{1}{i}\int \mathrm{d}^3x\mathrm{d}^d y\,\phi^* \overleftrightarrow{\partial_t}\,\phi\right),\end{aligned} \tag{35}$$

[1]We should remark that our findings contradict those of ref.[22] which concluded that different stability conditions and types of Q-balls can result. That analysis began with a decomposition of the action into Fourier modes, arriving at an infinite and intractable set of coupled differential equations for the Kaluza-Klein modes. However, the interactions among the different modes must be consistent with the fact that they come from higher dimensional interactions. Once this constraint is taken into account the stability condition must be as above. There is in effect one and only one kind of Q-ball however one chooses to squash it into extra dimensions, and at least in the large charge limit there are for example no special bounds on the mass per unit charge associated with the finite compactification radius.

where $\boldsymbol{\omega} = \{\omega_a\}$ and $\boldsymbol{P}$ are now d-vectors. The square in the kinetic terms is completed as

$$\begin{aligned}
\varepsilon_{\omega,\omega'} &= \int \mathrm{d}^3x\mathrm{d}^dy \left(|\boldsymbol{\omega}\cdot\nabla_d\phi - \partial_t\phi + i\omega'\phi|^2 \right. \\
&\quad +(\delta_{ab}-\omega_a\omega_b)\left(\partial_a\phi - \frac{i\omega_a\omega'}{1-\boldsymbol{\omega}.\boldsymbol{\omega}}\phi\right)\left(\partial_b\phi^* + \frac{i\omega_b\omega'}{1-\boldsymbol{\omega}.\boldsymbol{\omega}}\phi^*\right) \\
&\quad \left. +\partial_i\phi^*\partial_i\phi + \hat{U}(\phi\phi^*)\right) + \boldsymbol{\omega}\cdot\boldsymbol{P} + \omega' Q\,,
\end{aligned} \tag{36}$$

where

$$\hat{U}(\phi\phi^*) = U_{4+d}(\phi\phi^*) - \frac{\omega'^2}{(1-\boldsymbol{\omega}\cdot\boldsymbol{\omega})}\phi\phi^*. \tag{37}$$

As long as $\boldsymbol{\omega}\cdot\boldsymbol{\omega} < 1$ so that the "dual metric" is positive definite, the previous arguments go through unchanged, and the energy is minimised where

$$\theta(y,t) = \frac{\omega'}{1-\boldsymbol{\omega}\cdot\boldsymbol{\omega}}(\boldsymbol{\omega}\cdot\boldsymbol{y}+t), \tag{38}$$

and

$$\varphi = \varphi(\boldsymbol{\omega}\cdot(\boldsymbol{y}+\boldsymbol{\omega}t)). \tag{39}$$

Inserting these into the constraints with the large volume solution minimised at φ_0, explicitly gives

$$\begin{aligned}
Q &= \frac{2\omega'}{1-\boldsymbol{\omega}\cdot\boldsymbol{\omega}} V_{3+d}\varphi_0^2 \\
\boldsymbol{P} &= \frac{2\boldsymbol{\omega}\omega'^2}{(1-\boldsymbol{\omega}\cdot\boldsymbol{\omega})^2} V_{3+d}\varphi_0^2\,,
\end{aligned} \tag{40}$$

with

$$E = V_{3+d}\tilde{U} + \frac{1}{4}\frac{Q^2}{V_{3+d}\varphi_0^2} \tag{41}$$

where

$$\tilde{U} = U(\varphi_0^2) + \frac{\boldsymbol{P}\cdot\boldsymbol{P}}{Q^2}\varphi_0^2. \tag{42}$$

3 Small Q-balls in even smaller boxes

3.1 An exact solution

In the previous section we saw that Q-balls in the large charge limit are energetically independent of the size of the compactification, and in the thin wall approximation it is only the total volume they occupy in the bulk that matters.

