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ALGEBRAICITY OF L -VALUES FOR ELLIPTIC CURVES IN A FALSE TATE CURVE TOWER

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ABSTRACT. Let E be an elliptic curve over \mathbb{Q} , and τ an Artin representation over \mathbb{Q} that factors through the non-abelian extension $\mathbb{Q}(\sqrt[p]{m}, \mu_{p^n})/\mathbb{Q}$, where p is an odd prime and n, m are positive integers. We show that $L(E, \tau, 1)$, the special value at $s = 1$ of the L -function of the twist of E by τ , divided by the classical transcendental period $\Omega_+^{d^+} |\Omega_-^{d^-}| \epsilon(\tau)$ is algebraic and Galois-equivariant, as predicted by Deligne's conjecture.

1. INTRODUCTION

Recently, there has been great interest in generalising classical Iwasawa theory of elliptic curves to a non-commutative setting. That is, instead of considering cyclotomic extensions of number fields, one considers infinite extensions K_∞/K , whose Galois group is a compact p -adic Lie group (see [4]). A fundamental concept in the theory is that of a p -adic L -function. Roughly speaking, this should interpolate the special values at $s = 1$ of the L -functions of twists of the elliptic curve E by Artin representations that factor through K_∞ .

One of the most basic non-abelian p -adic Lie extension is the false Tate curve extension over \mathbb{Q} , that is

$$\mathbb{Q}_{FT} = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{p^n}, \sqrt[p]{m}),$$

where p is an odd prime, $m > 1$ is an integer that is not a p -th power, and where we write μ_k for the set of k -th roots of 1. The Galois group of \mathbb{Q}_{FT} over \mathbb{Q} is a semidirect product of \mathbb{Z}_p^* by \mathbb{Z}_p . In this paper, we prove Deligne's period conjecture (see [5]) for the L -function of an elliptic curve over \mathbb{Q} twisted by any Artin representation that factors through \mathbb{Q}_{FT} (Theorem 4.2). In particular, we show (Corollary 4.3) that if E is an elliptic curve over \mathbb{Q} , and if F is a number field contained in \mathbb{Q}_{FT} , then

$$\frac{L(E/F, 1) \sqrt{|\Delta_F|}}{\Omega_+(E)^{r_1+r_2} |\Omega_-(E)|^{r_2}} \in \mathbb{Q},$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of F , Δ_F is the discriminant of F , and Ω_\pm are the usual periods attached to E (see §2.2). Note that this is precisely the quotient that appears in the Birch–Swinnerton-Dyer conjecture.

The essential characteristic of the false Tate curve extension is that every irreducible Artin representation is induced from a 1-dimensional representation over $\mathbb{Q}(\mu_{p^n})$, for some $n \geq 0$ (see [6]). Note that, for $n > 0$, this is a CM field, i.e. it is a totally imaginary quadratic extension of the totally real field $\mathbb{Q}(\mu_{p^n})^+$. This enables us to write the L -function of the twist of E by such an irreducible Artin

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representation as a Rankin-Selberg product over $\mathbb{Q}(\mu_{p^n})^+$. In [12], Shimura has established results concerning the algebraicity of the corresponding L -values. The periods appearing in Shimura's work involve the Petersson inner product of Hilbert modular forms, while the periods in non-commutative Iwasawa theory involve the classical periods $\Omega_{\pm}(E)$. The main issue in the proof of Theorem 4.2 is how to relate these periods (Theorem 3.4).

2. NOTATION AND BACKGROUND RESULTS

2.1. Fields and Artin representations. We fix, once and for all, an embedding of $\overline{\mathbb{Q}}$ into \mathbb{C} , and an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_l$ and of $\overline{\mathbb{Q}}_l$ into \mathbb{C} for each prime l , in such a way that the composition $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_l \rightarrow \mathbb{C}$ agrees with $\overline{\mathbb{Q}} \rightarrow \mathbb{C}$.

For every prime v of a number field K , we fix a copy of the decomposition group at v . We write I_v for the inertia group at v , and Φ_v for a geometric Frobenius at v (i.e. its image modulo I_v is the inverse of arithmetic Frobenius).

An Artin representation ρ over a number field K will always be assumed to take algebraic values, i.e. $\rho : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{GL}_n(\overline{\mathbb{Q}})$. We write $\mathbb{Q}(\rho)$ for the finite Galois extension of \mathbb{Q} generated by the coefficients of ρ . We will abuse notation and also write ρ for $\rho \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_l$, the representation with scalars extended to $\overline{\mathbb{Q}}_l$ for a prime l .

If χ is a Hecke character of finite order over K , we also write χ for the corresponding ideal and Galois characters. (For the latter, we set our local reciprocity maps in unramified extensions to take a uniformiser to the geometric Frobenius element.)

We fix the following notation:

Δ_K	absolute discriminant of the number field K .
$\Delta_{K/K'}$	relative discriminant of K over K' .
\mathbb{Q}_{ab}	maximal abelian extension of \mathbb{Q} .
\mathbb{Q}_{ab}^{Σ}	for a finite set of rational primes Σ , this is the maximal abelian extension of \mathbb{Q} , unramified outside $\Sigma \cup \{\infty\}$.
K^+	for an abelian extension K of \mathbb{Q} , this is its maximal real subfield.
ρ^{\pm}	subspace of the Artin representation ρ over \mathbb{Q} on which complex conjugation acts by ± 1 . Its dimension does not depend on the choice of the complex conjugation element.
ρ^{σ}	for $\rho : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{GL}_n(\overline{\mathbb{Q}})$ an Artin representation over K , and $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this is the Artin representation $\sigma \circ \rho$, i.e. σ acts on the coefficients of ρ .

