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Three concepts or one? Students' understanding of basic limit concepts

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Abstract In many mathematics curricula, the notion of limit is introduced three times: the limit of a sequence, the limit of a function at a point and the limit of a function at infinity. Despite the use of very similar symbols, few connections between these notions are made explicitly and few papers in the large literature on student understanding of limit connect them. This paper examines the nature of connections made by students exposed to this fragmented curriculum. The study adopted a phenomenographic approach and used card sorting and comparison tasks to expose students to symbols representing these different types of limit. The findings suggest that, while some students treat limit cases as separate, some can draw connections, but often do so in ways which are at odds with the formal mathematics. In particular, while there are occasional, implicit uses of neighbourhood notions, no student in the study appeared to possess a unifying organisational framework for all three basic uses of limit.

1 Introduction

Limit is fundamental to the standard formal foundations of many aspects of calculus: derivative as the limit of slopes of secants of smaller widths; Riemann integral as the limit of Riemann sums of finer meshes etc.

In many cases, different *applications* of limits are represented with different symbols, such as Σ, $\int$, $\frac{d}{dx}$. Oehrtman (2008) argued that in these applications and symbols, the "role of limits is typically suppressed" [p. 68] so students' informal notions of limit play a larger role. However, in order to develop the foundations of these applications, students are introduced to three basic notions of limits: limit of a sequence and limit of a function at a point and limit of a function 'at infinity'. These have near identical symbols: $\lim_{n\to\infty} a_n$, $\lim_{x\to x_0} f(x)$ and $\lim_{x\to+\infty} f(x)$, and the given definitions have often very similar structure. We argue that, while there is a mathematical coherence evidenced by the similarity of the symbols and definitions, an underlying organising conceptual framework is rarely made explicit and so students may fail to connect these basic notions.

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A review suggests that existing literature has not explored the extent to which students recognise similarities and differences between these uses of limit. The study reported here explores whether students are able to see them as instances of a core underlying idea, as related but distinct ideas, or as distinct ideas which happen to share similar symbols.

The calculus and analysis curriculum challenges the portrayal of mathematics as hierarchical in its development in a way Raman (2004) described as "spiralling back": upper school pupils are introduced to techniques of calculus (such as simple derivatives) using limit notions which, if justified at all, are grounded on examples and informal descriptions. The formal grounding which justifies these techniques often only takes place at university. Even then, key logical steps (such as the construction of $\mathbb{R}$ and checking crucial properties like completeness) are omitted in all but the most specialised courses.

In university analysis courses, the approach often taken is to present the definition of limits of sequences, then limits of functions at a point and limits of functions at infinity. In some cases, the definitions for limits of functions are given in terms of limits of a sequence and in other cases a separate definition is given (contrast Bryant (1990) with Spivak (2006), for example).

It is possible to provide an overarching framework for all three basic concepts with the notion of "neighbourhood" defined to include infinity and sequences seen as functions on $\mathbb{N}$. $\lim_{X\to A} f(X) = L$ if for every neighbourhood V of L there is a neighbourhood U of A with $f(U \setminus \{A\}) \subset V$. However, such unifying concepts are rarely encountered before courses on metric spaces or topology. Instead the three basic concepts of limit tend to be defined separately, sometimes some distance apart in the course and with few links. This leaves open the extent to which students see them as manifestations of a unified notion of limit, as distinct but related or as separate notions. Hence the need to explore students' views of them.

2 Background

One of the key areas which has been explored in students' understanding of limit concerns the use of dynamic imagery. Lakoff and Núñez (2000) argued that limits of real functions (at a point) are *necessarily* conceived as co-ordinated dynamic sequences (in the domain and range) even though they note that such limits are often defined without using sequences. In fact, classic texts in real analysis (such as Spivak, 2006) provide an $\epsilon - \delta$ definition of limit then deal with continuity and differentiability before the formal $(\epsilon - N)$ definition of limits of a sequence. Moreover, there are equaly rigourous formulations of analysis in which this does not hold: the non-standard definition of limit of a function at a point need not involve sequences or dynamic imagery (see Keisler, 1986), nor does the topological definition above.

Rather than being indicative of some necessary cognitive process, it may be that students' understanding of limit notions reflects how they are taught: that is, dynamic imagery may be a useful cultural construct, not a necessary cognitive one. Indeed, Borovik and Katz (2012) argued that the prevalence of standard treatments of analysis may ignore a "dual history" of rigorous calculus post-Cauchy which allows (static) infinitesimal objects. Roh (2008) also argued that the dynamic imagery used by many students may come from the way they are introduced to the concepts and the issue is "not whether to use dynamic images in instruction, but rather, how to induce dynamic images that are compatible with the definition of limit" [p. 234]. Sierpínska (1987), amongst others, noted that some students have static conceptions of infinity and Ely (2010) detailed one student with an apparently consistent view of infinite and infinitesimal numbers which, while at odds with prevailing foundations of limits (and the "basic metaphor of infinity" of Lakoff and Núñez), fits with equally rigorous notions of non-standard analysis. These studies appear to contradict the assumption that understanding of limits of function at a point necessarily involves thinking about sequences or using dynamic imagery.

While research on understanding limits has a long history, it is clear that little existing research explores how students link different basic limit concepts and this paper aims to address this. We first explore existing research into the basic notions and we distinguish between papers which consider only one basic types of limit and those which consider more than one.

2.1 Limits of Sequences

McDonald, Mathews, and Strobel (2000) looked at students' understanding of the underlying concept of a sequence and whether it is conceived as a process or an object. While they did ask students about the limits of sequences, the focus was on the sequence itself. They noted students tended to focus on surface representational features (such as commas between terms in a sequence) and had difficulties with reconciling a list-based view of sequences with a functional view and seeing a sequence as a cognitive whole. Such problems also manifest themselves in students making sense of sequences defined in unfamiliar ways: for example, the sequence (a_n) where $a_n = 1 - (\frac{1}{n})$ for n odd and $a_n = \frac{1}{n}$ for n even, may be seen as two sequences, with two limits (Tall, Thomas, Davis, Gray, & Simpson, 1999).