In this section, we wish to get some idea of surface effects. We therefore turn to a logarithmic potential for which exact Q-ball solutions can be found in certain limits; continuing with the definition $\phi = \varphi e^{i\theta}$, the particular $U(1)$ invariant potential of interest is

$$U_5 = \mu^2\varphi^2\left(1 - \log\frac{\varphi^2}{\varphi_0^2}\right) + \mathcal{O}(\varphi^n). \tag{43}$$

This potential is particularly interesting for studying surface effects because it admits exact Q-ball solutions whose 'surfaces' constitute the whole Q-ball, whatever the charge. It has found use in a limited number of related works in the past, most recently [23].

The last term in the potential, $\mathcal{O}(\varphi^n)$ (where n is some large power), is added to lift the potential at large field values $\varphi \gtrsim \sqrt{e}\varphi_0$ thereby ensuring that it satisfies the requirement that the origin be the global minimum. However, the modified potential for finding the Q-ball can be written,

$$\hat{U} = \mu^2\varphi^2\left(1 - \log\frac{\varphi^2}{\varphi_0'^2}\right) + \mathcal{O}(\varphi^n), \tag{44}$$

where

$$\varphi_0' = e^{-\frac{\omega'^2}{\mu^2(1-\omega^2)}}\varphi_0. \tag{45}$$

Even for modest value of α the potential goes negative at field values that are exponentially smaller than the values at which φ^n dominates and consequently, for the purposes of finding the Q-ball solution, the latter is negligible.

Neglecting this term allows one to solve the equations of motion in eq.(9) by separation of variables. Of course the analysis regarding the phases of ϕ goes through as before, but now the solution for its modulus φ can be written,

$$\varphi = \varphi_0' Y(y)\Pi_i X_i(x_i), \tag{46}$$

giving

$$X_i = e^{\frac{1}{2}(1-\mu^2 x_i^2)} \tag{47}$$

and

$$\frac{\ddot{Y}}{Y} = -\frac{\mu^2}{1-\omega^2}\log Y^2. \tag{48}$$

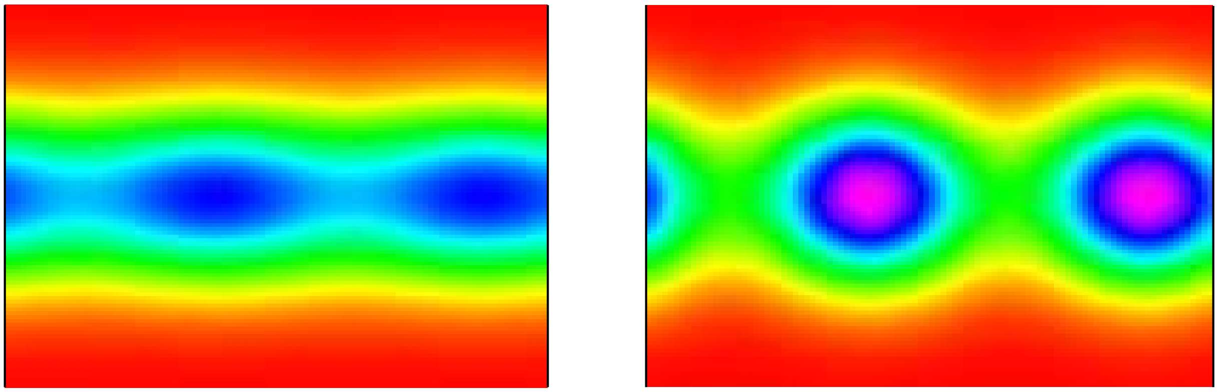
The problem is reduced to the one dimensional task represented by this last equation. In the large radius limit it naturally just gives the expected Lorentz boosted version of the solutions in the x_i directions,

$$Y \to \exp\left(\frac{1}{2} - \frac{\mu^2}{2}\frac{1}{1-\omega^2}(y+\omega t)^2\right). \tag{49}$$

In more general cases it can easily be solved numerically imposing the boundary conditions of periodicity in $y \to y + 2\pi R$. Note that the typical width of the Q-ball in the y-direction, $\sqrt{1-\omega^2}/\mu$ has the expected Lorentz contraction.

