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## ON THE COMPLEXITY OF THE MODEL CHECKING PROBLEM\*

FLORENT R. MADELAINE<sup>†</sup> AND BARNABY D. MARTIN<sup>‡</sup>

**Abstract.** The complexity of the model checking problem for various fragments of first-order logic (FO) has attracted much attention over the last two decades, in particular for the fragment induced by  $\exists$  and  $\wedge$  and that induced by  $\forall, \exists$ , and  $\wedge$ , which are better known as the *constraint satisfaction problem* and the *quantified constraint satisfaction problem*, respectively. The former was conjectured to follow a dichotomy between P and NP-complete by Feder and Vardi [*SIAM J. Comput.*, 28 (1998), pp. 57–104]. For the latter, there are several partial trichotomy results between P, NP-complete, and Pspace-complete, and Chen [*Meditations on quantified constraint satisfaction*, in Logic and Program Semantics, Springer, Heidelberg, 2012, pp. 35–49] ventured a conjecture regarding Pspace-completeness vs. membership in NP in the presence of constants. We give a comprehensive account of the whole field of the complexity of model checking similar syntactic fragments of FO. The above two fragments are in fact the only ones for which there is currently no known complexity classification. Indeed, we consider all other similar syntactic fragments of FO, induced by the presence or absence of quantifiers and connectives, and fully classify the complexities of the parameterization of the model-checking problem by a finite model  $\mathcal{D}$ , that is, the expression complexities for certain finite  $\mathcal{D}$ . Perhaps surprisingly, we show that for most of these fragments, “tractability” is witnessed by a generic solving algorithm which uses quantifier relativization. Our classification methodology relies on tailoring suitably the algebraic approach pioneered by Jeavons, Cohen, and Gyssens [*J. ACM*, 44 (1997), pp. 527–548] for the constraint satisfaction problem and by Börner et al. [*Inform. and Comput.*, 207 (2009), pp. 923–944] for the quantified constraint satisfaction problem. Most fragments under consideration can be relatively easily classified, either directly or using Schaefer’s dichotomy theorems for SAT and QSAT, with the notable exception of the *positive equality-free fragment* induced by  $\exists, \forall, \wedge$ , and  $\vee$ . This outstanding fragment can also be classified and enjoys a *tetrachotomy*: according to the model, the corresponding model checking problem is either tractable, NP-complete, co-NP-complete, or Pspace-complete.

**Key words.** Galois connection, quantified constraints, universal algebra, computational complexity, logic in computer science

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**1. Introduction.** The *model checking problem* over a logic  $\mathcal{L}$  takes as input or parameter a structure  $\mathcal{D}$  and a sentence  $\varphi$  of  $\mathcal{L}$  and asks whether  $\mathcal{D} \models \varphi$ . Vardi studied its complexity mostly for logics which subsume first-order logic (FO) [34]. In this paper, we will be interested in taking syntactic fragments  $\mathcal{L}$  of FO, induced by the presence or absence of quantifiers and connectives, and studying the complexities of the parameterization of the model checking problem by the model  $\mathcal{D}$ , that is, the expression complexities for certain  $\mathcal{D}$ . When  $\mathcal{L}$  is the *primitive positive* fragment of FO,  $\{\exists, \wedge\}$ -FO, the model checking problem is equivalent to the much-studied *constraint satisfaction problem* (CSP). The parameterization of this problem by the model  $\mathcal{D}$  is equivalent to what is sometimes described as the *nonuniform* constraint

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<sup>†</sup>Current addresses: LIMOS, Université Clermont-Auvergne, 63001 Clermont-Ferrand, France, and GREYC, Université Caen Normandie, 14000 Caen, France (florent.madelaine@unicaen.fr). Future address: LACL, Université Paris-Est Créteil, 94010 Créteil Cedex, France (florent.madelaine@upec.fr).

<sup>‡</sup>School of Engineering and Computing Sciences, University of Durham, DH1 3LE, UK (barnaby.d.martin@durham.ac.uk).

satisfaction problem,  $\text{CSP}(\mathcal{D})$  [20]. It has been conjectured by Feder and Vardi [14] that the class of CSPs exhibits *dichotomy*; that is,  $\text{CSP}(\mathcal{D})$  is always either in P or is NP-complete, depending on the model  $\mathcal{D}$ . While in general this conjecture remains open, it has been proved for substantial classes and various methods. Combinatorial (graph-theoretic), logical, and universal-algebraic approaches have been brought to bear on this classification project, with many remarkable consequences. Schaefer founded this area and provided a dichotomy for Boolean structures using a logico-combinatorial approach [33]. Further dichotomies were obtained, e.g., for structures of size at most three [6], for undirected graphs [16], and for smooth digraphs [1]. A conjectured delineation for the dichotomy was given in the algebraic language in [5].

When  $\mathcal{L}$  is *positive Horn*,  $\{\exists, \forall, \wedge\}$ -FO, the model checking problem is equivalent to the well-studied *quantified constraint satisfaction problem* (QCSP). No overarching polychotomy has been conjectured for the nonuniform QCSP( $\mathcal{D}$ ), although the only known attainable complexities<sup>1</sup> are P, NP-complete, and Pspace-complete. Schaefer announced a dichotomy in the Boolean case [33] between P and Pspace-complete in the presence of constants, a dichotomy which was proved to hold even when constants are not present [13, 12]. Some partial classifications were obtained, algebraically [8, 9, 4] or combinatorially [31, 30]. A conjecture delineating the border between NP and Pspace-complete was ventured by Chen in the algebraic language for structures with all constants [11].