Sierpínska (1987) focussed on students' understandings of limits of sequences (most notably in considering infinite decimals like $0.999\ldots$). These included the intuitive indefinitist ("all sequences are finite but sometimes it is impossible to determine the number of terms; the true limit is its last term" [p. 384]) or the infinitesimalist ("g is the limit of a sequence A if the difference between A and g is infinitely small" [p. 389]).

Sierpínska's students had not encountered the definition of the limit of a sequence. Mamona-Downs (2001), by contrast, focussed on the form of the $\epsilon - N$ definition, noting difficulties with the core inequality linked to complex quantification (Dubinsky, Elterman, & Gong, 1988). Roh (2008) took this further, introducing students to "ϵ-strip" activities to help make sense the arbitrary nature of ϵ and its relationship to N in the definition.

2.2 Limits of Functions at a Point

Some research focusses solely on limits of functions at a point. Following Sierpínska, Williams (1991) found students with different models for limit of a function at a point. Predominately, those models appeared to be based on informal, dynamic notions in which the limit is something unattained, though the context of instruction was one with less emphasis on formal definitions.

Cottrill et al. (1996) argued that limit is a complex schema with important dynamic aspects involving the coordination of two processes. They focussed on limit of a function at a point which co-ordinates the process of x approaching the given point and, applying f to that process, the process of $f(x)$ approaching L. However, as with Lakoff and Núñez (2000), the authors presumed the limit of a function at a point is conceived in terms of limit of a sequence: the activities introduced in their instructional treatment translated processes in the schema for limit of a function at a point into processes involved in the limits of sequences. The students wrote computer code for a sequence (a_n) approaching a and then examined the sequence $(f(a_n))$. However, as Swinyard (2011) noted, no students at the end of the intervention detailed in Cottrill et al. were apparently "coherently reasoning about the formal process of validating limits" [p. 94].

In contrast to literature which focuses on dynamic imagery, Oehrtman (2009) found that amongst five basic metaphors in students' reasoning about the limit of a function at a point, motion images were relatively rare. Instead, students held ideas such as approximation, closeness, atomic smallness, infinity-as-a-number and collapse of dimension.

However, the difference between the models students develop in these studies may have some foundation in how they have been taught. Güçler (2013) focussed on teacher and student discourse during lessons on limit of a function at a point and continuity, finding the teacher tending to focus on the limit as an object, but shifting to the metaphor of limit as a process when working informally. The students appeared to absorb these metaphors, but used them less coherently and struggled to cope with an over-reliance on the dynamic in understanding limit as an object.

Szydlik (2000) also investigated students' sources of conviction in mathematics in the context of limits of functions at a point. She noted that those with external sources of conviction (where value of mathematical statements comes from authority) were less likely to have a clear definition of limit, or understand why limit results hold, than those with internal sources of authority.

2.3 Limits of Functions at Infinity

In contrast to the rich research on limits of sequences and limits of a function at a point, little research focusses solely on limits of functions at infinity (or, the limits of functions as x increases without bound). Kidron (2011) followed a student's developing knowledge of the definition and noted the need for two constructions: asymptotes can be crossed by the graph and the limit is a number, not a process of "getting close". However, Jones (2015) argued that, for students learning calculus for science and engineering, developing a dynamic concept of limits of functions at infinity might be the goal of instruction, rather than a route to formal understanding.

2.4 Combining Types of Limit

Much research deals with more than one limit concept. However, most papers appear to deal with them as separate issues or conflate them. For example, Elia, Gagatsis, Panaoura, Zachariades, and Zoulinaki (2009) used a questionnaire with items on all three notions. The analysis detailed students use of algebraic or geometrical representations, but results and conclusions were drawn only from items on limits of functions at a point.

In their seminal paper, Tall and Vinner (1981) introduced the ideas of concept image and concept definition in the context of limits and continuity. They considered both sequences and limits of functions at a point. Interestingly they noted that, when asked to give a formal definition, students could get confused between different limits (e.g. including references to N in the definition of limit of a function at a point). In general, however, Tall and Vinner treated limits of sequences and limits of functions at a point separately and did not address the perceptions of similarities or differences between them.

Monaghan (1991) considered the influence of the language of limits. For example, he asked students to consider $(0.9, 0.99, 0.999, \ldots)$ and graphs of functions defined on the positive reals all having limit 0 at infinity. He focussed primarily on the language used to discuss limits, such as "tends to", "converges to" and "approaches", noting that these terms can cause confusion between everyday and formal meanings. While considering both limits of sequences and limits of functions at infinity, he made no explicit reference to similarities or differences between them.

However, some research does make some indirect link between basic limit concepts. Despite a main focus on functions, Przenioslo (2004) included items on all three basic limit types and explicitly considered the nature of their definitions. It appears the curriculum included explicit teaching of neighbourhood definitions and she found around 10% of students held neighbourhood conceptions. She also noted conceptions focussed on the graph approaching a point, values approaching a point, the idea of a function defined at a given point, the limit of a function at a necessarily equalling $f(a)$ and purely algorithmic conceptions. But, while all of these were research items, the aim of the work was on limits of functions at a point, so little attention was given to an analysis across the different types of limit.

One particularly interesting piece of research in the context of this paper concerned two high achieving students with no previous experience of limit definitions, working over a 10 week period to develop a formal definition of limit of a function at a point (Swinyard, 2011). Towards the end of this period, the researcher (commenting that it may be "less cognitively taxing"[p. 104]) drew attention to limit of a function at infinity. With considerable interaction with the researcher, they articulated a conventional form of the definition of limit of a function at infinity and used this to define limit of a function at a point. The structural similarity of these definitions of these concepts of limit was clearly noted.

