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On the expected diameter of planar Brownian motion

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Abstract

Known results show that the diameter d_1 of the trace of planar Brownian motion run for unit time satisfies $1.595 \leq \mathbb{E}d_1 \leq 2.507$. This note improves these bounds to $1.601 \leq \mathbb{E}d_1 \leq 2.355$. Simulations suggest that $\mathbb{E}d_1 \approx 1.99$.

Keywords: Brownian motion; convex hull; diameter.

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Let $(b_t, t \in [0, 1])$ be standard planar Brownian motion, and consider the set $b[0, 1] = \{b_t : t \in [0, 1]\}$. The Brownian convex hull $\mathcal{H}_1 := \text{hull } b[0, 1]$ has been well-studied from Lévy [5, §52.6, pp. 254–256] onwards; the expectations of the perimeter length ℓ_1 and area a_1 of \mathcal{H}_1 are given by the exact formulae $\mathbb{E}\ell_1 = \sqrt{8\pi}$ (due to Letac and Takács [4, 6]) and $\mathbb{E}a_1 = \pi/2$ (due to El Bachir [1]).

Another characteristic is the *diameter*

$$d_1 := \text{diam } \mathcal{H}_1 = \text{diam } b[0, 1] = \sup_{x, y \in b[0, 1]} \|x - y\|,$$

for which, in contrast, no explicit formula is known. The exact formulae for $\mathbb{E}\ell_1$ and $\mathbb{E}a_1$ rest on geometric integral formulae of Cauchy; since no such formula is available for d_1 , it may not be possible to obtain an explicit formula for $\mathbb{E}d_1$. However, one may get bounds.

By convexity, we have the almost-sure inequalities $2 \leq \ell_1/d_1 \leq \pi$, the extrema being the line segment and shapes of constant width (such as the disc). In other words,

$$\frac{\ell_1}{\pi} \leq d_1 \leq \frac{\ell_1}{2}.$$

The formula of Letac and Takács [4, 6] says that $\mathbb{E}\ell_1 = \sqrt{8\pi}$, so we get:

Proposition 1. $\sqrt{8/\pi} \leq \mathbb{E}d_1 \leq \sqrt{2\pi}$.

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Note that $\sqrt{8/\pi} \approx 1.5958$ and $\sqrt{2\pi} \approx 2.5066$. In this note we improve both of these bounds.

For the lower bound, we note that $b[0, 1]$ is compact and thus, as a corollary of Lemma 6 below, we have the formula

$$d_1 = \sup_{0 \leq \theta \leq \pi} r(\theta), \quad (1)$$

where r is the parametrized range function given by

$$r(\theta) = \sup_{0 \leq s \leq 1} (b_s \cdot \mathbf{e}_\theta) - \inf_{0 \leq s \leq 1} (b_s \cdot \mathbf{e}_\theta),$$

with \mathbf{e}_θ being the unit vector $(\cos \theta, \sin \theta)$. Feller [2] established that

$$\mathbb{E}r(\theta) = \sqrt{8/\pi} \quad \text{and} \quad \mathbb{E}(r(\theta)^2) = 4 \log 2, \quad (2)$$

and the density of $r(\theta)$ is given explicitly as

$$f(r) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp\{-k^2 r^2/2\}, \quad (r \geq 0). \quad (3)$$

Combining (1) with (2) gives immediately $\mathbb{E}d_1 \geq \mathbb{E}r(0) = \sqrt{8/\pi}$, which is just the lower bound in Proposition 1. For a better result, a consequence of (1) is that $d_1 \geq \max\{r(0), r(\pi/2)\}$. Observing that $r(0)$ and $r(\pi/2)$ are independent, we get:

Lemma 2. $\mathbb{E}d_1 \geq \mathbb{E} \max\{X_1, X_2\}$, where X_1 and X_2 are independent copies of $X := r(0)$.

It seems hard to explicitly compute $\mathbb{E} \max\{X_1, X_2\}$ in Lemma 2, because although the density given at (3) is known explicitly, it is not very tractable. Instead we obtain a lower bound. Since

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$$

we get

$$\mathbb{E} \max\{X_1, X_2\} = \mathbb{E}X + \frac{1}{2}\mathbb{E}|X_1 - X_2|. \quad (4)$$

Thus with Lemma 2, the lower bound in Proposition 1 is improved given any non-trivial lower bound for $\mathbb{E}|X_1 - X_2|$. Using the fact that for any $c \in \mathbb{R}$, if m is a median of X , $\mathbb{E}|X - c| \geq \mathbb{E}|X - m|$, we see that

$$\mathbb{E}|X_1 - X_2| \geq \mathbb{E}|X - m|.$$

Again, the intractability of the density at (3) makes it hard to exploit this. Instead, we provide the following as a crude lower bound on $\mathbb{E}|X_1 - X_2|$.

Lemma 3. For any $a, h > 0$,

$$\mathbb{E}|X_1 - X_2| \geq 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h).$$

Proof. We have

$$\begin{aligned} \mathbb{E}|X_1 - X_2| &\geq \mathbb{E}[|X_1 - X_2| \mathbf{1}\{X_1 \leq a, X_2 \geq a + h\}] \\ &\quad + \mathbb{E}[|X_1 - X_2| \mathbf{1}\{X_2 \leq a, X_1 \geq a + h\}] \\ &\geq h \mathbb{P}(X_1 \leq a) \mathbb{P}(X_2 \geq a + h) + h \mathbb{P}(X_2 \leq a) \mathbb{P}(X_1 \geq a + h) \\ &= 2h \mathbb{P}(X \leq a) \mathbb{P}(X \geq a + h), \end{aligned}$$

which proves the statement. \square

This lower bound yields the following result.