Owing to the natural duality between  $\exists, \vee$  and  $\forall, \wedge$ , we also consider various dual fragments. For example, the dual of  $\{\exists, \wedge\}$ -FO is *positive universal disjunctive* FO,  $\{\forall, \vee\}$ -FO. It is straightforward to see that this class of expression complexities exhibits a dichotomy between P and co-NP-complete if and only if the class of CSPs exhibits a dichotomy between P and NP-complete.

Table 1 summarizes the complexity of the model checking for syntactic fragments of FO, up to this duality. With the exception of the complexity of fragments corresponding to the CSP and QCSP, which are still open and active fields of research, we prove in this paper all such polychotomies. We study the fragments along the following four classes. Fragments of the first class exhibit a trivial complexity (in L) for any model  $\mathcal{D}$ , and the proof is simple. A fragment of the second class has trivial complexity if all quantifiers can be relativized to the same constant and hard otherwise (e.g., NP-complete for existential fragments and Pspace-complete for fragments that allow both quantifiers). The third class exhibits the most richness complexity-wise and cannot be explained by relativization of quantifiers. The two polychotomies we establish for the third class are corollaries of Schaeffer's dichotomy theorems. The fourth class consists solely of *positive equality-free first-order logic*  $\{\exists, \forall, \wedge, \vee\}$ -FO and is rich complexity-wise, though we will see that a drop in complexity is always witnessed by relativization of quantifiers. For this outstanding fragment, the corresponding model checking problem can be seen as an extension of the QCSP in which disjunction is returned to the mix.

We undertook studying the complexity of the model checking of  $\{\exists, \forall, \wedge, \vee\}$ -FO through the algebraic method that has been so fruitful in the study of the CSP and QCSP [33, 19, 6, 4, 9]. To this end, we defined *surjective hyper-endomorphisms* and used them to define a Galois connection that characterizes definability under  $\{\exists, \forall, \wedge, \vee\}$ -FO and prove that it suffices to study the complexity of problems associated with the closed sets of the associated lattice, the so-called *down-closed monoids*

<sup>1</sup>Some finer results are known within P for the CSP, but so far there has been no attempt to systematically refine cases within P for the QCSP.

TABLE 1

Complexity of the model checking according to the model for syntactic fragments of FO. L stands for logarithmic space, P for polynomial time, NP for nondeterministic polynomial time, co-NP for its dual, and Pspace for polynomial space.

Polychotomies for the expression complexity of the model checking problem			
Fragment	Dual	Complexity	Class
$\{\exists, \forall\}$ $\{\exists, \forall, =\}$ $\{\exists, \forall, \neq\}$	$\{\forall, \wedge\}$ $\{\forall, \wedge, \neq\}$ $\{\forall, \wedge, =\}$	Trivial (in L).	first
$\{\exists, \wedge, \forall\}$ $\{\exists, \wedge, \forall, =\}$ $\{\exists, \wedge, \forall, \neq\}$	$\{\forall, \wedge, \vee\}$ $\{\forall, \wedge, \vee, \neq\}$ $\{\forall, \wedge, \vee, =\}$	Trivial (in L) if the core of $\mathcal{D}$ has one element and NP-complete otherwise.	second
$\{\exists, \wedge, \forall, \neq\}$	$\{\forall, \wedge, \vee, =\}$	Trivial (in L) if $ D  = 1$ and NP-complete otherwise.	
$\{\exists, \wedge\}$ $\{\exists, \wedge, =\}$	$\{\forall, \vee\}$ $\{\forall, \vee, \neq\}$	<b>CSP dichotomy</b> conjecture: P or NP-complete.	third
$\{\exists, \wedge, \neq\}$	$\{\forall, \vee, =\}$	Trivial if $ D  = 1$ ; in P if $ D  = 2$ and $\mathcal{D}$ is affine or bijunctive; and NP-complete otherwise.	
$\{\exists, \forall, \wedge\}$ $\{\exists, \forall, \wedge, =\}$	$\{\exists, \forall, \vee\}$ $\{\exists, \forall, \vee, \neq\}$	Some <b>QCSP trichotomy</b> results between P, NP-complete, or Pspace-complete.	
$\{\exists, \forall, \wedge, \neq\}$	$\{\exists, \forall, \vee, =\}$	Trivial if $ D  = 1$ ; in P if $ D  = 2$ and $\mathcal{D}$ is affine or bijunctive; and Pspace-complete otherwise.	fourth
$\{\forall, \exists, \wedge, \vee\}$		<b>Positive equality-free tetrachotomy:</b> P, NP-complete, co-NP-complete, or Pspace-complete	
$\{\neg, \exists, \forall, \wedge, \vee\}$		Trivial when $\mathcal{D}$ contains only trivial relations (empty or all tuples), and Pspace-complete otherwise.	second
$\{\forall, \exists, \wedge, \vee, =\}$ $\{\forall, \exists, \wedge, \vee, \neq\}$ $\{\neg, \exists, \forall, \wedge, \vee, =\}$	$\{\forall, \exists, \wedge, \vee, \neq\}$	Trivial when $ D  = 1$ , and Pspace-complete otherwise.	

of unary surjective hyper-operations [25]. Unlike the case of the CSP where the corresponding lattice, the so-called *clone lattice*, is infinite and essentially uncharted when the domain size exceeds two, our lattice is finite for any fixed domain. This has meant that we were able to compute the lattice for modest domain sizes, or chart parts relevant to our classification project, whether by hand for a domain of up to three elements [23] or using a computer for up to four elements [32]. These papers culminate in a full classification—a tetrachotomy—as  $\mathcal{D}$  ranges over structures with up to four element domains. Specifically, the problems  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) are either in L, are NP-complete, are co-NP-complete, or are Pspace-complete. It is a pleasing consequence of our algebraic approach that we can give a quite simple explanation to the delineation of our subclasses. A “drop in complexity” arises precisely when we may relativize w.l.o.g. all quantifiers of one type to a single domain element. Moreover, for membership of L, NP, and co-NP, it is proved in [25] that it is sufficient that  $\mathcal{D}$  has certain special surjective hyper-endomorphisms.