3 Methods

Given the lack of other research into how limit concepts are linked, the aim of this study was to see what understandings students hold about these three uses of limit and, in particular, whether they see them as manifestations of a unified limit concept, distinct concepts with links between them or as disjoint concepts.

We accept the contention of Marton (1986) that there are a limited number of qualitatively different ways of understanding a phenomenon and, in this case, we wanted to know what ways there are of understanding limit in these different guises. In particular, Marton makes the distinction between researchers' approaches which examine perceptual processes in general terms — abstracted from the content that is being perceived — with phenomenography where the thinking is related to what is being perceived. That is, "research is never separated from the object of perception or the content of thought" (Marton, 1986, p. 32)

Since we were looking to retain the link to the limit concept while seeking a sense of the possible variation in its perception, we took a phenomenographic approach to data collection and analysis as described below.

A task was developed to encourage students to make explicit tacit connections through the development of categories. The task took the form of a think-aloud card sorting task, done individually with a researcher present. The approach has a long history in researching conceptual organisation (see, for example, Chi, Feltovich, & Glaser, 1981) and Fincher and Tenenberg (2005) argue that card sort techniques are "effective in eliciting ... semi-tacit understanding about objects in the world and their relationships" [p. 90].

The study involved undergraduate mathematics students with considerable experience of the "spiralling back" curriculum discussed above. The students were studying at a research intensive UK university with entry conditions ensuring all of them had the highest possible grades in pre-university mathematics where they had met informal calculus notions including limits of the gradient of smaller chords as an introduction to derivative.

The tasks were piloted with physical cards in four individual interviews with first year students. This demonstrated that the task fulfilled its role in encouraging students to talk about the different uses of limit: by grouping them, participants made explicit perceived connections between examples and, by directly comparing cards, participants demonstrated whether they could form links. However, the pilot exposed two difficulties. First, despite repeated reminders, participants stacked cards obscuring their references in gesturing towards cards. Second, at the time of the study, first year students had met all three limit concepts informally (through pre-university study) but had only met one (limit of a sequence) formally.

Thus, for the main study, a computer interface was developed which allowed students to drag cards into groups but prevented overlapping and aided recording categorisation. Bussolon, Russi, and Missier (2006) indicated there is no apparent difference between categorisations obtained from physical cards and those from a computerised card sorting task. In addition, participants were chosen from the second year of the degree programme. These students had met all three concepts formally and had seen them in advanced analytic ideas such as differentiation and integration. By this stage, the participants had taken a number of pure mathematics courses, including three compulsory analysis modules covering differentiation, (Riemann) integration, power series, analysis in higher dimensions and complex analysis.

Fourteen students responded to an email sent to all mathematics second years requesting volunteers for a mathematics education research project. Each interview was conducted one-to-one by the first author and participants were allowed as much time as they wished to complete the task, generally taking between forty minutes and one hour. The task involved both repeated single-criterion open card sorts and closed card comparison tasks. The first part, repeated single-criterion open card sorting, involved participants grouping the cards according to any criterion they wished. The instructions were deliberately neutral with respect to criteria, the interviewer began by asking them to sort the cards in to groups "in any way you want".

Since we were primarily interested in the connection they were making between cards, they were asked to explicitly state the criterion if they did not do so unprompted. They were asked to repeat the task using different criteria until they were no longer able to generate new ways of sorting the cards. Again, prompting was neutral with respect to criteria, with the interviewer asking if they could sort them '"in another way".

The second part of the task, closed card comparison, looked to see if participants could make connections when they were shown small card groups. In this part of the task, the interviewer first invited students to choose pairs or small groups of cards to discuss, then selected other small groups of cards for discussion.

The pilot study had shown that some participants wished to make notes or sketches, so they were given paper for this, which was retained by the researcher. In each case, computer work was screen captured to obtain the groupings and video captured for student speech and gestures.

The cards formed the 'shared definition' of the phenomenographic interview and were designed to fulfil four criteria:

- they used the same representation (the common symbolism discussed above);
- the sequences and functions under consideration were simple enough that any interference from misunderstanding this aspect would be minimised;
- there were examples of many of the main behaviours expected of sequences, functions at a point and functions at infinity and
- the size of card set was manageable.

For example, in terms of sequence limits, the emphasis is generally on convergence or divergence; with limits of functions at a point or at infinity, there can be a finite limit, the function can tend to $\pm\infty$ or the function can have no limit. Given the need to keep the card set manageable, not every behaviour was included and, in particular, we did not look at functions at a point with differing limiting behaviour from left and right.

Table 1 shows each of the card designs (the letters by each example are for reference in this paper and were not on the cards on the screen).

Many cards were designed for similarities and differences in the basic uses of limit to emerge, for example cards (l) and (n) might draw attention to the range of the variable (the function in card (n) is defined almost nowhere, but the sequence in card (l) is well defined, if divergent). Cards (r), (s) and (t) were designed to explicitly bring together sequences and functions (drawing on the idea that limits of function at a point is sometimes defined in terms of the limit of sequences).