2.2. Elliptic curves and their L -functions. Let E be an elliptic curve over \mathbb{Q} . We write $N(E)$ for the conductor of E . Fix a minimal Weierstrass equation for E , and write ω_E for the Néron differential. Pick a generator γ_+ (respectively γ_-) of the subspace of $H_1(E(\mathbb{C}), \mathbb{Z})$ on which complex conjugation acts trivially (respectively by -1). Set

$$\Omega_+(E) = \int_{\gamma_+} \omega_E \quad \Omega_-(E) = \int_{\gamma_-} \omega_E.$$

Classically, one chooses γ_{\pm} so that $\Omega_+(E)$ and $-i\Omega_-(E)$ are positive real numbers. We write $T_{\ell}(E)$ for the ℓ -adic Tate module of E , and set $V_{\ell}(E) = T_{\ell}(E) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$ and $H_{\ell}^1(E) = \text{Hom}(V_{\ell}(E), \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}_{\ell}} \overline{\mathbb{Q}}_{\ell}$.

We now define the Euler factors of E , ρ and the twist of E by ρ as follows. For a prime v of K , set

$$\begin{aligned} P_v(E/K, X) &= \det(1 - X\Phi_v|H_\ell^1(E)^{I_v}), \\ P_v(\rho, X) &= \det(1 - X\Phi_v|\rho^{I_v}), \\ P_v(E/K, \rho, X) &= \det(1 - X\Phi_v|(H_\ell^1(E) \otimes_{\mathbb{Q}_\ell} \rho)^{I_v}), \end{aligned}$$

where ℓ is any prime not divisible by v . These definitions are independent of the choice of ℓ . Moreover, the coefficients of $P_v(E/K, X)$ are integral, and those of $P_v(\rho, X)$ and $P_v(E/K, \rho, X)$ lie in $\mathbb{Q}(\rho)$.

We define the L -functions

$$\begin{aligned} L(E/K, s) &= \prod_v P_v(E/K, (N_{K/\mathbb{Q}}v)^{-s})^{-1}, \\ L(\rho, s) &= \prod_v P_v(\rho, (N_{K/\mathbb{Q}}v)^{-s})^{-1}, \\ L(E/K, \rho, s) &= \prod_v P_v(E/K, \rho, (N_{K/\mathbb{Q}}v)^{-s})^{-1}, \end{aligned}$$

where the products are taken over the primes of K . These Euler products converge for $\operatorname{Re}(s)$ sufficiently large. The L -function of ρ is known to have meromorphic continuation to \mathbb{C} . The L -functions of E and its twists are conjectured to have analytic continuation to \mathbb{C} .

2.3. Modular forms. For Hilbert modular forms we follow the conventions of Shimura's article [12]. Let F be a totally real field, abelian over \mathbb{Q} . Let \mathbf{g} be a Hilbert modular form of parallel weight l , level \mathbf{n} and character ϕ . We write $L(\mathbf{g}, s) = \sum_{\mathbf{m}} C(\mathbf{m}, \mathbf{g})N(\mathbf{m})^{-s}$ for the Dirichlet series attached to \mathbf{g} , where \mathbf{m} will always be supposed to run over the non-zero integral ideals of \mathcal{O}_F . For a Hecke character $\chi : \mathbf{A}_F^*/F^* \rightarrow \mathbb{C}^*$, we write

$$D(\mathbf{g}, \chi, s) = \sum_{\mathbf{m}} \chi(\mathbf{m})C(\mathbf{m}, \mathbf{g})N(\mathbf{m})^{-s}.$$

Let \mathbf{f} be a cusp form of parallel weight k , level \mathbf{n} and character ψ . We write $\mathbf{D}(\mathbf{f}, \mathbf{g}, s)$ for the Rankin-Selberg product of \mathbf{f} and \mathbf{g} , which is defined by

$$\mathbf{D}(\mathbf{f}, \mathbf{g}, s) = L_{\mathbf{n}}(\psi\phi, 2s + 2 - k - l) \sum_{\mathbf{m}} C(\mathbf{m}, \mathbf{f})C(\mathbf{m}, \mathbf{g})N(\mathbf{m})^{-s},$$

where $L_{\mathbf{n}}(\psi\phi, s)$ denotes the classical L -function over F of $\psi\phi$, with the Euler factors at the primes dividing \mathbf{n} removed. Whenever we do not specify the level of \mathbf{f} and \mathbf{g} , we always consider \mathbf{n} to be least common multiple of their levels. We write $\langle \mathbf{f}, \mathbf{g} \rangle_F$ for the Petersson inner product of \mathbf{f} and \mathbf{g} , normalised as in [12] §2.

For an elliptic curve E over \mathbb{Q} of conductor $N(E)$, we write f_E for the associated rational new form (over \mathbb{Q}) of weight 2, trivial character and level $N(E)$ (using the modularity of elliptic curves, [15, 14, 3]). We write \mathbf{f}_E for the base change of f_E to F . Such a base change exists as F/\mathbb{Q} is abelian, [7, 1]. It is a Hilbert modular form that is a new form of parallel weight 2, trivial character, and of conductor dividing $N(E)\mathcal{O}_F$.

Remark 2.1. Note that, following Shimura's terminology in [12], \mathbf{f}_E is primitive, in the sense that it is a normalised new form of some level.

For a Hilbert modular form \mathbf{g} , we write $\mathbb{Q}(\mathbf{g})$ for the extension of \mathbb{Q} generated by the $C(\mathbf{m}, \mathbf{g})$, defined above. For E an elliptic curve over \mathbb{Q} , the field $\mathbb{Q}(\mathbf{f}_E)$ is just \mathbb{Q} . If $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and \mathbf{g} is a Hilbert modular form of parallel weight and with $C(\mathbf{m}, \mathbf{g}) \in \overline{\mathbb{Q}}$, we write \mathbf{g}^σ for the Hilbert modular form which has $C(\mathbf{m}, \mathbf{g}^\sigma) = C(\mathbf{m}, \mathbf{g})^\sigma$.