Some examples are shown in Figures 1a-e where the compactification radius is increased from $R\mu \approx \sqrt{(1-\omega^2)/2}$ to $R\mu = 2\sqrt{(1-\omega^2)/2}$. In figure 1a the value of the radius corresponds to the 'natural' period; that is $Y(y)$ is oscillating close to $Y = 1$ in the upturned potential $-\hat{U}$. The oscillation period is monotonically increasing with amplitude so that (uniquely for this potential) there is a hard cut-off below which the solution is completely three dimensional: *when* $R\mu < \sqrt{(1-\omega^2)/2}$ *there can be no solutions except the* $O(3)$ *symmetric trivial one*, $Y(y) = 1$.

As the radius increases so does the amplitude of oscillation in order to maintain the correct periodicity. Extending the oscillation period (*i.e.* compactification radius) significantly, forces Y to approach the origin of φ. In other words, the solution quickly collapses to the $O(4)$ symmetric one.



a) b)

Figure 2: *The second in a family of solutions first appearing at a radius that is twice the critical radius of the solutions in Figure 1. Increasing the radius of this solution results in the condensation of two isolated balls. Reducing it forces coalescence into one of the single Q-ball solutions in figure 1.*

At the radius $R\mu \gtrsim 2\sqrt{(1-\omega^2)/2}$, there are two available solutions. One is the isolated $O(4)$ symmetric configuration of figure 1e, and the other is the doubled solution in figures 2a-b. The latter corresponds to $Y(y)$ oscillating twice in the period $2\pi R$. Further expansion of the radius causes the doubled solution to condense into two isolated Q-balls in the bulk. However, the doubled solution of figure 2a is energetically unstable to decay into the single isolated Q-ball with the same charge and P_5. Similarly two completely isolated Q-balls of this type will coalesce into one large one[2]. Figures 2a,b show two cases of interest, the first being the solution with $Y \approx 1$ and the second the isolated solution in eq.(49).

We now present the charges, momentum and energy for these different configurations. To do so we define

$$\begin{aligned} \gamma &= (e\sqrt{\pi})^3 \frac{\varphi_0^2}{\mu^3} \\ \xi &= \frac{\omega'}{\mu\sqrt{1-\omega^2}}. \end{aligned} \tag{50}$$

The general expressions are (redefining $y + \omega t \to y$)

$$\begin{aligned} Q &= 2\frac{\omega'}{1-\omega^2}\int \mathrm{d}^3x\mathrm{d}y\varphi^2 \\ &= 2\frac{\omega'}{1-\omega^2}\gamma\int \mathrm{d}yY^2\,, \\ P_5 &= \frac{\omega\omega'}{1-\omega^2}Q + 2\omega\int \mathrm{d}^3x\mathrm{d}y(\partial_y\varphi)^2 \\ &= \frac{\omega\omega'}{1-\omega^2}Q + 2\omega\frac{\mu^2}{1-\omega^2}\gamma\int \mathrm{d}yY^2\log Y^2\,, \end{aligned} \tag{51}$$

[2]Note that these statements are all with the caveat that there are no relative phases between the Q-balls. In more general cases they can attract or repel and can exchange charge continuously [24]. One would expect this to be so with compact dimensions as well, although it would be of interest to extend the study of ref. [24] to this case.

where in the last line we used eq.(48) and integrated by parts for this example. This then gives for the squeezed and isolated limits respectively

$$\begin{aligned} Q_{sq} &= \frac{2R\mu\gamma}{\sqrt{1-\omega^2}}\xi e^{-\xi^2} \\ Q_{is} &= 2e\sqrt{\pi}\gamma\xi e^{-\xi^2} . \end{aligned} \tag{52}$$

This determines ξ while ω can be determined by the equations for P_5; defining

$$\omega = \frac{1}{\sqrt{1+\rho^2}} , \tag{53}$$

the latter lead to

$$\begin{aligned} \rho &= \frac{Q\mu\xi}{P_5} \\ \rho &= \frac{Q\mu\xi}{P_5}(1+1/2\xi^2) , \end{aligned} \tag{54}$$

and we can then parameterize the energy as

$$\begin{aligned} E_{sq} &= Q_{sq}\mu \left(\frac{\alpha}{\mu} + \frac{1}{2}\frac{\mu}{\alpha} \right) \\ E_{is} &= Q_{is}\mu \left(\frac{\alpha}{\mu} + \frac{1}{4}\frac{\mu}{\alpha} \right) , \end{aligned} \tag{55}$$

in the squeezed and isolated cases respectively, where as before $\alpha = \omega'/(1-\omega^2)$.