The converse, that it is necessary to have these special surjective hyper-endomorphisms, is more subtle and was initially only an indirect consequence of our exploration of the associated lattice. The main contribution of this paper is that we settle this converse direction and the tetrachotomy for any domain size. A key step is introducing the notion of a  $\forall U$ - $\exists X$ -core, which is the analogue for  $\{\exists, \forall, \wedge, \vee\}$ -FO of the core, so useful in the case of  $\{\exists, \wedge\}$ -FO and the CSP. This novel notion is fairly robust: one can relativize simultaneously quantifiers of any  $\{\exists, \forall, \wedge, \vee\}$ -FO-sentence to subsets of the domains  $U$  and  $X$ ; and equivalently, there are certain special surjective hyper-morphisms witnessing the special role of these subsets. The second key step consists of a careful analysis of the down-closed monoid of a  $\forall U$ - $\exists X$ -core, where we show that all surjective hyper-morphisms have a special form. Finally, we rely

heavily on this special form to reduce the complexity analysis to a few generic cases depending on the relationship and size of the sets  $U$  and  $X$ .

The paper is organized as follows. We recall basic definitions and state some known results in section 2. In particular, we introduce hyper-morphisms in subsection 2.4 and the Galois connection suitable for  $\{\exists, \forall, \wedge, \vee\}$ -FO in subsection 2.5. In section 3, we settle the complexity for all fragments except for  $\{\exists, \forall, \wedge, \vee\}$ -FO. In section 4, we concentrate on key algebraic and logical properties of  $\{\exists, \forall, \wedge, \vee\}$ -FO. In subsection 4.1, we recast the methodology used to derive the Galois connection for  $\{\exists, \forall, \wedge, \vee\}$ -FO to provide an algebraic characterization of containment for this fragment. In subsection 4.2, we refine this methodology to derive an algebraic characterization of relativization for  $\{\exists, \forall, \wedge, \vee\}$ -FO. In subsection 4.3, we introduce  $\forall U$ - $\exists X$ -cores, and in subsection 4.4, we prove some important properties of some surjective hyper-operations of the  $\forall U$ - $\exists X$ -core and derive its uniqueness up to isomorphism. In section 5, we set the lower bounds and prove the tetrachotomy for  $\{\exists, \forall, \wedge, \vee\}$ -FO. In particular, a key point presented in subsection 5.1 consists in showing that all surjective hyper-morphisms of the  $\forall U$ - $\exists X$ -core have a special form, which allows us to prove hardness in a generic way in subsequent sections. Finally, in section 6, we investigate the complexity of the meta-problem: given a finite structure  $\mathcal{D}$ , what is the complexity of evaluating positive equality-free sentences of FO over  $\mathcal{D}$ ? We establish that the meta-problem is NP-hard, even for a fixed and finite signature.

## 2. Preliminaries.

**2.1. Basic definitions.** Unless otherwise stated, we shall work with finite relational structures that have at least one element and share the same finite relational signature  $\sigma$ . Let  $\mathcal{D}$  be such a structure. We will denote its domain by  $D$ . We denote the size of such a set  $D$  by  $|D|$ . The *complement*  $\overline{\mathcal{D}}$  of a structure  $\mathcal{D}$  consists of relations that are exactly the set-theoretic complements of those in  $\mathcal{D}$ . In other words, for an  $a$ -ary  $R$ ,  $R^{\overline{\mathcal{D}}} := D^a \setminus R^{\mathcal{D}}$ . For graphs this leads to a slightly nonstandard notion of complement, as it includes self-loops.

A *homomorphism* (resp., *full homomorphism*) from a structure  $\mathcal{D}$  to a structure  $\mathcal{E}$  is a function  $h : D \rightarrow E$  that preserves (resp., preserves fully) the relations of  $\mathcal{D}$ ; i.e., for all  $a_i$ -ary relations  $R_i$ , and for all  $x_1, \dots, x_{a_i} \in D$ ,  $R_i(x_1, \dots, x_{a_i}) \in \mathcal{D}$  implies (resp., if and only if)  $R_i(h(x_1), \dots, h(x_{a_i})) \in \mathcal{E}$ .  $\mathcal{D}$  and  $\mathcal{E}$  are *homomorphically equivalent* if there are homomorphisms both from  $\mathcal{D}$  to  $\mathcal{E}$  and from  $\mathcal{E}$  to  $\mathcal{D}$ . Let  $\mathcal{L}$  be a fragment of FO. Let  $\langle \mathcal{D} \rangle_{\mathcal{L}}$  be the set of relations that may be defined on  $\mathcal{D}$  in  $\mathcal{L}$ .