Table 1 Limit cards

a) $\lim_{n\to\infty} \frac{1}{n}$	b) $\lim_{x\to+\infty} \sin(\frac{1}{x})$	c) $\lim_{x\to 0} \frac{1}{x}$	d) $\lim_{x\to 1} \frac{x^2-1}{x-1}$
e) $\lim_{x\to+\infty} 2^{-x}$	f) $\lim_{x\to+\infty} \frac{1}{x}$	g) $\lim_{x\to+\infty} \sin(2\pi x)$	h) $\lim_{n\to\infty} \sin(\frac{1}{n})$
i) $\lim_{x\to 1} (x+1)$	j) $\lim_{x\to 0} \sin(\frac{1}{x})$	k) $\lim_{n\to\infty} 2^{-n}$	l) $\lim_{n\to\infty} (-2)^n$
m) $\lim_{n\to\infty} \sin(2\pi n)$	n) $\lim_{x\to+\infty} (-2)^x$	o) $\lim_{x\to+\infty} g(x)$, $g(x) = \begin{cases} 1 & x \text{ is a natural number} \\ \frac{1}{x} & x \text{ is not a natural number} \end{cases}$	
p) $\lim_{n\to 12} a_n$, $a_n = \begin{cases} 28 & n = 12 \\ \frac{1}{n} & n \neq 12 \end{cases}$	q) $\lim_{x\to 12} h(x)$, $h(x) = \begin{cases} 28 & x = 12 \\ \frac{1}{x} & x \neq 12 \end{cases}$	r) $\lim_{n\to\infty} f(1+\frac{(-1)^n}{n})$, $f(x) = \begin{cases} 1 & x = 1 \\ \frac{x^2-1}{x-1} & x \neq 1 \end{cases}$	
s) $\lim_{x\to 1} f(x)$, $f(x) = \begin{cases} 1 & x = 1 \\ \frac{x^2-1}{x-1} & x \neq 1 \end{cases}$	t) $\lim_{n\to\infty} f(1+\frac{1}{n})$, $f(x) = \begin{cases} 1 & x = 1 \\ \frac{x^2-1}{x-1} & x \neq 1 \end{cases}$		

One other card worthy of note is card (p). Weber (2005) asked a student to give a description of a limit of a sequence and noted “she gave a calculus-style account of what constituted the limit of a function, and spoke of limits of sequences as n approached 0 and -2” [p. 354]. In Przenioslo (2004), there is a short but intriguing passage:

> [Some students] ...believed that it makes sense to speak of the limit of a sequence at a point belonging to a set of natural numbers. For some students the limit of a sequence (a_n) at the point n_0 equals to a_{n_0} ‘by definition’. ... Other students thought that the limit of a sequence (a_n) equals a_{n_0} only at a point n_0 such that “the points (n, a_n) get closer and closer to (n_0, a_{n_0}) from both sides”. [p. 121]

but she makes no other comment. These are the only places in the literature we found in which the concept of ‘limit of a sequence at a point’ was considered and, in both, there is evidence that students try to make some sense of it. But in both cases, the authors did not investigate further. So the study included a card which might evoke discussion on this issue.

The data gathered were analysed using the phenomenographic method outlined by Marton (1986). The aim is not to find a single way in which limits might be viewed, but how views of these different limits might vary and the structure of that variation. The videos were viewed repeatedly to identify points at which students reflected on their conceptions of limit, these formed a ‘pool of meanings’[p. 43]. Similarities and differences within that pool of meaning were identified and these were organised into preliminary categories of linked descriptions of the variations in the ways basic uses of limit are experienced. These categories were compared and contrasted, and the relational structure detailed below was identified. From this, the subcategories and central illustrations were chosen.

It is important to recognise that what follows is based on categories of phenomena; they are not levels and certainly not proxy for ability. Students undertook repeated rounds of sorting cards, so evidence that one student grouped on ‘surface’ criteria is not an indication that they lacked other ways of organising them. It simply demonstrates that one way in which one might make sense of the phenomena involves surface characteristics.

Also note that in the extracts presented here, the students' spoken mathematics has been formatted in symbols to aid readability. Extracts include the cards referred to and descriptions of gestures where it is important for contextualising the meaning of students' words.

4 Results

Our analysis of the data suggests that there were five main ways of making sense of the tasks and the symbols on the cards: (1) using 'non-analytic' characteristics of the symbols (that is, not considering issues of limiting processes, continuity, etc.); (2) connecting limits of functions and limits of sequences; (3) connecting limits of functions at a point and at infinity; (4) connecting limits of functions at a point and limits of sequences and (5) nascent neighbourhood notions. Each of these is elaborated below.

We reiterate that we do not see these categories as hierarchical but as different ways of experiencing the phenomena of the stimuli; nor are they chronological. Indeed, the initial response from eight students was to categorise on analytical properties, with the remaining six categorising first on 'surface' or non-analytic properties.

4.1 Non-analytic characteristics

One category of response relates to what might be called 'surface' or non-analytic characteristics: many students in the open sort appeared to focus on the form of the expression of the function or sequence, rather than on an analytic concept related to the limit or a limiting process.

For example, some students distinguished groups of cards on the format of the formulae given:

Student (S3): All of these ones have got sine in them so they're trig functions [*collects together all the cards in which the sine function appears*] and these ones don't have sine so they'll be normal functions.[*indicates the remaining cards*] All the ones that that have conditions, so either an input value or an output value, that must be satisfied [*working with the remaining set of cards, picks up card (r)*]

Researcher(R): Defined by conditions?

S3: Yeah.[*collects together all the cards in which a function is defined piecewise and writes "satisfied only by conditions" on paper, then puts cards back*] Hmm ... I'm going to go for ones that are exponential functions and tend to get faster as you go ... so ... exponential.... [*searches cards, collects* $\boxed{\lim_{x\to+\infty} 2^{-x}}$, $\boxed{\lim_{n\to\infty} 2^{-n}}$, $\boxed{\lim_{n\to\infty} (-2)^n}$ *and* $\boxed{\lim_{x\to 1} \frac{x^2-1}{x-1}}$, *pauses and then puts* $\boxed{\lim_{x\to 1} \frac{x^2-1}{x-1}}$ *back*]. ... I'm doing the ones so that as that number increases[*gestures '$-x$' in* $\boxed{\lim_{x\to+\infty} 2^{-x}}$]

In this extract, S3 talks about the cards as representing "functions" repeatedly and classifies entirely on the nature of the expression for which the limit is to be evaluated. At this point, he made no distinction between cards with different variables, limit points or deeper analytic concepts and, in particular, different forms of limit play no role in the classification. It is interesting to note that his choice of category appears based on an apparent typology of functions: so-called "normal" ones (presumably polynomial and rational), trigonometrical and exponential functions and then functions with piecewise definitions. This typology probably matches his familiarity with different types of function and the point at which they were introduced in his schooling.