Remark 2.2. If \mathbf{f}_E comes from an elliptic curve over \mathbb{Q} , and if \mathbf{g} corresponds to a 2-dimensional Artin representation ρ over F , the Rankin-Selberg product $D(\mathbf{f}_E, \mathbf{g}, s)$ is the L -function $L(E/F, \rho, s)$ up to a finite number of Euler factors. It has an Euler product

$$D(\mathbf{f}_E, \mathbf{g}, s) = \prod_v \det(1 - (N_{F/\mathbb{Q}v})^{-s} \Phi_v |H_\ell^1(E)^{I_v} \otimes \rho^{I_v})^{-1}.$$

The Euler factor at v can only differ from that of $L(E, \rho, s)$ when both $H_\ell^1(E)$ and ρ are ramified at v . Moreover, the functions $D(\mathbf{f}_E, \mathbf{g}, s)$ and $L(E, \rho, s)$ coincide provided that at every prime of bad reduction of E ,

$$(H_\ell^1(E) \otimes \rho)^{I_v} = H_\ell^1(E)^{I_v} \otimes \rho^{I_v}.$$

There is a similar Euler product if we replace \mathbf{g} by a character χ of finite order. Thus, once again, $L(E/F, \chi, s)$ may differ from $D(\mathbf{f}_E, \chi, s)$ at finitely many Euler factors, and there is a similar criterion for when $D(\mathbf{f}_E, \chi, s) = L(E, \chi, s)$.

Remark 2.3. With notation as in Remark 2.2, the functions $D(\mathbf{f}_E, \mathbf{g}, s)$ and $D(\mathbf{f}_E, \chi, s)$ are analytic on \mathbb{C} (see [12] Proposition 4.13). As the local polynomials $P_v(E/F, \rho, X)$ and $P_v(E/F, \chi, X)$ do not vanish at $X = (N_{F/\mathbb{Q}v})^{-1}$, it follows that the L -functions $L(E/F, \rho, s)$ and $L(E/F, \chi, s)$ are meromorphic on \mathbb{C} , and analytic at $s = 1$.

2.4. Gauss sums and ϵ -factors. Let K be a number field, and ρ an Artin representation over K . We write $\epsilon_K(\rho)$ for the global ϵ -factor of ρ (see [13, 5]). Recall that it can be written as a product of local ϵ -factors. We write $\epsilon_{K, \infty}(\rho)$ for the contribution from the archimedean places of K , where we have chosen the standard additive character and measure at these places (i.e. $\exp(2\pi i x), dx$ for real places and $\exp(2\pi i \text{Tr}_{\mathbb{C}/\mathbb{R}} z), |dz \wedge d\bar{z}|$ for complex ones). Global ϵ -factors are inductive, in the sense that, whenever K/L is a finite extension and ρ an Artin representation over K ,

$$\epsilon_K(\rho) = \epsilon_L(\text{Ind } \rho).$$

For ψ a character over K of finite order we define

$$\tau_K(\psi) = \sqrt{|\Delta_K|}^{-1} \frac{\epsilon_K(\psi)}{\epsilon_{K, \infty}(\psi)}.$$

This is Tate's $\epsilon_1(\psi)$ (see [13] §3.6), with the local epsilon factors at infinity removed. The quantity τ_K coincides with that used by Shimura in [12], where τ_K is defined for totally real fields K .

Remark 2.4. With our conventions, the local ϵ -factor at a complex place of an Artin representation is 1. In particular, if K is totally imaginary, then $\epsilon_{K, \infty}(\psi) = 1$. The local ϵ -factor at a real place v of an Artin representation ρ is $i^{\dim \rho^-}$, where ρ^- is the subspace on which complex conjugation at v acts by -1 .

We have already mentioned that ϵ_K is inductive. On the other hand, τ_K satisfies the following Galois equivariance property:

Lemma 2.5. (see also [12] Lemma 4.12) *Let K be a number field, and let ψ_1, ψ_2 be characters over K of finite order. Then for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\frac{\tau_K(\psi_1)^\sigma \tau_K(\psi_2)^\sigma}{\tau_K(\psi_1 \psi_2)^\sigma} = \frac{\tau_K(\psi_1^\sigma) \tau_K(\psi_2^\sigma)}{\tau_K(\psi_1^\sigma \psi_2^\sigma)}.$$

Proof. The formula holds prime-by-prime when we view τ_K as a product of local non-archimedean ϵ -factors (see [13] (3.2.3) and (3.2.6)). \square

2.5. Results from the theory of Hilbert modular forms. The following two results from the theory of modular forms, concerning the special values of L -functions, are fundamental to our study.

Theorem 2.6. (Shimura [12]) *Let F be a totally real field. Write $d = [F : \mathbb{Q}]$. Let \mathbf{f} be a primitive Hilbert modular form of parallel weight 2 and with trivial character. Let \mathbf{g} be a Hilbert modular of parallel weight 1 and character ϕ . Define*

$$T(\mathbf{f}, \mathbf{g}) = \frac{D(\mathbf{f}, \mathbf{g}, 1)}{(2i)^d \pi^{3d} \langle \mathbf{f}, \mathbf{f} \rangle_F \tau_F(\phi)}.$$

Then $T(\mathbf{f}, \mathbf{g}) \in \overline{\mathbb{Q}}(\mathbf{g})$. Moreover, for every $\sigma \in \text{Aut}(\mathbb{C})$,

$$T(\mathbf{f}, \mathbf{g})^\sigma = T(\mathbf{f}^\sigma, \mathbf{g}^\sigma).$$

Theorem 2.7. (Shimura [11], see also Birch [2]) *Let E be an elliptic curve over \mathbb{Q} , and let $\psi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{C}^*$ be a Dirichlet character. Define*

$$S(E, \psi) = \frac{L(E/\mathbb{Q}, \psi, 1)}{\Omega_{\text{sign}(\psi)}(E) \tau_{\mathbb{Q}}(\psi)}.$$

Then $S(E, \psi) \in \overline{\mathbb{Q}}$. Moreover, for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$S(E, \psi)^\sigma = S(E, \psi^\sigma).$$

3. PERIODS OF ELLIPTIC CURVES AND THE PETERSSON INNER PRODUCT

In this section we establish a relation between the quantity $\Omega_+ \Omega_-$, which appears in the Birch–Swinnerton-Dyer conjecture and its generalisations, and $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F$, which appears in the theory of Hilbert modular forms (Theorem 3.4).