We say that  $\mathcal{D}$  is  $\mathcal{L}$ -*contained* in  $\mathcal{E}$  if and only if, for any  $\varphi$  in  $\mathcal{L}$ ,  $\mathcal{D} \models \varphi$  implies  $\mathcal{E} \models \varphi$ . We say that  $\mathcal{D}$  and  $\mathcal{E}$  are  $\mathcal{L}$ -*equivalent* if and only if, for any  $\varphi$  in  $\mathcal{L}$ ,  $\mathcal{D} \models \varphi \Leftrightarrow \mathcal{E} \models \varphi$ . If  $\mathcal{E}$  is a minimal structure w.r.t. domain size such that  $\mathcal{D}$  and  $\mathcal{E}$  are  $\mathcal{L}$ -equivalent, then we say that  $\mathcal{E}$  is an  $\mathcal{L}$ -*core* of  $\mathcal{D}$ .

*Remark 1.* When  $\mathcal{L}$  is  $\{\exists, \wedge\}$ -FO, our definition coincides with the classical notion of a core of  $\mathcal{D}$ , which is a minimal substructure that is homomorphically equivalent to  $\mathcal{D}$ .

Given a sentence  $\varphi$  in  $\{\exists, \wedge\}$ -FO, we denote by  $\mathcal{D}_\varphi$  its *canonical database*, that is, the structure with its domain being the variables of  $\varphi$  and whose tuples are precisely those that are atoms of  $\varphi$ . In the other direction, given a finite structure  $\mathcal{A}$ , we write  $\varphi_{\mathcal{A}}$  for the so-called *canonical conjunctive query*<sup>2</sup> of  $\mathcal{A}$ , the quantifier-free formula

<sup>2</sup>Most authors consider the canonical query to be the sentence which is the existential closure of  $\varphi_{\mathcal{A}}$ .

that is the conjunction of the positive facts of  $\mathcal{A}$ , where the variables  $v_1, \dots, v_{|A|}$  correspond to the elements  $a_1, \dots, a_{|A|}$  of  $\mathcal{A}$ .

**2.2. Model checking fragments of FO.** Let  $\mathcal{L}$  be a fragment of FO. Let  $\mathcal{D}$  be a fixed structure. The decision problem  $\mathcal{L}(\mathcal{D})$  takes as input a sentence  $\varphi$  of  $\mathcal{L}$  and accepts if and only if  $\mathcal{D} \models \varphi$ . In this paper, we will be concerned with syntactic fragments  $\mathcal{L}$  of FO defined by allowing or disallowing symbols from  $\{\exists, \forall, \wedge, \vee, \neq, =, \neg\}$ . Given any sentence  $\varphi$  in  $\mathcal{L}$ , we may compute in logarithmic space an equivalent sentence  $\varphi'$  in prenex normal form, with negation pushed inwards to the atomic level. Since we will not be concerned with complexities beneath L, we assume hereafter that all inputs are in this form. In general, Pspace membership of  $\text{FO}(\mathcal{D})$  follows by a simple evaluation procedure inward through the quantifiers. Similarly, the expression complexity of the existential fragment  $\{\exists, \wedge, \vee, \neq, =\}$ -FO is at most NP, and that of its dual fragment  $\{\forall, \vee, \wedge, =, \neq\}$ -FO is at most co-NP (in both cases, we may even allow atomic negation) [34].

Let  $\mathcal{L}$  be a syntactic fragment of FO defined by allowing or disallowing symbols from  $\{\exists, \forall, \wedge, \vee, \neq, =\}$ . We denote by  $\overline{\mathcal{L}}$  its dual fragment by de Morgan's law:  $\wedge$  is dual to  $\vee$ ,  $\exists$  to  $\forall$ , and  $=$  to  $\neq$ .

**PROPOSITION 2** (principle of duality). *The problem  $\mathcal{L}(\mathcal{D})$  belongs to a complexity class C if and only if the problem  $\overline{\mathcal{L}}(\mathcal{D})$  belongs to the dual complexity class co-C.*

*Proof.* For any sentence  $\varphi$  in  $\mathcal{L}$ , we may rewrite its negation  $\neg\varphi$  by pushing the negation inwards until all atoms appear negatively, denoting the sentence hence obtained by  $\psi$  (which is logically equivalent to  $\neg\varphi$ ). Next, we replace every occurrence of a negated relational symbol  $\neg R$  by  $R$  to obtain a sentence of  $\overline{\mathcal{L}}$  which we denote by  $\overline{\varphi}$ . The following chain of equivalences holds:

$$\mathcal{D} \models \varphi \iff \mathcal{D} \models \neg(\neg\varphi) \iff \mathcal{D} \models \neg(\psi) \iff \mathcal{D} \not\models \psi \iff \overline{\mathcal{D}} \not\models \overline{\varphi}.$$

Clearly,  $\overline{\varphi}$  can be constructed in logspace from  $\varphi$  and the result follows.  $\square$

**2.3. Hintikka games.** Let  $\varphi$  be a sentence of FO in prenex form. A *strategy* for  $\exists$  in the (Hintikka)  $(\mathcal{A}, \varphi)$ -game is a set of mappings  $\sigma = \{\sigma_x : \text{"}\exists x\text{"} \in \varphi\}$  with one mapping  $\sigma_x$  for each existentially quantified variable  $x$  of  $\varphi$ . The mapping  $\sigma_x$  ranges over the domain  $A$  of  $\mathcal{A}$ , and its domain is the set of functions from  $Y_x$  to  $A$ , where  $Y_x$  denotes the universally quantified variables of  $\varphi$  preceding  $x$  ( $\sigma_x$  is nothing but a *Skolem function* associated with the existential variable  $x$ ). We say that  $\{\sigma_x : \text{"}\exists x\text{"} \in \varphi\}$  is *winning* if for any assignment  $\pi$  of the universally quantified variables of  $\varphi$  to  $A$ , when each existentially quantified variable  $x$  is set according to  $\sigma_x$  applied to  $\pi|_{Y_x}$  (the restriction of  $\pi$  to  $Y_x$ ), the quantifier-free part  $\psi$  of  $\varphi$  is satisfied. We write  $\sigma.\pi$  for this map from the variables of the sentence to the domain of  $\mathcal{A}$  and refer to this assignment as the *variable assignment*. It follows directly from our definitions that  $\mathcal{A} \models \varphi$  if and only if the existential player has a winning strategy in the  $(\mathcal{A}, \varphi)$ -game. For further details, the reader may consult [18].