This approach — categorising on the basis of the surface form of the expression for which the limit is to be evaluated — was common, but students often showed explicit awareness that such groupings were superficial:

S1: I'm focussing on whether the function contains a one or a two. So these functions all contain a two in them and these ones all contain, an important thing, a '+1' a '$\frac{1}{x}$', an '$x^2 - 1$' and so forth, a '$\frac{1}{x}$' again, ... Trivial to the extreme.

A second type of classification which, while not fully focussed on the analytic, involved noticing some mathematical properties of the objects referred to by the expressions at which the limit is to be evaluated or the variable used. For example, one student explained the difference between expressions with x and with n as representing real functions and sequences respectively:

S14: I think the most obvious one to go with to begin with is to separate ns and x. x is a continuous, it, it can take any value, normally it's associated with any value on the real number line, whereas n is just normally a natural number, so, zero or one. ... There are so many obvious differences between the cards, this is one of them.

Students did not always initially recognise the difference between sequence and functions (which, after all, is often only indicated here through use of conventional variable names):

S2: Well yeah, just those two. [*selects cards* $\boxed{\lim_{x\to+\infty} \frac{1}{x}}$, $\boxed{\lim_{n\to\infty} \frac{1}{n}}$] Those are the most obvious ones to me because, just because of the fact that one is n and one's x doesn't make any difference to how they operate, so they're the same thing.

R: You consider them the same thing? Any other cards?

S2: Yeah [*identifies cards* $\boxed{\lim_{x\to+\infty} \sin(2\pi x)}$ *and* $\boxed{\lim_{n\to\infty} \sin(2\pi n)}$] and those two [*points to* $\boxed{\lim_{x\to+\infty} 2^{-x}}$, $\boxed{\lim_{n\to\infty} 2^{-n}}$] and these two [*points to* $\boxed{\lim_{x\to+\infty} \sin(\frac{1}{x})}$, $\boxed{\lim_{n\to\infty} \sin(\frac{1}{n})}$]

In most cases, students did note this difference, using it in some choice of classification, but if students did consider n a real variable, the interviewer reminded them of the convention.

Just one further non-analytic classification occurred: the nature of the point at which the limit is to be evaluated:

S11: So, new grouping. We've got limits as x or n approaches infinity as one group, which we've got here [*gestures to a group of cards in which the* ∞ *symbol appears as the point at which the limit is to be evaluated*] As x approaches zero [*gestures to a group of cards in which the 0 symbol appears as the point at which the limit is to be evaluated*] and as x approaches some constant [*gestures to a group of cards in which a constant (1 or 12) appears as the point at which the limit is to be evaluated*]

Note the separation of zero from other constants and including the card with '$n \to 12$' in the group described as "as x approaches some constant".

4.2 Limits of functions at infinity and limits of sequences

Many students talked about limits of functions at infinity and limits of sequences interchangeably, even when aware of the naming convention:

S8: It looks different because that's [(g)] positive infinity and that's [(m)] infinity. But I think that [m] implies positive infinity, so they're the same.[*Identifies cards g:* $\boxed{\lim_{x\to+\infty}\sin(2\pi x)}$ *and m:* $\boxed{\lim_{n\to\infty}\sin(2\pi n)}$]

R: The same? What is the value of the sine of $2\pi n$ for different n?

S8: Zero. So I suppose if, n is all the natural numbers and so they're all zero. But infinity, I don't know if that's a natural number or ... If it's the natural numbers, then zero, but if n can be like 1.3 or like 19.3 then it can be, then it won't exist, but if n is strictly 1, 2, 3, 4, 15, 17. It depends on whether n is natural or not.

R: What about this? [*gestures to card (g)*]

S8: It's the same thing, we could just say "let x equal n" and then it becomes the same.

The student appeared to be aware of the convention and recognised the logical bound on the variable n, but still seems to have concluded the objects are identical on the basis of their visual similarity rather than considering the type of objects referred to by the expressions.

Other students did have a sense of a relationship between limits of functions at infinity and limits of sequences:

S7: Well, I can pick those two first [*identifies cards g:* $\boxed{\lim_{x\to+\infty}\sin(2\pi x)}$ *and m:* $\boxed{\lim_{n\to\infty}\sin(2\pi n)}$] ... there is no difference since if we're going to infinity it doesn't matter if we x or n because we're still going to infinity, so yeah, I have to draw a graph of how to explain properly [*draws a straight horizontal line*] Since we've got the real axis and we've got infinity, no matter how close we get to infinity, it's not a real number ....

R: So what do you, are they the same, not the same?

S7: Well this [*indicates (m)*] is a subsequence of this [*indicates (g)*] it contains only some values of the other, yeah if you have $a_1, a_2, \ldots a_n$ then a subsequence might only have a_2, a_4, a_6 and so on; it only takes some values ....

This student suggested an inclusion relationship between these situations, but phrased it as a relationship between the objects referred to in the expressions (that the sequence is a subsequence of the function) and seemed to link this situation to the sequence inclusion rule (that if a sequence converges then every subsequence converges to the same limit).

Other students identified a difference between the limit of a function at infinity and limit of a sequence and treated them separately even if noticing the similarity of the expressions:

S9: This one here ... [*indicates* $\boxed{\lim_{n\to\infty}\sin(2\pi n)}$] ... is interesting because 2π, erm $\sin(2\pi)$. Well $\sin(0) = 0$ so if you multiply it by any natural number it will always equal zero, no matter what and then, the same thing here [*indicates* $\boxed{\lim_{x\to+\infty}\sin(2\pi x)}$]: 2π multiplied by any number x is, as x tends to infinity it will oscillate between one and minus one and I think that's interesting.