The method is the following. Suppose that E/\mathbb{Q} is an elliptic curve, K an abelian extension of \mathbb{Q} not contained in \mathbb{R} , and F its maximal totally real subfield. Theorem 2.6 gives an expression for $L(E/K, 1)$ involving the Petersson inner product $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F$. On the other hand, we can decompose the L -function $L(E/K, 1)$ as a product of L -functions over \mathbb{Q} of 1-dimensional twists of E . Theorem 2.7 then provides an expression for $L(E/K, 1)$ involving the classical periods $\Omega_\pm(E)$.

The problem with the above method as it stands, is that the L -value $L(E/K, 1)$ may be zero. To correct this, we replace $L(E/K, s)$ in the above procedure by $L(E/K, \chi, s)$, for a suitable 1-dimensional Artin representation χ over K . Our proof uses the following well-known result of Rohrlich,

Theorem 3.1. (Rohrlich [8, 9]) *Let E be an elliptic curve over \mathbb{Q} . Let Σ be a finite set of primes. Then for all but finitely many Dirichlet characters ψ unramified outside Σ and infinity,*

$$L(E/\mathbb{Q}, \psi, 1) \neq 0.$$

Corollary 3.2. *Let E be an elliptic curve over \mathbb{Q} . Let Σ be a finite set of primes of \mathbb{Q} . Let $F \subset \mathbb{Q}_{ab}^\Sigma$. Then there exist an extension F_0/F , with $F_0 \subset \mathbb{Q}_{ab}^\Sigma$, such that for every character $\psi : \text{Gal}(\mathbb{Q}_{ab}^\Sigma/F) \rightarrow \mathbb{C}^*$ of finite order that does not factor through F_0/F ,*

$$L(E/F, \psi, 1) \neq 0.$$

Proof. Let $S = \{\phi_1, \dots, \phi_k\}$ be the set of all Dirichlet characters over \mathbb{Q} that are unramified outside $\Sigma \cup \{\infty\}$ and satisfy $L(E/\mathbb{Q}, \phi_j, 1) = 0$. Note that by Rohrlich's theorem 3.1, this set is finite.

Let us pick $F \subset F_0 \subset \mathbb{Q}_{ab}^\Sigma$ such that all the ϕ_j factor through F_0 . We claim that this field satisfies the conclusion of the corollary. Indeed, let $\psi : \text{Gal}(\mathbb{Q}_{ab}^\Sigma/F) \rightarrow \mathbb{C}^*$ be a character of finite order that does not factor through F_0/F . Then, by the inductive properties of L -functions,

$$L(E/F, \psi, s) = \prod_j L(E/\mathbb{Q}, \psi_j, s),$$

where the $\bigoplus_j \psi_j$ is the representation of $\text{Gal}(\mathbb{Q}_{ab}^\Sigma/\mathbb{Q})$ induced from ψ . By Frobenius Reciprocity, the ψ_j are precisely the characters of $\text{Gal}(\mathbb{Q}_{ab}^\Sigma/\mathbb{Q})$ that restrict to ψ . In particular, none of them factor through F_0/\mathbb{Q} , and hence $\psi_j \notin S$ for every j . By the definition of S we conclude that

$$L(E/F, \psi, 1) = \prod_j L(E/\mathbb{Q}, \psi_j, 1) \neq 0.$$

□

Corollary 3.3. *Let E be an elliptic curve over \mathbb{Q} . Let F be any abelian extension of \mathbb{Q} . Let $\epsilon : \text{Gal}(\mathbb{Q}_{ab}/F) \rightarrow \mathbb{C}^*$ be any character of finite order. Then there exists a character of finite order $\tilde{\chi} : \text{Gal}(\mathbb{Q}_{ab}/F) \rightarrow \mathbb{C}^*$, such that for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have*

$$L(E/F, \tilde{\chi}^\sigma, 1) \neq 0, \quad L(E/F, \tilde{\chi}^\sigma \epsilon, 1) \neq 0$$

and

$$L(E/F, \tilde{\chi}^\sigma, s) = D(\mathbf{f}_E, \tilde{\chi}^\sigma, s), \quad L(E/F, \tilde{\chi}^\sigma \epsilon, s) = D(\mathbf{f}_E, \tilde{\chi}^\sigma \epsilon, s).$$

Proof. Let Σ be a set of primes containing all primes of bad reduction of E , the divisors of Δ_F , and the primes of \mathbb{Q} below those dividing the conductor of ϵ . Let F_0 be the field given by Corollary 3.3 for the field F and the set Σ .

Pick a character $\tilde{\chi} : \text{Gal}(\mathbb{Q}_{ab}^\Sigma/F) \rightarrow \mathbb{C}^*$, such that for every prime v of F above Σ ,

$$\text{ord}_v N(\tilde{\chi}) > \max(\text{ord}_v(\Delta_{F_0/F}), \text{ord}_v N(\epsilon), \text{ord}_v N(E/F)),$$

where $N(\xi)$ denotes the conductor of ξ . Such a character can always be found in $F(\mu_m)/F$ for $m = \prod_{p \in \Sigma} p^{k_p}$, with the k_p sufficiently large (see e.g. [10], chapter IV).

We now have, for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and every such v ,

$$\text{ord}_v N(\tilde{\chi}^\sigma) = \text{ord}_v N(\tilde{\chi}^\sigma \epsilon) > \max(\text{ord}_v(\Delta_{F_0/F}), \text{ord}_v N(E/F))$$

Therefore, for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the characters $\tilde{\chi}^\sigma$ and $\tilde{\chi}^\sigma \epsilon$ do not factor through F_0/F . Moreover, for every prime v of F where E has bad reduction,

$$(H_l(E) \otimes \tilde{\chi}^\sigma)^{I_v} = 0 = H_l(E)^{I_v} \otimes (\tilde{\chi}^\sigma)^{I_v}$$

and

$$(H_l(E) \otimes \tilde{\chi}^\sigma \epsilon)^{I_v} = 0 = H_l(E)^{I_v} \otimes (\tilde{\chi}^\sigma \epsilon)^{I_v}.$$

Thus $\tilde{\chi}$ satisfies the requirements of the corollary. \square

Theorem 3.4. *Let E be an elliptic curve over \mathbb{Q} . Let F be a totally real field, with F/\mathbb{Q} abelian. Write $d = [F : \mathbb{Q}]$. Then*

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} \in \mathbb{Q}.$$

Proof. Pick $K \subset \mathbb{Q}_{ab}$ to be any totally imaginary quadratic extension of F (for instance $K = F(i)$ will do). Write ϵ for the quadratic character of K/F . Take $\tilde{\chi} : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \mathbb{C}^*$ to be the character of finite order provided by Corollary 3.3 in this setting.