Notice that the minimum value for the energy of the isolated Q-ball, i.e. $Q\mu$, is less than that for the squeezed Q-ball $\sqrt{2}Q\mu$, indicating that the effect of surface tension is for Q-balls energetically to favour large radius where they can assume a more symmetric configuration.

To complete this discussion, we should remark that the Q-balls considered in this example are not unstable to decay into free states despite the fact that $E > Q\mu$. This is because the parameter μ is not the physical mass of any asymptotic quantum at the origin. (Formally the mass at the origin is logarithmically infinite so there are no asympotic states there at all.) Indeed consider the physical system in which Q-balls with such a potential could appear, namely the F and D-flat directions corresponding to conserved $B-L$ current in supersymmetry, as was considered in for example ref.[8]. The one-loop improved tree-level potential of this system would typically be of the form

$$U = \mu^2\varphi^2 \left(1 - \log\frac{\varphi^2+m^2}{\varphi_0^2} \right) + \mathcal{O}(\varphi^n), \tag{56}$$

where now φ is the scalar denoting the VEV along the flat direction. The scale $\sqrt{\varphi^2+m^2}$ is the approximate renormalisation scale due to the field φ giving a mass to for example squarks and sleptons along the flat direction, and the scale m would therefore naturally correspond to the scale of supersymmetry breaking which would in a typical supersymmetric phenomenology be of order μ itself. Provided $\varphi_0 \gg m$ the Q-ball analysis goes through unchanged upto corrections of order $\mathcal{O}(m^2/\varphi_0^2)$, while the mass-squared of the asymptotic states at the origin, $\mu^2\,(1+\log\varphi_0^2/m^2)$, is now regulated by the infra-red cut-off m, and is parametrically larger than μ^2.

3.2 The thick wall / small charge approximation

We can also consider more general "small" Q-balls which would be more appropriate for the Q-balls on dielectric branes we discuss later. Following ref.[5] our task is to minimise

$$\varepsilon_{\omega,\omega'} \;=\; \int \mathrm{d}^3x\mathrm{d}y\,\left((\partial_y\varphi)^2+(\partial_i\varphi)^2+\hat{U}_{\omega\omega'}(\varphi^2)\right) \tag{57}$$

for fixed ω and ω', where in the thick wall limit we keep only the first two terms in an expansion of the potential,

$$\hat{U}_{\omega\omega'}=\frac{\mu'^2}{2}\varphi^2-A\varphi^3+\dots\,, \tag{58}$$

and the effective mass-squared is

$$\mu'^2=\mu^2-\frac{\omega'^2}{1-\omega^2}\,. \tag{59}$$

Note that in a 5D theory, μ has mass-dimension 1, but A has mass-dimension 1/2.

The bounce action can be related by a simple rescaling to the bounce action for the rescaled potential $V_\psi=\frac{1}{2}\psi^2-\psi^3$ which can be computed numerically in certain cases [5]. The rescaling is of the form $\psi=\varphi A/\mu'^2$ and $\hat{x}=\mu' x$, $\hat{y}=\mu'\sqrt{1-\omega^2}y$, so that the typical isolated solution would have $O(4)$ symmetry and width ~ 1 in the rescaled units, and again we infer squeezed solutions for $R\mu'\sqrt{1-\omega^2}<1$. It is not possible to obtain the solution in full generality, however we can again restrict ourselves to either squeezed ($O(3)$ symmetric) or isolated ($O(4)$ symmetric) solutions as in ref.[25]. Considering the former for definiteness gives $S_\psi=4.85$ and an energy of

$$\varepsilon_{\omega,\omega'} \;=\; S_\psi\frac{2\pi R\mu'^3}{A^2}+\omega' Q+\omega P_5\,. \tag{60}$$