This student drew these cards out together and noticed they have different analytic properties — even commenting that he found this interesting. However, few students commented on relationships between them. Indeed, only one student tried to make such a formal link:

S14: That doesn't have a limit [*indicates (g)* $\boxed{\lim_{x\to+\infty}\sin(2\pi x)}$] and that does [*indicates (m)* $\boxed{\lim_{n\to\infty}\sin(2\pi n)}$] and that shows how this [*(m)*] is necessary for that [*(g)*] but is not sufficient, necessary but not sufficient, because this has a limit but this does not. That proves it, that's a good example.

While not well expressed, the student appeared to have a sense that if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} f(n)$ exists (and then the limits are equal). This may be what S7 was attempting to express above, but she seemed to confuse the nature of the sequence and function objects.

4.3 Limits of functions at infinity and limits of functions at a point

One way in which some students connected these two forms of limit was by equating $f(x)$ for $x \to \infty$ with $f(\frac{1}{x})$ for $x \to 0$ (or vice versa) with some overgeneralising by applying techniques from a different limit context:

> **S11:** So, when that tends to infinity. I know you shouldn't split it up but, $\frac{1}{x}$ has a limit as x tends to infinity, which we know is going to be zero and we know that sine is defined at zero, whereas sine isn't defined at $\frac{1}{0}$, which is undefined. So we can always say that this has a limit, because we know what sine is defined at, at zero, zero, whereas sine doesn't have a definition as, as x gets very small, so if we say that x isn't equal to zero, x is very very small, $\sin(x)$ could be anything between plus and minus one because it's an oscillating function. [*selects cards* $\boxed{\lim_{x\to+\infty} \sin(\frac{1}{x})}$ *and* $\boxed{\lim_{x\to 0} \sin(\frac{1}{x})}$]

This student appeared, initially, to take a similar approach to both situations, by considering the limit of the argument of the function. So she concluded that as $\lim_{x\to\infty} \frac{1}{x} = 0$ she could infer that $\lim_{x\to\infty} \sin(\frac{1}{x}) = \sin(0) = 0$, but she tried to apply the same approach to $\lim_{x\to 0} \sin(\frac{1}{x})$ by considering $\lim_{x\to 0} \frac{1}{x}$. This strategy appeared in other contexts, shown below. However, the second part of the extract shows her thinking, perhaps, in terms of neighbourhoods of x around 0 mapping to 'neighbourhoods' of infinity.

While rare, this neighbourhood thinking did appear in one other student's interview:

> **S4:** There is a calculation tool which means I can change the variable. Which means if I was to say that x equals erm, $\frac{1}{2\pi y}$ or something. It's not equivalent. OK, in this case, because the sine function is continuous it's equivalent to saying that x tends to, oh, change the variable, as y tends to zero of $\sin(\frac{1}{x})$ so because the sine function is continuous. All the trig functions are continuous. [*considers card* $\boxed{\lim_{x\to+\infty} \sin(2\pi x)}$] I'm still getting my head around this. ... because this limit is only dealing with x tending to something from one direction... as x tends to infinity there is always a value bigger than x for which the value of the function is one or minus one likewise with this as x tends to zero there is always a value of x smaller than x for which the function is either one or minus one [*considers* $\boxed{\lim_{x\to 0} \sin(\frac{1}{x})}$] which means that neither of these limits exist

This student appeared to both use the $x \to \infty / \frac{1}{x} \to 0$ connection and, later in the extract, the neighbourhood notion. However, his phrase "because this limit is only dealing with x tending to something from one direction" may show he realised that $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(\frac{1}{x})$ are not always interchangeable.

4.4 Limits of functions at a point and limits of sequences

In the example above, S11 appeared to think about $\lim_{x\to\infty} \sin(\frac{1}{x})$ as $\sin(\lim_{x\to\infty} \frac{1}{x})$. This was particularly common when students were considering cards across the *limits of functions at a point* and *limit of sequences* areas.

S9: As n tends to infinity, that's going to equal $1+\frac{1}{\infty}$, practically zero, so that's going to be $f(1)$ and $f(1)$ when x is one is one, so I think that is going to equal one. [*indicates card*

$$\boxed{\begin{array}{l}\lim_{n\to\infty} f(1+\frac{1}{n})\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

] Then the same here 'cos $(-1)^n$ as n tends to infinity, it will oscillate between positive and negative values, but because it's over n, n beats $(-1)^n$, so it will be $f(1)$ again and it will equal one. [*indicates card*

$$\boxed{\begin{array}{l}\lim_{n\to\infty} f(1+\frac{(-1)^n}{n})\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

]

There is no sense here of connecting limits of functions at a point with the limits of sequences; this student seemed to consider the argument of f as a sequence for which a limit must be found and, as a separate step, f is evaluated. This separation is seen more explicitly in students who were prepared to accept different values for the expressions on these cards:

S11: [*Considering cards*

$$\boxed{\begin{array}{l}\lim_{x\to 1} f(x)\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

and

$$\boxed{\begin{array}{l}\lim_{n\to\infty} f(1+\frac{(-1)^n}{n})\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

]

So, if we have x which is a continuous variable tending to 1, we know that ... if we were to draw the graph of this one, we would get, we know that we would get, this is $f(x)$ and this is x and we know x is continuous. We know that when $x=1$ we know that $f(x)=1$ but when $f(x)=\frac{x^2-1}{x-1}$, we know that that equals $\frac{(x+1)(x-1)}{(x-1)}$, so we know that this equals $x+1$ which is just a straight line, which will go.