We write χ for the restriction of $\tilde{\chi}$ to $\text{Gal}(\overline{\mathbb{Q}}/K)$. Note that the induction of χ to $\text{Gal}(\overline{\mathbb{Q}}/F)$ is $\tilde{\chi} \oplus \tilde{\chi}\epsilon$. We let \mathbf{g}_χ be the Hilbert modular form of parallel weight 1 and character $\epsilon \tilde{\chi}^2$ that is the automorphic induction of χ , see e.g. [12] §5. (This is the Hilbert modular form associated to the Artin representation $\tilde{\chi} \oplus \tilde{\chi}\epsilon$.)

With notation as in Theorem 2.6, with $\mathbf{f} = \mathbf{f}_E$ and $\mathbf{g} = \mathbf{g}_\chi$,

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = T(\mathbf{f}_E, \mathbf{g}_\chi)(2i)^d \pi^{3d} \tau_F(\tilde{\chi}^2 \epsilon) \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F.$$

By our choice of $\tilde{\chi}$

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = L(E/F, \tilde{\chi} \oplus \tilde{\chi}\epsilon, 1) = L(E/F, \tilde{\chi}, 1)L(E/F, \tilde{\chi}\epsilon, 1).$$

Let $\theta_1^{\tilde{\chi}}, \dots, \theta_d^{\tilde{\chi}}$ and $\theta_1^{\tilde{\chi}\epsilon}, \dots, \theta_d^{\tilde{\chi}\epsilon}$ be the characters of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that restrict to $\tilde{\chi}$ and $\tilde{\chi}\epsilon$, respectively. In particular, by Frobenius reciprocity, the induction of $\tilde{\chi}$ to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is $\bigoplus_j \theta_j^{\tilde{\chi}}$, and similarly for $\tilde{\chi}\epsilon$. Inductive properties of L -functions allow us to rewrite the above formula as

$$D(\mathbf{f}_E, \mathbf{g}_\chi, 1) = \prod_j L(E/\mathbb{Q}, \theta_j^{\tilde{\chi}}, 1) \prod_j L(E/\mathbb{Q}, \theta_j^{\tilde{\chi}\epsilon}, 1).$$

We obtain, using the notation of 2.7,

$$\frac{T(\mathbf{f}_E, \mathbf{g}_\chi) \tau_F(\tilde{\chi}^2 \epsilon) \cdot (2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{\prod_j \Omega_{\text{sign}(\theta_j^{\tilde{\chi}\epsilon})}(E) \Omega_{\text{sign}(\theta_j^{\tilde{\chi}})}(E)} = \prod_j S(E, \theta_j^{\tilde{\chi}}) \tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}}) \prod_j S(E, \theta_j^{\tilde{\chi}\epsilon}) \tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}\epsilon}).$$

Note that, by the choice of $\tilde{\chi}$, the above does not read $0 = 0$, since the corresponding L -functions do not vanish at $s = 1$. We rewrite the above as

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} = \frac{\prod_j \tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}}) \tau_{\mathbb{Q}}(\theta_j^{\tilde{\chi}\epsilon})}{\tau_F(\tilde{\chi}^2 \epsilon)} \frac{\prod_j S(E, \theta_j^{\tilde{\chi}}) S(E, \theta_j^{\tilde{\chi}\epsilon})}{T(\mathbf{f}_E, \mathbf{g}_\chi)},$$

where we have used the fact that exactly d of $\theta_1^{\tilde{\chi}}, \dots, \theta_d^{\tilde{\chi}}, \theta_1^{\tilde{\chi}\epsilon}, \dots, \theta_d^{\tilde{\chi}\epsilon}$ have sign $+1$.

Using the definition of τ , the inductive properties of global epsilon factors and the formula for the local ϵ -factor at a real place (see Remark 2.4), we obtain

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} = |\Delta_F| \frac{\tau_F(\tilde{\chi}) \tau_F(\tilde{\chi}\epsilon) \prod_j S(E, \theta_j^{\tilde{\chi}}) S(E, \theta_j^{\tilde{\chi}\epsilon})}{\tau_F(\tilde{\chi}^2 \epsilon) T(\mathbf{f}_E, \mathbf{g}_\chi)}.$$

Let us call the quantity on the right-hand-side $R(\tilde{\chi})$. To obtain the above equality we have only used the fact that $L(E/F, \tilde{\chi} \oplus \tilde{\chi}\epsilon, 1)$ coincides with $D(\mathbf{f}_E, \mathbf{g}_\chi, 1)$, and that this L -value is non-zero. Thus, by our choice of $\tilde{\chi}$, the above equality also holds if we replace $\tilde{\chi}$ by $\tilde{\chi}^\sigma$ for any $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and so we have $R(\tilde{\chi}) = R(\tilde{\chi}^\sigma)$.