**2.4. Hyper-morphisms.** A *hyper-operation*  $f$  from a set  $A$  to a set  $B$  is a function from  $A$  to the power set of  $B$ . For a subset  $S$  of  $A$ , we define its image  $f(S)$  under the hyper-operation  $f$  as  $\bigcup_{s \in S} f(s)$ . When we wish to stress that an element may be sent to  $\emptyset$ , we speak of a *partial hyper-operation*; otherwise we assume that  $f$  is *total*, that is, for any  $a$  in  $A$ ,  $f(a) \neq \emptyset$ . We say that  $f$  is *surjective* whenever  $f(A) = B$ . To clarify the exposition and better convey our intuition, for elements  $a$

in  $A$  and  $b$  in  $B$ , we may write that  $a$  reaches  $b$  or that  $b$  is reached by  $a$  under  $f$  whenever  $b \in f(a)$ . The inverse of a (total) hyper-operation  $f$  from  $A$  to  $B$ , denoted by  $f^{-1}$ , is the partial hyper-operation from  $B$  to  $A$  defined for any  $b$  in  $B$  as  $f^{-1}(b) := \{a \in A \mid b \in f(a)\}$ . We call an element of  $f^{-1}(b)$  an *antecedent* of  $b$  under  $f$ . Let  $f$  be a hyper-operation from  $A$  to  $B$  and  $g$  be a hyper-operation from  $B$  to  $C$ . The hyper-operation  $g \circ f$  is defined naturally as  $g \circ f(x) := g(f(x))$  (recall that  $f(x)$  is a set).

When  $f$  is a (total) surjective hyper-operation from  $A$  to  $A$ , we say that  $f$  is a *shop* of  $A$ . Note that the inverse of a shop is a shop and that the composition of two shops is also a shop. Observing further that shop composition is associative and that the identity shop (which sends an element  $x$  of  $A$  to the singleton  $\{x\}$ ) is the identity w.r.t. composition, we may consider the monoid generated by a set of shops. A shop  $f$  is a *subshop* of a shop  $g$  whenever, for every  $x$  in  $A$ ,  $f(x) \subseteq g(x)$ . A *down-shop-monoid* (DSM) is a set of shops that contains the identity shop and is closed under both composition and taking subshops.<sup>3</sup> We denote by  $\langle F \rangle_{DSM}$  the DSM generated by a set  $F$  of shops, that is, the smallest DSM containing  $F$ . Let  $\mathcal{M}$  and  $\mathcal{M}'$  be two DSMs. We say that  $\mathcal{M}$  is a *sub-DSM* of  $\mathcal{M}'$  or that  $\mathcal{M}'$  is a *super-DSM* of  $\mathcal{M}$  whenever  $\mathcal{M}$  is included in  $\mathcal{M}'$ .

Let  $f$  be a shop of  $A$ . When for a subset  $U$  of  $A$  we have  $f(U) = A$ , we say that  $f$  is  *$U$ -surjective*. Observing that the totality of  $f$  may be rephrased as  $f^{-1}(A) = A$ , we say more generally that  $f$  is  *$X$ -total* for a subset  $X$  of  $A$  whenever  $f^{-1}(X) = A$ . Note that for shops,  $U$ -surjectivity and  $X$ -totality are dual to one another; that is, the inverse of a  $U$ -surjective shop is an  $X$ -total shop with  $X = U$  and vice versa. If  $f$  is a  $U$ -surjective shop and  $U$  (resp.,  $X$ -total and  $X$ ) is a singleton, then we will write that  $f$  is a singleton-surjective (resp., singleton-total) shop.<sup>4</sup>

LEMMA 3. Let  $f$  and  $g$  be two shops:

- (i) If  $f$  is a  $U$ -surjective shop, then  $g \circ f$  is a  $U$ -surjective shop.
- (ii) If  $g$  is an  $X$ -total shop, then  $g \circ f$  is an  $X$ -total shop.
- (iii) If  $f$  is a  $U$ -surjective shop and  $g$  is an  $X$ -total shop, then  $g \circ f$  is a  $U$ -surjective  $X$ -total shop.
- (iv) If  $f$  and  $g$  are  $U$ -surjective  $X$ -total shops, then  $g \circ f$  is a  $U$ -surjective  $X$ -total shop.
- (v) The iterate of a  $U$ -surjective  $X$ -total shop is a  $U$ -surjective  $X$ -total shop.

*Proof.* We prove (i). Since  $f(U) = A$ , we have  $g(f(U)) = g(A)$ . By the surjectivity of  $g$ , we know that  $g(A) = A$ . It follows that  $g(f(U)) = A$ , and we are done. (ii) is dual to (i), and (iii) follows directly from (i) and (ii). (iv) is a restriction of (iii) and is only stated here, as we shall use it often. (v) follows by induction on the order of iteration using (iv).  $\square$

A *hyper-morphism* (resp., *full hyper-morphism*)  $f$  from a structure  $\mathcal{A}$  to a structure  $\mathcal{B}$  is a hyper-operation from  $A$  to  $B$  such that  $R(a_1, \dots, a_i) \in \mathcal{A}$  implies (resp., if and only if)  $R(b_1, \dots, b_i) \in \mathcal{B}$  for all  $b_1 \in f(a_1), \dots, b_i \in f(a_i)$ . When  $\mathcal{A}$  and  $\mathcal{B}$  are the same structure, we speak of a *hyper-endomorphism*. Note that the inverse of a full surjective hyper-morphism is also a full surjective hyper-morphism.