Proposition 4. For $a, h > 0$ define

$$g(a, h) := h \left(\frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{2a^2} \right\} \right) \left(1 - \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8(a+h)^2} \right\} \right).$$

Then $\mathbb{E}d_1 \geq \sqrt{8/\pi} + g(1.492, 0.337) \approx 1.6014$.

Proof. Consider

$$Z := \sup_{0 \leq s \leq 1} |b_s \cdot \mathbf{e}_0|.$$

Then it is known (see [3]) that for $x > 0$,

$$\frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8x^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{8x^2} \right\} \leq \mathbb{P}(Z < x) \leq \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8x^2} \right\}. \quad (5)$$

Moreover, we have

$$Z \leq X \leq 2Z.$$

Since $X \leq 2Z$, we have

$$\mathbb{P}(X \leq a) \geq \mathbb{P}(Z \leq a/2) \geq \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{2a^2} \right\} - \frac{4}{3\pi} \exp \left\{ -\frac{9\pi^2}{2a^2} \right\},$$

by the lower bound in (5). On the other hand,

$$\mathbb{P}(X \geq a+h) \geq \mathbb{P}(Z \geq a+h) \geq 1 - \frac{4}{\pi} \exp \left\{ -\frac{\pi^2}{8(a+h)^2} \right\},$$

by the upper bound in (5). Combining these two bounds and applying Lemma 3 we get $\mathbb{E}|X_1 - X_2| \geq 2g(a, h)$. So from (4) and the fact that $\mathbb{E}X = \sqrt{8/\pi}$ by (2) we get $\mathbb{E}d_1 \geq \sqrt{8/\pi} + g(a, h)$. Numerical evaluation using MAPLE suggests that $(a, h) = (1.492, 0.337)$ is close to optimal, and this choice gives the statement in the proposition. \square

We also improve the upper bound in Proposition 1.

Proposition 5. $\mathbb{E}d_1 \leq \sqrt{8 \log 2} \approx 2.3548$.

Proof. First, we claim that

$$d_1^2 \leq r(0)^2 + r(\pi/2)^2. \quad (6)$$

It follows from (6) and (2) that

$$\mathbb{E}(d_1^2) \leq \mathbb{E}(X_1^2 + X_2^2) = 2\mathbb{E}(X^2) = 8 \log 2.$$

The result now follows by Jensen's inequality.

It remains to prove the claim (6). Note that the diameter is an increasing function, that is, if $A \subseteq B$ then $\text{diam } A \leq \text{diam } B$. Note also, that by the definition of $r(\theta)$, $b[0, 1] \subseteq \mathbf{z} + [0, r(0)] \times [0, r(\pi/2)] =: R_{\mathbf{z}}$ for some $\mathbf{z} \in \mathbb{R}^2$. Since the diameter of the set $R_{\mathbf{z}}$ is attained at the diagonal,

$$\text{diam } R_{\mathbf{z}} = \sqrt{r(0)^2 + r(\pi/2)^2},$$

for all $\mathbf{z} \in \mathbb{R}^2$, and we have $\text{diam } b[0, 1] \leq \text{diam } R_{\mathbf{z}}$, the result follows. \square

We make one further remark about second moments. In the proof of Proposition 5, we saw that $\mathbb{E}(d_1^2) \leq 8 \log 2 \approx 5.5452$. A bound in the other direction can be obtained from the fact that $d_1^2 \geq \ell_1^2/\pi^2$, and we have (see [7, §4.1]) that

$$\mathbb{E}(\ell_1^2) = 4\pi \int_{-\pi/2}^{\pi/2} d\theta \int_0^\infty du \cos \theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh\left(\frac{(2\theta + \pi)u}{4}\right) \approx 26.1677,$$

which gives $\mathbb{E}(d_1^2) \geq 2.651$.

Finally, for completeness, we state and prove the lemma which was used to obtain equation (1).

Lemma 6. *Let $A \subset \mathbb{R}^d$ be a nonempty compact set, and let $r_A(\theta) = \sup_{x \in A}(x \cdot \mathbf{e}_\theta) - \inf_{x \in A}(x \cdot \mathbf{e}_\theta)$. Then*

$$\text{diam } A = \sup_{0 \leq \theta \leq \pi} r_A(\theta).$$

Proof. Since A is compact, for each θ there exist $x, y \in A$ such that

$$\begin{aligned} r_A(\theta) &= x \cdot \mathbf{e}_\theta - y \cdot \mathbf{e}_\theta \\ &= (x - y) \cdot \mathbf{e}_\theta \leq \|x - y\|. \end{aligned}$$

So $\sup_{0 \leq \theta \leq \pi} r_A(\theta) \leq \sup_{x, y \in A} \|x - y\| = \text{diam } A$.

It remains to show that $\sup_{0 \leq \theta \leq \pi} r_A(\theta) \geq \text{diam } A$. This is clearly true if A consists of a single point, so suppose that A contains at least two points. Suppose that the diameter of A is achieved by $x, y \in A$ and let $z = y - x$ be such that $\hat{z} := z/\|z\| = \mathbf{e}_{\theta_0}$ for $\theta_0 \in [0, \pi]$. Then

$$\begin{aligned} \sup_{0 \leq \theta \leq \pi} r_A(\theta) &\geq r_A(\theta_0) \geq y \cdot \mathbf{e}_{\theta_0} - x \cdot \mathbf{e}_{\theta_0} \\ &= z \cdot \hat{z} = \|z\| = \text{diam } A, \end{aligned}$$

as required. □

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