Non abelian p-adic L-functions and Eisenstein series of unitary groups

Thanasis Bouganis

In [6, 7] a vast generalization of the Main Conjecture of the classical (abelian) Iwasawa theory to a non-abelian setting was proposed. However, the evidences for this non-abelian Main Conjecture are still very modest. One of the central difficulties of the theory seems to be the construction of non-abelian p-adic L-functions. Actually, the only known results in this direction are mainly restricted to the Tate motive thanks to the works of Ritter and Weiss in [14, 15] and Kakde [13].

For other motives besides the Tate motive not much is known (but see [1, 5]). Our aim is to tackle the question of the existence of non-abelian p-adic L-functions for "motives", whose classical L-functions can be studied through L-functions of automorphic representations of definite unitary groups. Our first goal is to prove the so called "torsion congruences" (to be explained below) for these motives and then use our approach to tackle the so called Möbious-Wall congruences (see [15]).

Let p be an odd prime number. We write F for a totally real field and F' for a totally real Galois extension with $\Gamma := Gal(F'/F)$ of order p. We assume that the extension is unramified outside p. We write $G_F := Gal(F(p^\infty)/F)$ where $F(p^\infty)$ is the maximal abelian extension of F unramified outside p (may be ramified at infinity). We make the similar definition for F'. Our assumption on the ramification of F'/F implies that there exist a transfer map $ver : G_F \to G_{F'}$, which induces also a map $ver : \mathbb{Z}_p[[G_F]] \to \mathbb{Z}_p[[G_{F'}]]$, between the Iwasawa algebras of G_F and $G_{F'}$, both of them taken with coefficients in $\mathbb{Z}_p$.

Let us now consider a motive M/F (by which we really mean the usual realizations of it and their compatibilities) defined over F such that its p-adic realization has coefficients in $\mathbb{Z}_p$. Then under some assumptions on the critical values of M and some ordinarity assumptions at p (to be made more specific later) it is conjectured that there exists an element $\mu_F \in \mathbb{Z}_p[[G_F]]$ that interpolates the critical values of M/F twisted by characters of G_F. Similarly we write $\mu_{F'}$ for the element in $\mathbb{Z}_p[[G_{F'}]]$ associated to M/F', the base change of M/F to F'. Then the so-called torsion congruences read

$$ver(\mu_F) \equiv \mu_{F'} \mod T,$$

where T is the trace ideal in $\mathbb{Z}_p[[G'_F]]^\Gamma$ generated by the elements $\sum_{\gamma\in\Gamma} \alpha^\gamma$ with $\alpha \in \mathbb{Z}_p[[G'_F]]$. These congruences have been proved by Ritter and Weiss [14] for the Tate motive and under some assumptions by the author [2] for M/F equal to the motive associated to an elliptic curve with complex multiplication. In this work we prove these congruences for motives that their L-functions can be studied by automorphic representations of definite unitary groups.

We now write K for a totally imaginary quadratic extension of F, that is K is a CM field. On our prime number p we put the following ordinary assumption: all primes above p in F are split in K. We write $K' := F'K$, a CM field with $K'^+ = F'$. Next we pick an ordinary CM type Σ of K and denote this pair by

(Σ, K). We consider the inflated type Σ' of Σ to K' and write (Σ', K') for this CM type which is also an ordinary CM type.

The motives M/F that we will consider are of the form $M(\psi)/F \otimes M(\pi)/F$ where $M(\psi)/F$ and $M(\pi)/F$ are to be defined as next. Let ψ be a Hecke character of K and assume that its infinite type is $k\Sigma$ for some integer $k \geq 1$. We write $M(\psi)/F$ for the motive over F that is obtained by "Weil Restriction" to F from the rank one motive over K associated to ψ. We consider now a hermitian space (W, θ) over K and write n for its dimension. We write $U(\theta)/F$ for the corresponding unitary group. We consider now a motive $M(\pi)/F$ over F such that there exists an automorphic representation π of some unitary group $U(\theta)(\mathbb{A}_F)$ with the property that the L-function $L(M(\pi)/K, s)$ of $M(\pi)/K$ over K is equal to $L(\pi, s)$.

We prove the torsion congruences for the motive $M/F := M(\psi)/F \otimes M(\pi)/F$ under the following three assumptions: (1) The p-adic realizations of $M(\pi)$ and $M(\psi)$ have $\mathbb{Z}_p$-coefficients, (2) the motive $M(\pi)/F$ is associated to an automorphic representation π of a definite unitary group in n variables and we have $k \geq n$.

Theorem: Assume that the prime p is unramified in F (but may ramify in F'). Then we have: (1) For $n = 1$: The torsion congruences hold true. (2) For $n = 2$: We write (π, π) for the standard normalized Peterson inner product of π. If (π, π) has trivial valuation at p then the torsion congruences hold true.