**2.5. The Galois connection  $\text{Inv} - \text{shE}$ .** Let  $\text{shE}(\mathcal{B})$  be the set of surjective hyper-endomorphisms of  $\mathcal{B}$ . For a set  $F$  of shops on the same finite domain  $B$ , let

<sup>3</sup>The “down” comes from *down-closure*, here under subshops; it is a nomenclature inherited from [3].

<sup>4</sup>We wrote about A-shops and E-shops in [25].

$\text{Inv}(F)$  be the set of relations on  $B$  of which each  $f$  in  $F$  is a hyper-endomorphism (a relation  $R$  on  $B$  being viewed as the structure  $\langle B; R \rangle$ ). Let  $\langle \mathcal{B} \rangle_{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$  be the sets of relations that may be defined on  $\mathcal{B}$  in  $\{\exists, \forall, \wedge, \vee\}$ -FO.

THEOREM 4 (see [25, 29]).

- (i) For any structure  $\mathcal{B}$ ,  $\langle \mathcal{B} \rangle_{\{\exists, \forall, \wedge, \vee\}\text{-FO}} = \text{Inv}(\text{shE}(\mathcal{B}))$ .
- (ii) For any set of shops  $F$ ,  $\langle F \rangle_{DSM} = \text{shE}(\text{Inv}(F))$ .

COROLLARY 5. Let  $\mathcal{B}$  and  $\mathcal{B}'$  be structures over the same domain  $B$ . If  $\text{shE}(\mathcal{B}) \subseteq \text{shE}(\mathcal{B}')$ , then  $\{\exists, \forall, \wedge, \vee\}\text{-FO}(\mathcal{B}') \leq_L \{\exists, \forall, \wedge, \vee\}\text{-FO}(\mathcal{B})$ .

**2.6. Some complexity results.** We start with two well-known theorems due to Schaefer. For definitions and further details regarding their proofs, the reader may consult the nice survey by Chen [10].

THEOREM 6 (see [33]). Let  $\mathcal{D}$  be a Boolean structure. Then  $\text{CSP}(\mathcal{D})$  (equivalently,  $\{\exists, \wedge\}\text{-FO}(\mathcal{D})$ ) is in P if all relations of  $\mathcal{D}$  are simultaneously 0-valid, 1-valid, Horn, dual-Horn, bijunctive, or affine, and otherwise it is NP-complete.

A similar result holds when universal quantifiers are added to the mix.

THEOREM 7 (sketched in the presence of constants [33] and then proved in the absence of constants in [12, 13]). Let  $\mathcal{D}$  be a Boolean structure. Then  $\text{QCSP}(\mathcal{D})$  (equivalently,  $\{\exists, \forall, \wedge\}\text{-FO}(\mathcal{D})$ ) is in P if all relations of  $\mathcal{D}$  are simultaneously Horn, dual-Horn, bijunctive, or affine, and otherwise it is Pspace-complete.

Example 8. The canonical example of a relation that does not fall in any of the tractable cases is  $\text{NAE} := \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$ . Let  $\mathcal{B}_{\text{NAE}}$  be the Boolean structure with this relation. It follows from the above theorems that  $\text{CSP}(\mathcal{B}_{\text{NAE}})$  is NP-complete and that  $\text{QCSP}(\mathcal{B}_{\text{NAE}})$  is Pspace-complete.

Example 9. For larger domains, though the classification remains open, the canonical hard problem is induced by the relation  $\neq$ . Let  $\mathcal{K}_n$  denote the clique of size  $n$  (we view an undirected graph as a structure with a single binary predicate  $E$  that is symmetric). For  $n \geq 3$ ,  $\text{CSP}(\mathcal{K}_n)$  is a reformulation of the  $n$ -colorability problem and is NP-complete. It is also known that for  $n \geq 3$ ,  $\text{QCSP}(\mathcal{K}_n)$  is Pspace-complete [4].

The lattice of Boolean DSMs is drawn in Figure 1. We write  $\frac{0}{1} \mid \frac{01}{1}$  for the surjective hyper-operation that sends 0 to  $\{0, 1\}$  and 1 to  $\{1\}$ .

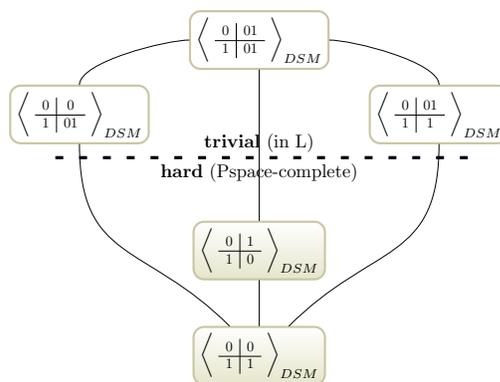


FIG. 1. The Boolean lattice of DSMs with their associated complexity (L or Pspace-complete).