Minimising in ω and ω' gives

$$\begin{aligned} Q &= \frac{3S_\psi}{A^2}\frac{\omega'_{min}}{1-\omega^2_{min}}2\pi R\mu'_{min} \\ P_5 &= \frac{\omega_{min}\,\omega'_{min}}{1-\omega^2_{min}}Q\,, \end{aligned} \tag{61}$$

with of course the second relation following from eq.(51). The energy can then be written

$$E=Q\mu\left(\frac{\alpha}{\mu}+\frac{1}{3}\frac{\mu}{\alpha}-\frac{1}{3}\frac{\omega'}{\mu}\right)\,. \tag{62}$$

The usual thick-wall solution of [5] has $w\equiv 0$ and hence

$$E=Q\mu\left(\frac{2}{3}\frac{\omega'}{\mu}+\frac{1}{3}\frac{\mu}{\omega'}\right)\,.$$

This gives $E>\frac{2\sqrt{2}}{3}Q\mu$ so the mass cannot be made *arbitrarily* small with respect to a collection of asymptotic quanta of the same charge. With non-zero P_5 a similar situation obtains, but with non-zero ω acting to increase the mass, such that

$$E>\frac{2\sqrt{2+\omega^2}}{3}Q\mu. \tag{63}$$

We conclude that thick-wall Q-balls with $\omega>1/2$ are always unstable to decay.

4 Q-branes

4.1 Background: the dielectric brane potential

We now turn to our particular application of the previous discussion, Q-balls as deformations of stacks of Dp-branes.

Let us briefly recap the Lagrangian for this system. As is well known, the massless modes of the open string form a supersymmetric $U(1)$ gauge theory with a vector A_μ, $\mu = 0, 1, .., p$, $9-p$ "collective coordinate" scalars Φ^i, $i = p+1, ..., 9$ and their fermionic partners. The dynamics of a single Dp-brane is described by the DBI-action

$$\begin{aligned} S_{DBI} &= -T_p \int d^{p+1}\sigma e^{-\phi_s} \sqrt{\det\left([G + 2\pi\alpha' B]_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} \\ &\qquad\qquad + \mu_p \int \sum C^{(n)} \wedge e^{2\pi\alpha'(B+F)}\,, \end{aligned} \tag{64}$$

where T_p, μ_p are respectively the tension and the RR charge of the Dp-brane, $C^{(n)}$ is the $n+1$-form RR potential and ϕ_d is the string theory dilaton. We denote by [..] the pull-back of spacetime tensors to the Dp worldvolume; for example

$$[G]_{\mu\nu} = G_{\mu\nu} + 4\pi\alpha' G_{i(\mu}\partial_{\nu)}\Phi^i + 4\pi\alpha' G_{ij}\partial_\mu\Phi^i\partial_\nu\Phi^j. \tag{65}$$

A collection of N coincident Dp-branes supports a supersymmetric $U(N)$ gauge theory with gauge field A_μ, and scalars Φ^i in the adjoint of $U(N)$. The latter act as the collective coordinates of the branes. The action which describes the dynamics of such a collection of coincident Dp-branes is not completely known. For example, replacing the abelian $U(1)$ in the action (64) and taking the symmetrized trace over the gauge group as was suggested in [26] does not capture the full infrared dynamics [27], and in fact additional commutators of the field-strength are neeeded at sixth order [28]. Some progress can be made especially for the structure of the Chern-Simons term, the second term in (64), in the non-abelian case [20]. By using T-duality arguments, Myers showed that a Dp-brane couples not only to the $p+1$-form RR potential but also to the RR potential with form degree higher than $p+1$ [20]. A collection of N D0-branes for example in an electric RR four-form flux develops a dipole moment under the three-form potential. This is a "dielectric" property of the Dp-branes similar to the dielectric properties of neutral materials in electric fields. Indeed, in general, the Chern-Simons term for N coincident Dp-branes is modified to [20]