By Theorems 2.6 and 2.7, $R(\tilde{\chi}) \in \overline{\mathbb{Q}}$. To prove our theorem, we need to show that $R(\tilde{\chi})$ actually lies in \mathbb{Q} . So let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Lemma 2.5, Theorem 2.6 and Theorem 2.7,

$$\left(\frac{\tau_F(\tilde{\chi})\tau_F(\tilde{\chi}\epsilon)}{\tau_F(\tilde{\chi}^2\epsilon)} \right)^\sigma = \frac{\tau_F(\tilde{\chi}^\sigma)\tau_F(\tilde{\chi}^\sigma\epsilon)}{\tau_F((\tilde{\chi}^2)^\sigma\epsilon)},$$

and

$$\left(\frac{\prod_j S(E, \theta_j^{\tilde{\chi}})S(E, \theta_j^{\tilde{\chi}\epsilon})}{T(\mathbf{f}_E, \mathbf{g}_\chi)} \right)^\sigma = \frac{\prod_j S(E, (\theta_j^{\tilde{\chi}})^\sigma)S(E, (\theta_j^{\tilde{\chi}\epsilon})^\sigma)}{T(\mathbf{f}_E, \mathbf{g}_\chi^\sigma)}.$$

Observe that $\{\theta_1^{\tilde{\chi}^\sigma}, \dots, \theta_d^{\tilde{\chi}^\sigma}\} = \{(\theta_1^{\tilde{\chi}})^\sigma, \dots, (\theta_d^{\tilde{\chi}})^\sigma\}$, and similarly for $\theta_j^{\tilde{\chi}\epsilon}$. Furthermore, as $\mathbf{g}_\chi^\sigma = \mathbf{g}_{\chi^\sigma}$, we deduce that $R(\tilde{\chi})^\sigma = R(\tilde{\chi}^\sigma) = R(\tilde{\chi})$. As σ was arbitrary, it follows that

$$\frac{(2i)^d \pi^{3d} \langle \mathbf{f}_E, \mathbf{f}_E \rangle_F}{(\Omega_+(E)\Omega_-(E))^d} \in \mathbb{Q}.$$

□

Corollary 3.5. *Let E be an elliptic curve over \mathbb{Q} . Let K be an abelian extension of \mathbb{Q} , which is not a subfield of \mathbb{R} . Let $\psi : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \mathbb{C}^*$ be a character of finite order, and write ρ for the Artin representation over \mathbb{Q} induced by ψ . Then*

$$\frac{L(E, \rho, 1)}{\Omega_+(E)^d \Omega_-(E)^d} \in \overline{\mathbb{Q}},$$

where $d = [K : \mathbb{Q}]/2$.

Proof. This follows from Theorems 2.6 and 3.4, and the inductive properties of L -functions. For \mathbf{g} in Theorem 2.6 we take the Hilbert modular form over K^+ induced from ψ (as a Hecke character). Notice that the Rankin-Selberg product $D(\mathbf{f}, \mathbf{g}, s)$ may differ from $L(E, \rho, s)$ only at finitely many Euler factors, that take algebraic values and do not vanish at $s = 1$. □

Remark 3.6. In view of Theorems 2.6 and 2.7, the result in Theorem 3.4 is equivalent to saying that $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F / \langle f_E, f_E \rangle_{\mathbb{Q}}^d \in \mathbb{Q}$. This result can be easily generalised to deal with any primitive cusp form f over \mathbb{Q} that has even weight, trivial character and $C(m, f) \in \mathbb{Q}$. Indeed, the results of Rohrlich and Shimura that we used in the proof apply in this setting. The rôle of Ω_\pm can be played by the quantities u_1^\pm in [11].

4. THE FALSE TATE CURVE EXTENSION

Let p be an odd prime, and let $m > 1$ be an integer, that is not a p -th power. We consider the false Tate curve extension of \mathbb{Q} ,

$$\mathbb{Q}_{FT} = \cup_n \mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n}).$$

We write $K_n = \mathbb{Q}(\mu_{p^n})$. For a discussion of the representation theory in the false Tate curve extension see [6].

Let ρ_n be the irreducible Artin representation over \mathbb{Q} , that is induced from any 1-dimensional representation of $\text{Gal}(K_n(\sqrt[p^n]{m})/K_n)$ of exact order p^n . Every irreducible Artin representation that factors through \mathbb{Q}_{FT} has the form $\rho_n \phi$, where ϕ is a 1-dimensional Artin representation over \mathbb{Q} that factors through K_k , for some $k \geq 0$. If ρ_n is induced from $\chi : \text{Gal}(K_n(\sqrt[p^n]{m})/K_n) \rightarrow \mathbb{C}^*$, then $\rho_n \phi$ is induced from $\chi \text{Res } \phi : \text{Gal}(\overline{\mathbb{Q}}/K_n) \rightarrow \mathbb{C}^*$. Note that, in particular, every irreducible

representation of dimension larger than 1 is induced from a 1-dimensional Artin representation over an abelian extension of \mathbb{Q} , not contained in \mathbb{R} . We immediately obtain from Corollary 3.5 and Theorem 2.7

Corollary 4.1. *Let E be an elliptic curve over \mathbb{Q} . Let p be an odd prime and let $m > 1$ be an integer that is not a p -th power. Let ρ be an Artin representation over \mathbb{Q} that factors through $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})$, for some $n \geq 0$. Then*

$$\frac{L(E, \rho, 1)}{\Omega_+(E)^{\dim \rho^+} \Omega_-(E)^{\dim \rho^-}} \in \overline{\mathbb{Q}}.$$

Our aim is to make a more precise statement about the field of definition of the special value in the above corollary. First, we recall some more terminology.

Let F be a totally real field of degree d , and let K/F be a totally imaginary quadratic extension. We fix an embedding $K \rightarrow \mathbb{C}$, and write c for the complex conjugation element. We say that a character of finite order $\psi : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \mathbb{C}^*$ is cyclotomic if $\psi(cgc) = \psi(g)$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/K)$. We call ψ anticyclotomic if $\psi(cgc) = \psi(g)^{-1}$. Note that when ψ is cyclotomic, it can be extended to a character $\tilde{\psi}$ of $\text{Gal}(\overline{\mathbb{Q}}/F)$.

Write ϵ for the quadratic character of K/F . If ψ is cyclotomic over K , then the representation induced by ψ to F is $\tilde{\psi} \oplus \tilde{\psi}\epsilon$. Moreover, if χ is anticyclotomic and ψ is cyclotomic, then the determinant of the representation induced by $\chi\psi$ to F is $\tilde{\psi}^2\epsilon$.