THEOREM 10 (see [25]). *Let  $\mathcal{D}$  be a Boolean structure. If either  $\frac{0}{1} \mid \frac{01}{1}$  or  $\frac{0}{1} \mid \frac{0}{01}$  is a surjective hyper-endomorphism of  $\mathcal{D}$ , then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is in L. Otherwise,  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is Pspace-complete.*

Example 11.  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_n$ ) is Pspace-hard for all  $n \geq 2$ . When  $n \geq 3$ , it follows from the hardness of the QCSP [4]. When  $n = 2$ , the QCSP is actually in P by Theorem 7 since  $\mathcal{K}_2$  is bijunctive. We appeal to the above to deduce the hardness of  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_2$ ) from the fact that  $\mathcal{K}_2$  has DSM  $\text{shE}(\mathcal{B}) = \langle \frac{0}{1} \mid \frac{1}{0} \rangle$ .

**3. Complexity classification of the model checking problem.** We assume at least one quantifier and one binary connective since weaker fragments are trivial in our context, where we only consider sentences. By the duality principle, we may consider only purely existential fragments or fragments with both quantifiers.

Regarding connectives, we have three possibilities: purely disjunctive fragments, purely conjunctive fragments, and fragments with both connectives. Regarding equality and disequality, we should have the four possible subsets of  $\{=, \neq\}$ , but it will become clear that cases with both follow the same complexity delineation as the case with  $\neq$  only. Moreover, for fragments with both quantifiers, we may use the duality principle between  $\{\exists, \forall, \wedge\}$  and  $\{\forall, \exists, \vee\}$  to simplify our task. This means that we would need to consider  $3 \times 3$  positive existential fragments and  $2 \times 3$  positive fragments with both quantifiers. Actually, we can decrease this last count by one, due to the duality between  $\{\exists, \forall, \wedge, \vee, \neq\}$ -FO and  $\{\exists, \forall, \wedge, \vee, =\}$ -FO. Regarding fragments with  $\neg$ , since we necessarily have both connectives and both quantifiers, we only have to consider two fragments: FO and  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO. However, we shall see that the complexity of FO agrees with that of  $\{\exists, \forall, \wedge, \vee, \neq\}$ -FO (and its dual  $\{\exists, \forall, \wedge, \vee, =\}$ -FO).

We proceed through the relevant fragments of Table 1 according to their class.

### 3.1. First class.

PROPOSITION 12.

- (i) *When  $\mathcal{D}$  has a single element, the model checking problem for FO is in L.*
- (ii) *The model checking problem  $\{\exists, \forall, \neq, =\}$ -FO is in L.*

*Proof.* A Boolean sentence is an expression formed from propositional connectives  $\wedge, \vee, \neg$ , etc., from constants 1 and 0 (for true and false, respectively), and from parentheses. The *Boolean sentence value problem* (BSVP) is the decision problem of, given a Boolean sentence, determining whether the value of the sentence is 1 (i.e., true). It is known to be in L [22].

(i) In the case where  $|D| = 1$ , every relation is either empty or contains all tuples (one tuple), and the quantifiers  $\exists$  and  $\forall$  are semantically equivalent. Hence, the problem can be reduced in logspace to the BSVP (under the substitution of 0 and 1 for the empty and nonempty relations, respectively).

(ii) We may assume by the previous point that  $|D| > 1$ . We only need to check whether one of the atoms that occurs as a disjunct in the input sentence holds in  $\mathcal{D}$ . Since  $|D| > 1$ , a sentence with an atom like  $x = y$  or  $x \neq y$  is always true in  $\mathcal{D}$ . For sentences of  $\{\exists, \vee\}$ -FO, the atoms may have implicit equality as in  $R(x, x, y)$  for a ternary predicate  $R$ : in any case, each atom may be checked in constant time since  $\mathcal{D}$  is a fixed structure, resulting in overall logspace complexity.  $\square$

### 3.2. Second class.

PROPOSITION 13. *The class of problems  $\{\exists, \forall, \wedge, \vee, \neq\}$ -FO( $\mathcal{D}$ ) exhibits a dichotomy: if  $|D| = 1$ , then the problem is in L; otherwise it is Pspace-complete. Consequently, the fragment extended with  $=$  follows the same dichotomy.*

*Proof.* When  $|D| \geq 2$ , Pspace-hardness may be proved using no extensional relation of  $D$  other than  $\neq$ . The formula  $\varphi_{\mathcal{K}_{|D|}}(x, y) := (x \neq y)$  simulates the edge relation of the clique  $\mathcal{K}_{|D|}$ , and the problem  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_{|D|}$ ) is Pspace-complete (see Example 11).

Note that we have not used  $=$  in our hardness proof, and, in the case  $|D| = 1$ , we may allow  $=$  without affecting tractability (triviality). Thus, the fragment extended with  $=$  follows the same delineation.  $\square$

FACT 14. *Any sentence of  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO is logically equivalent to false or to a prenex sentence whose quantifier-free part is a disjunction of conjunction-of-atoms  $\psi_i$  such that for each  $r$ -ary symbol  $R$  and each choice of  $r$  variables occurring in  $\psi_i$  exactly one of  $R(v_1, v_2, \dots, v_r)$  or  $\neg R(v_1, v_2, \dots, v_r)$  is an atom of  $\psi_i$ .*

*Proof.* First, push negations to the atomic level and write the quantifier-free part  $\psi$  of  $\varphi$  in disjunctive normal form. If a disjunct  $\psi_i$  has contradictory positive and negative atoms (as in  $E(x, y) \wedge \neg E(x, y)$ ), then  $\psi_i$  is false. If this holds for all disjuncts, then the original sentence is false, and otherwise we may discard such disjuncts and complete the other disjuncts as follows. If neither  $R(v_1, v_2, \dots, v_r)$  nor  $\neg R(v_1, v_2, \dots, v_r)$  occurs in  $\psi_i$ , then replace  $\psi_i$  by the logically equivalent  $(\psi_i \wedge R(v_1, v_2, \dots, v_r)) \vee (\psi_i \wedge \neg R(v_1, v_2, \dots, v_r))$ .  $\square$