$$S_{SC} = \mu_p \int \mathrm{Tr}\left(e^{i2\pi\alpha' i_\Phi i_\Phi} \sum \left(C^{(n)} \wedge e^{2\pi\alpha'(B+F)}\right)\right)\,, \tag{66}$$

where i_Φ denotes the interior product by Φ^i if the latter is considered as a vector in the transverse space. The existence of these additional couplings in turn modifies the scalar potential of the world-volume theory. In the case of N Dp-branes, for flat world-volume metric and vanishing RR and B-fields, the DBI-action, in lowest order in α', turns out to be

$$S_{DBI} = \int d^{p+1}\sigma \left[-4\pi^2\alpha'^2 T_p e^{-\phi_s}\,\mathrm{Tr}\left(D_\mu\Phi^i D^\mu\Phi^i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - V\right], \tag{67}$$

where the scalar potential is

$$V = -T_p e^{-\phi_s} \pi^2 \alpha'^2 \mathrm{Tr}([\Phi^A, \Phi^B])^2 \,. \tag{68}$$

Then by turning on an electric $p+3$-form potential $C_{01\dots pAB}$, an additional coupling of the Dp-brane appears as can be seen from the Chern-Simons term (66), so that the total potential turns out to be

$$V = -T_p e^{-\phi_s} \pi^2 \alpha'^2 \mathrm{Tr}([\Phi^A, \Phi^B])^2 - \frac{i}{3} 4\pi^2 \alpha'^2 \mu_p \mathrm{Tr}(\Phi^A \Phi^B \Phi^C) f_{ABC} \,, \tag{69}$$

where $f_{ABC} = \frac{1}{(p+1)!} \epsilon^{\mu_0 \cdots \mu_p} (\partial_A C_{\mu_0 \dots \mu_p BC} + cyclic)$.

4.2 Q-balls on dielectric branes

Now let us consider simple Q-ball configurations on such dielectric branes. For definiteness we will take N coincident D3-branes; as the collective coordinate of the D3-branes plays the role of the (non-abelian) internal Q-charge, the Q-lumps will describe the physical displacement of the D-branes within the compact dimensions, with the D3-branes oriented so their internal Neumann dimensions fill space-time. The results extend trivially to other Dp-branes.

Generally speaking, non-abelian Q-ball solutions can be found in theories that have scalar fields ϕ_{ab} in a real $M \times M$ matrix representation of some non-abelian symmetry group [4, 6]. We should remark that the latter will turn out to be a subgroup of the $U(N)$ gauge symmetry described by the Dp-branes so that the result will be gauged Q-balls rather than global. As discussed in [3] such objects are subject to a further constraint on their size coming from Coulomb repulsion of the charge which distributes itself over the surface of what is effectively a superconductor; however we will work in the small coupling limit in which this effect is negligible and the solutions are the same as the global ones.

In the absence of gauge fields VEVs then, the action to lowest order after appropriate field redefinitions is

$$S = \int \mathrm{d}^{3+d} x \mathrm{d}t \left(\frac{1}{2} (\partial \phi)^2 - U(\phi) \right) \,, \tag{70}$$

where $U(\phi)$ is the scalar potential, and traces over the $M \times M$ matrix indices are implied. Generalising the results of the earlier sections, reparameterization invariance leads to the following conserved energies and momenta;

$$\begin{aligned} E &= \int \mathrm{d}^{3+d} x \, \left(\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_i \phi)^2 + U(\phi) \right) \\ P_i &= \int \mathrm{d}^{3+d} x \; \partial_t \phi \partial_i \phi \,, \end{aligned} \tag{71}$$

where $i = 1, \dots 3+d$, again traces are inferred, and we are assuming canonical kinetic terms. The conserved charges of the non-abelian symmetry are

$$Q^k = \frac{1}{i} \int \mathrm{d}^{3+d} x \; \partial_t \phi \left[T^k, \phi \right] \,, \tag{72}$$

where T^k are the relevant generators. To each charge we can associate a Lagrange multiplier, ω_k, so that we must minimise the expression

$$\varepsilon_{\omega_k} = E + \omega_k \left(Q^k - \frac{1}{i} \int \mathrm{d}^{3+d}x \; \partial_t \phi \left[T^k, \phi \right] \right) . \tag{73}$$