It goes up to there and we have a gap here when x equals one, so the limit of this will be 2, because it doesn't matter that the x jumps, we have a jump discontinuity, I don't think that matters. So the limit will be at whatever x would be, x would be two. But what is the difference between that and this? [*sketches a graph of $y=x+1$ with a jump discontinuity highlighted at $x=1$*] ... Because if this, with the continuous value you can get as close as you want without getting there, but with the natural numbers you can't get that close to one, you can only get from nought and two. ...

The two sequences will tend to the same thing, but the continuous function will tend to a different thing. ... So what else can we say about it? These two sequences must tend to the same thing, because f of whatever it is as n tends to infinity, tends to one, so if these both tend to $f(1)$ which is one.

There seems to be some confusion about the argument of the function f. The phrase "with the natural numbers you can't get that close to one, you can only get from nought and two" suggests that he was thinking of a sequence *consisting of* natural numbers tending to 1, but he was also clear earlier that the argument is a sequence, *indexed by* the naturals and which tends to one. In addition, as he did earlier in the case of the sine function, he seemed to use an implicit rule like '$\lim f(\text{expression}) = f(\lim \text{expression})$'.

In these cases, the students appeared to treat the limit of the function at a point and the limit of a sequence as disconnected, but not all students did this. Some saw an analytic connection between the expressions on the cards:

S14: That [*indicates card (d)* $\boxed{\lim_{x\to 1}\frac{x^2-1}{x-1}}$] is similar to that [*indicates card (r)*

$$\boxed{\begin{array}{l}\lim_{n\to\infty} f(1+\frac{(-1)^n}{n})\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

] is similar to that. [*indicates card (t)*

$$\boxed{\begin{array}{l}\lim_{n\to\infty} f(1+\frac{1}{n})\\ f(x)=\begin{cases}1 & x=1\\ \frac{x^2-1}{x-1} & x\neq 1\end{cases}\end{array}}$$

] That is one-sided [*(t)*] that is

this from above, ok. [*(d)*] It took me a while at the beginning to work out what was going on. [*drags in card (s)* $\boxed{\lim_{x\to 1} f(x) \quad f(x) = \begin{cases} 1 & x = 1 \\ \frac{x^2-1}{x-1} & x \neq 1 \end{cases}}$] These are very similar because they're all determining the behaviour [*gestures over cards (r), (s) and (t)*] of this function as it goes to a limit [*(d)*] But this and this, these two are the most similar because they're the same. They're limits from both sides [*(d) and (r)*] This is the limit of the function. [*(d)*] This is the same as the limit as x tends to one from above [*(t)*] and this is like the, apart from, apart from this is like a step discontinuity so [*(s)*] ah no, this also has a step discontinuity. [*(r)*]

The student appeared to have co-ordinated his understanding of the sequence and function situation, at least in seeing that the behaviour of the sequences which are the arguments for f in cards (r) and (t) are different and that this appears to be related to the limit of the function at 1. However, as with many other students, he seemed to argue that the convergence of $f(a_n)$ for a sequence which converges to a from above is equivalent to the convergence of f at a.

4.5 Neighbourhoods

We saw in section 4.3 that some students did appear to use some neighbourhood reasoning when comparing limits of functions at infinity and limits of functions at a point; apparently mapping neighbourhoods of zero to neighbourhoods of infinity. The notion of neighbourhood came up more often and clearly when students came to consider the meaning of the 'limit of a sequence at a point'.

For the most part, students dismissed the expression

$$\lim_{n\to 12} a_n \text{ where } a_n = \begin{cases} 28 & n = 12 \\ \frac{1}{n} & n \neq 12 \end{cases}$$

as nonsense, or simply treated n as a real variable. But others, as they tried to make sense of it in relation to the other cards, made explicit reference to neighbourhood notions:

S14: The limit of n towards twelve, that's really strange. Sorry, you're right. I appreciate the intricacy of this. It's n towards 12. That's weird because n basically takes values if you have a line, it goes 10, 11, 12, so it would very much depend on what you define as a limit, erm, I just don't think that expression makes any sense. Because when you're considering a limit, you shouldn't consider the value at n equals 12, but at the same time, you've got no, it's not defined on an open subset around 12, unless you go all the way to 11.

This student tried to make sense of the limit of a sequence at a point in terms of neighbourhoods or open sets which he appeared to have adopted from his understanding of the limit of a function at a point. In both cases he dismiss the expression as meaningless.

5 Discussion

The aim of the study was to investigate what understandings students might hold about the three basic uses of limit: limits of functions at infinity, limits of functions at a point and limits of sequences. In particular, we wanted to see whether they see them as manifestations of a unified concept, distinct concepts with links between them or as disjoint concepts. There is little doubt that some students made useful and powerful links between different uses of limit. While many categorised cards according to syntactical similarity or non-analytic concepts such as a personal typology of functions, they all showed, to differing extents, awareness of analytic properties. For the most part, however, each different limit situation was treated separately: cards which might be compared with each other on the basis of superficial similarity were evaluated as limits of sequences or functions and the equality or difference of their limit values seen as a curiosity.

In some cases, however, connections between different types of limit were made. Not all of these fit with formal mathematics. For example, many equated limits of function in x as x tends to infinity, with limits of the function in $\frac{1}{x}$ as x tends to zero. Others equated the limit of a function with the function evaluated at the limit of the argument. Quite commonly, an instance of $a_n \to a$ with $f(a_n) \to l$ was seen as a guarantee of the limit of f at a, with only one student attempting a logical relationship between different limit types.

There was only occasional evidence of students having a unified limit concept through the notion of neighbourhood: in the case of linking x in the neighbourhood of zero with $\frac{1}{x}$ in the neighbourhood of infinity (which, in these cases, presumed $x > 0$) and when students attempted to make sense of a limit of a sequence at a point, where they attempted to generalise their notions from limit of a function at a point.