Theorem 4.2. *Let E be an elliptic curve over \mathbb{Q} . Let p be an odd prime and let $m > 1$ be an integer, that is not a p -th power. Let ρ be an Artin representation over \mathbb{Q} that factors through $\mathbb{Q}(\sqrt[p^n]{m}, \mu_{p^n})$, for some $n \geq 0$. Define*

$$R(E, \rho) = \frac{L(E, \rho, 1)\epsilon(\rho)^{-1}}{\Omega_+(E)^{\dim \rho^+} |\Omega_-(E)|^{\dim \rho^-}}.$$

Then $R(E, \rho) \in \overline{\mathbb{Q}}$. Moreover, for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$R(E, \rho)^\sigma = R(E, \rho^\sigma).$$

In particular,

$$R(E, \rho) \in \mathbb{Q}(\rho).$$

Proof. It is sufficient to prove the Galois-equivariance formula for irreducible representations ρ . If ρ is 1-dimensional, the result follows from Theorem 2.7, observing that in this case $i^{\dim \rho^-} \tau_{\mathbb{Q}}(\rho) = \epsilon_{\mathbb{Q}}(\rho)$.

Otherwise ρ is induced from a character $\chi\phi$ over K_n for some $n \geq 1$, where χ factors through $\text{Gal}(K_n(\sqrt[p^n]{m})/K_n)$ and ϕ factors through $\text{Gal}(K_k/K_n)$, for some $k \geq n$. In this case $\dim \rho^- = \dim \rho^+ = p^{n-1}(p-1)/2$. The field K_n is a quadratic totally imaginary extension of the totally real field $F = K_n^+$. Note that χ is anticyclotomic and ϕ is cyclotomic. We write ϵ for the quadratic character of K_n/F , and $\tilde{\phi}$ for an extension of ϕ to $\text{Gal}(\overline{\mathbb{Q}}/F)$.

Let \mathbf{g} be the Hilbert modular form over F of parallel weight 1 and character $\epsilon\tilde{\phi}^2$ that is the automorphic induction of $\chi\phi$ to F (see e.g. [12] §5). This Hilbert modular form corresponds to the Artin representation $\text{Ind } \chi\phi$ over F . We apply Theorem 2.6 to the pair \mathbf{f}_E, \mathbf{g} . Eliminating the Petersson inner product $\langle \mathbf{f}_E, \mathbf{f}_E \rangle_F$ using Theorem 3.4, we deduce that the expression

$$\frac{D(\mathbf{f}_E, \mathbf{g}, 1)}{\Omega_+(E)^{\dim \rho^+} \Omega_-(E)^{\dim \rho^-} \tau_F(\epsilon\tilde{\phi}^2)}$$

is algebraic and Galois-equivariant. The term $D(\mathbf{f}_E, \mathbf{g}, 1)$ can be replaced by the L -value $L(E, \rho, 1)$ by Remark 2.2 and Lemma 4.4, and the inductive property of L -functions. Finally, the Gauss sum $i^{\dim \rho^-} \tau_F(\epsilon \tilde{\phi}^2)$ can be replaced by $\epsilon_{K_n}(\chi \phi) = \epsilon_{\mathbb{Q}}(\rho)$ by Proposition 4.5. \square

Corollary 4.3. *Let E be an elliptic curve over \mathbb{Q} . Let p be an odd prime and let $m, n > 1$ be integers. For any subfield F of $\mathbb{Q}(\sqrt[n]{m}, \mu_{p^n})$,*

$$\frac{L(E/F, 1) \sqrt{|\Delta_F|}}{|\Omega_+(E)^{r_1+r_2}| |\Omega_-(E)^{r_2}|} \in \mathbb{Q},$$

where r_1 (resp. r_2) is the number of real (resp. complex) places of F .

Proof. Take ρ to be the Artin representation induced from the trivial representation of $\text{Gal}(\overline{\mathbb{Q}}/F)$. Then ρ can be realised over \mathbb{Q} , and $\epsilon(\rho) = \pm \sqrt{|\Delta_F|}$. The result follows from Theorem 4.2. Note that in the resulting formula, $\sqrt{|\Delta_F|}^{-1}$ can be replaced by $\sqrt{|\Delta_F|}$, as their quotient is rational. \square

The Rankin-Selberg product. In the lemma below, we justify the assertion in the proof of Theorem 4.2, that the quotient of the Rankin-Selberg product by the classical L -function is Galois equivariant. More precisely, if E/\mathbb{Q} is an elliptic curve, F a totally real field and ρ a (suitable) two dimensional Artin representation over F , then $\mathbf{D}(\mathbf{f}_E, \mathbf{g}, 1)/L(E/F, \rho, 1)$ is Galois equivariant, where we view this quotient as the finite product over primes where the local Euler factors of \mathbf{D} and L differ (so that it is well-defined even when both L -values are zero.)

Lemma 4.4. *Let K be a number field, and let E be an elliptic curve over K . Let ρ be an Artin representation over K with coefficients in $\overline{\mathbb{Q}}$. Let v be a prime of K , and let ℓ be a rational prime not divisible by v . Then, for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\det(1 - X \Phi_v | (H_\ell^1(E) \otimes \rho)^{I_v})^\sigma = \det(1 - X \Phi_v | (H_\ell^1(E) \otimes \rho^\sigma)^{I_v})$$

and

$$\det(1 - X \Phi_v | (H_\ell^1(E)^{I_v} \otimes \rho^{I_v}))^\sigma = \det(1 - X \Phi_v | (H_\ell^1(E)^{I_v} \otimes (\rho^\sigma)^{I_v})).$$

Proof. Recall that the above characteristic polynomials have algebraic coefficients. Let L be a finite Galois extension of \mathbb{Q} that contains these coefficients, and such that ρ is realised over L . It is clearly sufficient to prove the assertion in the lemma for $\sigma \in \text{Gal}(L/\mathbb{Q})$.

Let $\sigma \in \text{Gal}(L/\mathbb{Q})$. Write M for the subfield of L fixed by σ . The coefficients in the above characteristic polynomials are independent of the choice of ℓ , with $v \nmid \ell$. We can therefore take ℓ to be a rational prime not divisible by v , that splits in M/\mathbb{Q} and is inert in L/M . Such primes exist by Chebotarev's density theorem.