Completing the square and minimising as before we find

$$\begin{aligned} \phi &= e^{i\omega_k T^k t} \varphi(x) \, e^{-i\omega_k T^k t} \\ \varepsilon_{\omega_k} &= \int \mathrm{d}^{3+d}x \; \left(\frac{1}{2} (\partial_i \varphi)^2 + \hat{U}_{\omega_k}(\varphi) \right) + \omega_k Q^k, \end{aligned} \tag{74}$$

where

$$\hat{U} = U + \frac{1}{2} Tr \left(\omega_k \left[T^k, \varphi \right] \right)^2 , \tag{75}$$

and of course now φ is also $M \times M$ matrix-valued. Note that we could have found the same result by using the equations of motion, as we did for the $U(1)$ case discussed earlier.

So far the discussion applies completely generally for non-abelian Q-balls. Our task now is to find a local minimum of the dielectric potential that preserves such a global non-abelian symmetry, and to determine $U(\phi)$ there. To do this let us turn on a background field,

$$f_{ABC} = f \varepsilon_{ABC} , \tag{76}$$

where we will take the Φ^A_{ab} to be three $N \times N$ matrix-valued fields transforming under the $U(N)$, with $A = 1..3$, $a, b = 1..N$. As before, A labels the three arbitrarily chosen extra dimensions in which we turn on the background field. As an ansatz, let the three fields Φ^A fall into an irreducible $SU(2)$ multiplet as follows:

$$\Phi^A(t,x) = \beta \, \hat{\phi}(t,x) \otimes \alpha^A , \tag{77}$$

where the α^A form an $N/M \times N/M$ irreducible representation of $SU(2)$, $\hat{\phi}(t,x)$ is an arbitrary $M \times M$ real matrix and $\beta^{-1} = 2\pi\alpha' T_p^{1/2} e^{-\phi_s/2}$ is a parameter that ensures canonical kinetic terms for $\hat{\phi}$. In particular we have that

$$[\alpha^A, \alpha^B] = 2i \varepsilon_{ABC} \alpha^C , \tag{78}$$

and

$$\mathrm{Tr}(\alpha^A \alpha^B) = \frac{n}{3}(n^2 - 1)\delta^{AB}, \quad n = N/M . \tag{79}$$

Inserting this ansatz into eq.(69) and using the BPS condition for the tension and RR charge of the Dp-branes, $T_p = \mu_p$, we find that V becomes

$$V = 8\pi^2 T_p e^{-\phi_s} \alpha'^2 \beta^4 n(n^2 - 1) \left(Tr \hat{\phi}^4 + \frac{\hat{f}}{3} Tr \hat{\phi}^3 \right) , \tag{80}$$

where $\hat{f} = f e^{\phi_s} / \beta$. A local minimum exists at

$$\langle \hat{\phi} \rangle = \phi_0 I_M , \tag{81}$$

where $\phi_0 = -\hat{f}/4$ and I_M is the $M \times M$ unit matrix. This is the usual dielectric minimum, representing a configuration in which the N D3-branes are bound to the surface of a D5-brane (forming a sphere in the non-space-time dimensions with radius $r_0 = \frac{\pi}{2}\alpha' \hat{f} N$).

We may then define the D-brane displacements with respect to the shifted centres of mass of the blocks of M D-branes as

$$\hat{\phi}(t,x) = -\frac{\hat{f}}{4} I_M + \phi(t,x)\,. \tag{82}$$

The $M \times M$ matrix-valued field $\phi(t,x)$, corresponding to the displacement around the minimum, is precisely our desired non-abelian Q-ball field. Substituting into V gives a potential for it of the form (ignoring a vacuum energy term)

$$U(\phi) = \frac{1}{2}\mu^2 \mathrm{Tr}\phi^2 + \frac{g}{3!}\mathrm{Tr}\phi^3 + \frac{\lambda}{4!}\mathrm{Tr}\phi^4\,, \tag{83}$$