Of course, many ideas from the literature were reflected in the language used by the students: certainly many students focused on surface features, matching the findings of McDonald et al. (2000). However, as noted above, the students in our study were asked to group and regroup the cards and none of them *only* used surface features. They were all able to shift attention to analytic properties of the objects represented on the cards: both static and dynamic notions of infinity were apparent (Sierpínska, 1987); while some students appeared to use co-ordinated processes of $x \to a$ and $f(x) \to l$ (Cottrill et al., 1996), others did not and we saw many of the metaphors listed by Oehrtman (2009) including closeness, infinity-as-number and approximation.

The task the students were given, of sorting and comparing cards, involves deciding on some similarity/difference criteria on the basis of meaning made of the symbols on the card and classifying according to those criteria. Nosofsky (1986) argues that such a classification requires the "selective attention to component dimensions" [p. 53].

We can consider the expressions on the cards as both a collection of separate symbols, but also as a single, compound symbol. This compound limit symbol was introduced by Weierstrass and developed into its modern form by Hardy. It can be thought of as having two unvarying components and three varying ones, illustrated in figure 1. The two constant components are the letters "lim" (1) and the arrow (2) (which Font, Bolite, & Acevedo, 2010 argue is a fossilised metaphor for dynamic notions of movement).

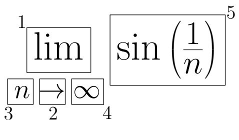

Fig. 1 The limit symbol as a compound of sub symbols

The three varying components are the variable (3), the limit indicator (4) and the expression (5). The variable carries with it conventions which, while sometimes confusing, seemed well understood amongst the students — that n indicates a natural number and x a real. The limit indicator — the point at which the limit is to be evaluated — is normally either a number (given explicitly or as variable-as-a-general-number in the sense of Trigueros & Ursini, 2003) or ∞.

The expression usually involves the variable: if not, students may consider the compound symbol as being trivial (as in $\lim_{x\to 3} 4$) or as improperly formed (as in $\lim_{x\to 3} 4n+1$, where the variable n is unbound). Indeed, one student in our sample takes the view that limit notions don't apply in some contexts, such as constant expressions: looking to classify $\lim_{x\to 1}(x+1)$ he said "Part of the function has a limit, but the other part doesn't ... the x has a limit, but the 1 doesn't". Of course, the expression (5) might itself be further composed of multiple symbols.

In making sense of the compound symbol, one could attend in various degrees to subsets of these components. At the simplest level, one may attend predominately to just one component. Those attending predominately to these may categorise according to 'surface' features: cards belong to the same group if they share the same variable, share the same limit indicator or share the same expression (with or without recognition of the independence of unbound variables).

However, shifting focus to the variable and limit indicator might lead one to focus on the nature of limit being considered: n and ∞ as the limit of a sequence; x and ∞ as the limit of a function at infinity; x and a finite limit indicator as the limit of a function at a point and n and a finite limit indicator as ill-defined. Thus this focus may lead to a categorisation according to these distinct limit types. This seems to have been one of the main approaches to thinking about the cards amongst the students in this study: these parts of the compound limit symbol identify distinct types of limit which are approached in distinct ways.

It appears that only when attention was drawn to cards with similar features, but representing different limit types, that students began to make connections. Indeed, it was often evident that some students were attempting to make these connections for the first time: a student working for a long time before recognising that the limit notion was different in the case $\sin(2\pi x)$ and $\sin(2\pi n)$ said "I can't believe I just realised that". Another looking at limits of expressions of the form $f(a_n)$ noted how unsure he was: "I'm changing my mind all over the place". This suggests the students may not have previously thought about different uses of limit.

There are, of course, a number of limitations with this study: the participants were well qualified and had had at least three lecture courses (of standard 'definition-theorem-proof' style) involving analysis. It would only be expected that the connections students do, or do not, make, will be heavily influenced by the particular teaching they have encountered, but the approach encountered by these students is typical of teaching analysis in the UK, as well as many other countries. Similarly, the stimuli used were deliberately chosen to have one consistent representation (the symbolic) and it may be that graphical or other representations might have exposed different ways students connect limit notions. One limitation of phenomenography (as with other inductive approaches) is that we can never be certain that we have uncovered all possible ways in which the phenomena must be experienced. However, we have shown ways in which the limit notions *might* be experienced, particularly for students with similar backgrounds.

In addition, this research was not designed to correlate different views of different limits against each other, against whether people do or don't visualise (in the way Alcock and Simpson (2004, 2005) did for learners' beliefs) or against ability. The study wasn't designed to identify how students' images of different types of limit may be affected by teaching, as various authors have for single notions of limit for example Sierpínska (1987); Cottrill et al. (1996); Roh (2008). That research still needs to be done.

However, this research suggests a lack of an overarching mechanism for drawing together different uses of limit amongst a group of high achieving students and so does, at least, allow some initial suggestions for curricular change. The findings may be manifestations of even more fundamental concerns relating to students' understanding of the completeness of the reals and of continuity. However, the curriculum these students are exposed to introduces limit at three different points and makes few connections between them. It certainly appears that some students are able to adapt neighbourhood notions to unfamiliar limit situations and it may be that, as teachers, we need to make links more explicit or to provide a framework like 'neighbourhood' as an organising principle across first encounters with different limit notions, as appears to be the case in some other curricula, see Przenioslo (2004). The approach which emerged in Swinyard (2011) where his students defined limit in one context to make sense of it in another, may suggest tasks for larger classes. Introducing organising metaphors such as 'nearness' and 'farness', focussing on what it means to quantify over sequences in the definition of limit of a function at a point (if that is the definition introduced) and drawing students' attention explicitly to the common and unique features of different limit objects may not only help them link these notions, but may help them overcome the obstacles they face when they encounter yet more sophisticated limit ideas in later contexts.

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