Let α be an element in L with $L = M(\alpha)$. For a fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, write $\tilde{\alpha} \in \overline{\mathbb{Q}}_\ell$ for the image of α . We obtain an explicit isomorphism $\text{Gal}(L/M) \cong \text{Gal}(\mathbb{Q}_\ell(\tilde{\alpha})/\mathbb{Q}_\ell)$. We write σ also for its image in $\text{Gal}(\mathbb{Q}_\ell(\tilde{\alpha})/\mathbb{Q}_\ell)$ under this isomorphism.

If R is any representation of the Weil group at v with coefficients in $\mathbb{Q}_\ell(\tilde{\alpha})$, we have

$$\det(1 - \Phi_v X | (R^\sigma)^{I_v}) = \det(1 - \Phi_v X | R^{I_v})^\sigma.$$

Indeed, letting V be the $\mathbb{Q}_\ell(\bar{\alpha})$ -vector space on which R acts, if v_1, \dots, v_n is a basis of $V^{R(I_v)}$ then $v_1^\sigma, \dots, v_n^\sigma$ is a basis of $V^{R^\sigma(I_v)}$. If the matrix of $R(\Phi_v)$ with respect to v_1, \dots, v_n is (λ_{ij}) , then the matrix of $R^\sigma(\Phi_v)$ with respect to $v_1^\sigma, \dots, v_n^\sigma$ is (λ_{ij}^σ) .

Applying this observation with $R = H_\ell^1(E), \rho$ and $H_\ell^1(E) \otimes \rho$ we obtain the asserted Galois-equivariance. \square

Comparing Gauss sums and ϵ -factors. We now complete the proof of Theorem 4.2 by showing that the quotient of the Gauss sum τ_F , that appears in the definition of $T(E/F, \rho)$, by the ϵ -factor that appears in Theorem 4.2, is Galois equivariant.

Proposition 4.5. *Let F be a totally real field of degree d , and let K be a totally imaginary quadratic extension of F . Write ϵ for the quadratic character of K/F . Let χ be an anticyclotomic character of K , and let ψ be a cyclotomic character. Then, for every $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have*

$$\left(\frac{\epsilon_K(\chi\psi)}{i^d \tau_F(\tilde{\psi}^2\epsilon)} \right)^\sigma = \frac{\epsilon_K(\chi^\sigma\psi^\sigma)}{i^d \tau_F((\tilde{\psi}^\sigma)^2\epsilon)},$$

where $\tilde{\psi}$ denotes an extension of ψ to $\text{Gal}(\bar{\mathbb{Q}}/F)$.

Proof. This is a straight-forward computation using the definition of τ and Lemma 2.5. We have

$$\left(\frac{\epsilon_K(\chi\psi)}{i^d \tau_F(\tilde{\psi}^2\epsilon)} \right)^\sigma = \left(\frac{\sqrt{|\Delta_K|} \tau_K(\chi\psi)}{i^d \tau_F(\tilde{\psi}^2\epsilon)} \right)^\sigma$$

Applying Lemma 2.5, we rewrite the latter expression as

$$\begin{aligned} & \frac{\sqrt{|\Delta_K|}^\sigma}{(i^d)^\sigma \tau_F(\tilde{\psi}^2\epsilon)^\sigma} \frac{\tau_K(\chi^\sigma\psi^\sigma)}{\tau_K(\chi^\sigma)\tau_K(\psi^\sigma)} \tau_K(\chi)^\sigma \tau_K(\psi)^\sigma = \\ & = \frac{\sqrt{|\Delta_K|}^\sigma}{(i^d)^\sigma \sqrt{|\Delta_K|}} \cdot \frac{\tau_F(\tilde{\psi}^\sigma\epsilon) \tau_F(\tilde{\psi}^\sigma)}{\tau_F(\tilde{\psi}\epsilon)^\sigma \tau_F(\tilde{\psi})^\sigma} \cdot \frac{\epsilon_K(\chi^\sigma\psi^\sigma)}{\tau_F((\tilde{\psi}^2)^\sigma\epsilon)} \cdot \frac{\tau_K(\chi)^\sigma}{\tau_K(\chi^\sigma)} \cdot \frac{\tau_K(\psi)^\sigma}{\tau_K(\psi^\sigma)}. \end{aligned}$$

As χ is anticyclotomic, the term $\tau_K(\chi)^\sigma/\tau_K(\chi^\sigma)$ in the above expression is 1. We now rewrite the last term in the formula

$$\begin{aligned} \frac{\tau_K(\psi)^\sigma}{\tau_K(\psi^\sigma)} &= \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}} \frac{\epsilon_K(\psi)^\sigma}{\epsilon_K(\psi^\sigma)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}} \frac{\epsilon_F(\tilde{\psi} \oplus \tilde{\psi}\epsilon)^\sigma}{\epsilon_F(\tilde{\psi}^\sigma \oplus \tilde{\psi}^\sigma\epsilon)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}} \frac{\epsilon_F(\tilde{\psi})^\sigma \epsilon_F(\tilde{\psi}\epsilon)^\sigma}{\epsilon_F(\tilde{\psi}^\sigma) \epsilon_F(\tilde{\psi}^\sigma\epsilon)} = \\ &= \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}} \cdot \frac{\tau_F(\tilde{\psi})^\sigma \tau_F(\tilde{\psi}\epsilon)^\sigma}{\tau_F(\tilde{\psi}^\sigma) \tau_F(\tilde{\psi}^\sigma\epsilon)} \cdot \frac{\epsilon_{F,\infty}(\tilde{\psi})^\sigma \epsilon_{F,\infty}(\tilde{\psi}\epsilon)^\sigma}{\epsilon_{F,\infty}(\tilde{\psi}^\sigma) \epsilon_{F,\infty}(\tilde{\psi}^\sigma\epsilon)} = \frac{\sqrt{|\Delta_K|}^\sigma}{\sqrt{|\Delta_K|}} \cdot \frac{\tau_F(\tilde{\psi})^\sigma \tau_F(\tilde{\psi}\epsilon)^\sigma}{\tau_F(\tilde{\psi}^\sigma) \tau_F(\tilde{\psi}^\sigma\epsilon)} \cdot \frac{(i^d)^\sigma}{i^d}. \end{aligned}$$

The proposition follows. \square

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