where

$$\mu^2 = \frac{\hat{f}^2\lambda}{96}, \quad g = -\frac{\hat{f}\lambda}{6}, \tag{84}$$

and

$$\lambda = \frac{12n(n^2-1)e^{\phi_s}}{\pi^2 T_p \alpha'^2}. \tag{85}$$

Note that λ has engineering dimension $3-p$ as required for the action defined over the $p+1$ dimensional world volume. Clearly eq.(83) is invariant under transformations $\phi \to e^{i\Omega}\phi e^{-i\Omega}$, where Ω are elements of $GL(M,\mathbb{R})$, and we may therefore contemplate precisely the same Q-ball solutions as described above. Although the general case can be worked out, let us consider the simplest case in which Ω are elements of $SO(3)$ (indeed minimal stable Q-balls are all unitarily equivalent to the $SO(3)$ Q-ball [4]),

$$\phi(t,x) = e^{i\omega_k T_k t}\varphi(x) e^{-i\omega_k T_k t}\,, \tag{86}$$

where $\varphi(x)$ is in the adjoint of $SO(3)$, and T_k ($k = 1,2,3$) are $SO(3)$ generators in the fundamental representation. The potentials U and $\hat{U}$ are shown in figure 3. This case was explicitly worked out in [4, 6], with the result that the necessary and sufficient condition for the existence of Q-balls [4] is

$$1 \leq \frac{g^2}{\mu^2\lambda} < 9. \tag{87}$$

The lower bound is the energetic condition for the existence of the Q-balls (ensuring that the Q-ball will not decay into free mesons), whereas the upper bound is the condition that the cubic coupling is not very large so that $\phi = 0$ is the global minimum. For the case at hand we have

$$\frac{g^2}{\mu^2\lambda} = \frac{8}{3}\,, \tag{88}$$

and thus eq.(87) is satisfied. Therefore, dielectric branes support stable Q-balls in their world-volume.

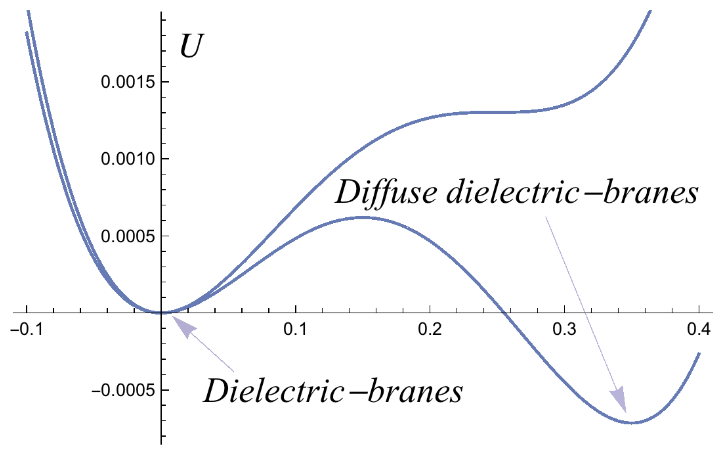


Figure 3: *The original dielectric potential U (upper), and the potential around its minimum $\hat{U}$ (lower) for non-abelian Q-ball deformations.*

Given this, it is interesting to ask what their mass can be. Adopting the small Q-ball approximation, eq.(63) gives $E > \frac{2\sqrt{2}}{3}Q\mu$ if we, for simplicity, take the lump to be non-relativistic in the compact dimensions, $\omega \ll 1$. In this limit our 5 dimensional system (which would correspond to dielectric D4 branes wrapped on a dimension of size $2\pi R$) would give Q-balls with a mass less than $Q\mu = 81\pi S_\psi \frac{\omega' R}{\lambda}$, so the minimum Q-ball mass is proportional to the compactification radius measured in units of the Compton wavelength $1/\omega'$. As we saw this number can in principle be less than unity. (Precisely how small it can be depends on the complicated dynamics of the Q-charge exchange which is beyond the scope of this paper to discuss, but would require further studies along the lines of [24, 23].) In addition λ scales as n^3 and can therefore be large. We conclude that such fundamental Q-balls could be significantly less massive than the fundamental scale.

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