The key idea of the proof is the following: Special values of L functions of unitary representations can be realized with the help of the doubling method of Shimura, Piateski-Shapiro and Rallis [16, 17, 8] either (i) as values of hermitian Siegel-type Eisenstein series on CM points of Hermitian domains or (ii) as constant terms of hermitian Klingen-type Eisenstein series for some proper Fourier-Jacobi expansion. In the first approach (see [4]) we consider Siegel-type Eisenstein series of the group $U(n, n)$ with the property that their values at particular CM points are equal to the special L-values that we want to study. The CM points are obtained from the doubling method as indicated by the embedding $U(n, 0) \times U(0, n) \hookrightarrow U(n, n)$. Then we make use of the fact that the CM pairs (K, Σ) and (K', Σ') that we consider are closely related (i.e. the second is induced from the first) which allows us to relate the various CM points over K and K'. Then we use the diagonal embedding, induced from the embedding $K \hookrightarrow K'$, between the symmetric space of $U(n, n)_{/F}$ and that of $Res_{F'/F} U(n, n)_{/F'}$ to relate the Eisenstein series over the different fields.

In the other approach (see [3]) we obtain Klingen-type Eisenstein series of the group $U(n+1, 1)$ with the property that the constant term of their Fourier-Jacobi expansion is related with the special values that we want to study. Then again we use the embedding $K \hookrightarrow K'$ to relate these Klingen-type Eisenstein series over the different fields and hence also to obtain a relation between their constant terms. The main difficulty here is that the Klingen-type Eisenstein series have a rather complicated Fourier-Jacobi expansion, which makes hard the direct study of the arithmetic properties of these Eisenstein series. However the Klingen-type

Eisenstein series are obtained with the help of the pull-back method from Siegel-type Eisenstein series of the group $U(n+1, n+1)$ using the embedding $U(n+1,1) \times U(0,n) \hookrightarrow U(n+1, n+1)$. The Siegel-type Eisenstein series have a much better understood Fourier expansion, which turns out it suffices to study also the Klingen-type Eisenstein series.

Of course both approaches should happen in a p-adic setting. The needed theory for all these has been developed in the papers [9, 10] of Harris, Li and Skinner on the Eisenstein measure on unitary groups and in the works of Ming-Lun Hsieh [11, 12].

References

[1] Th. Bouganis, Special values of L-fucntions and false Tate curve extenions (with an appendix by Vl. Dokchitser), *J. Lond. Math. Soc.* vol 82 (3) (2010) .

[2] Th. Bouganis, Non-abelian congruences between special values of L-fucntions of elliptic curves; the CM case, to appear in the *International Journal of Number Theory* (IJNT).

[3] Th Bouganis, Non abelian p-adic L-functions and Eisenstein series of unitary groups; the Constant Term Method. *In preparation.*

[4] Th Bouganis, Non abelian p-adic L-functions and Eisenstein series of unitary groups; the CM-method. *preprint.*

[5] Th. Bouganis and O. Venjakob, On the non-commutative Main Conjecture for Elliptic Curves with Complex Multiplication, *Asian J. Math.* Vol. 14, No. 3, (2010), pp 385-416

[6] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, The GL_2-main conjecture for elliptic curves without complex multiplication, Publ. Math. IHES. 101 (2005), no. 1, 163-208

[7] T. Fukaya and K. Kato, A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory, Proceedings of the St. Petersburg Mathematical Society, vol XII (Providence RI), Amer. Math. Soc. Transl. Ser. 2 vol 219, 2006

[8] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, Explicit Constructions of Automorphic L-function, *Lecture Notes in Mathematics 1254*, Springer-Verlag 1987.

[9] M. Harris, J.-S. Li, and C. Skinner, The Rallis inner product formula and p-adic L-functions, *Automorphic Representations, L-Functions and Applications: Progress and Prospects*, Volume in honor of S.Rallis, Berlin:de Gruyter, 225-255 (2005)

[10] M. Harris, J.-S. Li, and C. Skinner, p-adic L-functions for unitary Shimura varieties, I: Construction of the Eisenstein measure, *Documenta Math.*, Extra Volume: John H.Coates' Sixtieth Birthday 393-464 (2006)

[11] M-L. Hsieh, Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields, preprint.

[12] M-L. Hsieh, p-adic oridnary Eisenstein series and p-adic L-functions for unitary groups, *Annales de l'Institut Fourier*, Vol. 60 (2010)

[13] M. Kakde, Proof of the main conjecture of non-commutative Iwasawa theory of totally real number fields, *preprint* 2011

[14] J. Ritter and A. Weiss, Congruences between abelian pseudomeasures, *Math. Res. Lett.* 15 (2008) 715-725

[15] J. Ritter and A. Weiss, Congruences between abelian pseudomeasures II, available at the arXiv.

[16] G. Shimura, Euler Products and Eisenstein Series, *CBMS Regional Conference Series in Mathematics*, No. 93, Amer. Math. Soc. 1997

[17] G. Shimura, Arithmeticity in the Theory of Automorphic Forms, *Mathematical Surveys and Monographs*, Vol 82, Amer. Math. Soc. 2000