LEMMA 15. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two structures such that there is a full surjective hyper-morphism from  $\mathcal{A}$  to  $\mathcal{B}$ . Then, for every sentence  $\varphi$  in  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO, if  $\mathcal{A} \models \varphi$ , then  $\mathcal{B} \models \varphi$ .*

*Proof.* Assume w.l.o.g. that  $\varphi$  is in the form of Fact 14. Every conjunction of atoms  $\psi_i$  corresponds to a structure  $\mathcal{D}_{\psi_i}$  induced naturally by the positive part of  $\psi_i$ . Observe that, for any structure  $\mathcal{C}$ , winning the  $(\mathcal{C}, \varphi)$ -game is now equivalent to the variable assignment being a full homomorphism from some  $\mathcal{D}_{\psi_i}$  to  $\mathcal{C}$ .

Let  $h$  be a full surjective hyper-morphism from  $\mathcal{A}$  to  $\mathcal{B}$  and  $\varphi$  be a sentence of  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO such that  $\mathcal{A} \models \varphi$ . We fix arbitrary linear orders over  $A$  and  $B$  and denote the minimum of a set using a subscript as in  $\min_A$ .

Let  $\{\sigma_x : \text{"}\exists x \text{"} \in \varphi\}$  be a winning strategy in the  $(\mathcal{A}, \varphi)$ -game. We construct a strategy  $\{\sigma'_x : \text{"}\exists x \text{"} \in \varphi\}$  in the  $(\mathcal{B}, \varphi)$ -game as follows. Let  $\pi_B : Y_x \rightarrow B$  be an assignment to the universal variables  $Y_x$  preceding an existential variable  $x$  in  $\varphi$ ; then we set  $\sigma'_x(\pi_B) := \min_B h(\sigma_x(\pi_A))$ , where  $\pi_A : Y_x \rightarrow A$  is the assignment such that for any universal variable  $y$  preceding  $x$ , we have  $\pi_A(y) := \min_A h^{-1}(\pi_B(y))$ .

This strategy is well defined since  $h$  is surjective (which means that  $\pi_A$  is well defined) and total (which means that  $h(\sigma_x(\pi_A)) \neq \emptyset$ ). It remains to prove that  $\{\sigma'_x : \text{"}\exists x \text{"} \in \varphi\}$  is winning.

Let  $w_A$  be the full homomorphism from some  $\mathcal{D}_{\psi_i}$  to  $\mathcal{A}$  induced by the assignment  $\pi_A$  to the universal variables and the winning strategy  $\{\sigma_x : \text{"}\exists x \text{"} \in \varphi\}$ . Viewing  $w_A$  as a full hyper-morphism with singleton range, let  $g := h \circ w_A$ . Note that  $g$  is a full hyper-morphism from  $\mathcal{D}_{\psi_i}$  to  $\mathcal{B}$ . Note also that the map  $w_B$  from the domain of  $\mathcal{D}_{\psi_i}$  to  $\mathcal{B}$  induced by  $\pi_B$  and the strategy  $\{\sigma'_x : \text{"}\exists x \text{"} \in \varphi\}$  is a range restriction of this full hyper-morphism and is therefore a full homomorphism. This proves that  $\{\sigma'_x : \text{"}\exists x \text{"} \in \varphi\}$  is indeed a winning strategy for  $\exists$  in the  $(\mathcal{B}, \varphi)$ -game.  $\square$

As a corollary, we note the fact that containment and equivalence coincide for  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO since this logic is closed under negation. At the algebraic level, this argument becomes that the inverse of a full surjective hyper-morphism is a full surjective hyper-morphism.

For  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO, we define an equivalence relation  $\sim$  over the structure elements in the spirit of the Leibnitz rule for equality. For propositions  $P$  and  $Q$ , let  $P \leftrightarrow Q$  be an abbreviation for  $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ . For each  $r$ -ary symbol  $R$ , let  $\psi_R(x, y)$  stand for

$$\begin{aligned} & (R(x, z_1, z_2, \dots, z_{r-1}) \leftrightarrow R(y, z_1, z_2, \dots, z_{r-1})) \\ & \wedge (R(z_1, x, z_2, \dots, z_{r-1}) \leftrightarrow R(z_1, y, z_2, \dots, z_{r-1})) \\ & \wedge \dots \\ & \wedge (R(z_1, z_2, \dots, z_{r-1}, x) \leftrightarrow R(z_1, z_2, \dots, z_{r-1}, y)). \end{aligned}$$

Let  $\varphi_{\sim}(x, y) := \bigwedge_{R \in \sigma} \forall z_1, z_2, \dots, z_{r-1} \psi_R(x, y)$ . It is straightforward to verify that  $\varphi_{\sim}$  induces an equivalence relation  $\sim$  over the vertices. Let  $\mathcal{A}/\sim$  be the quotient structure defined in the natural way. Mapping an element to its equivalence class results in a full surjective homomorphism from  $\mathcal{A}$  to  $\mathcal{A}/\sim$ . Its inverse (viewing the homomorphism as a hyper-morphism) is a full surjective hyper-morphism from  $\mathcal{A}/\sim$  to  $\mathcal{A}$ . Thus, it follows from Lemma 15 that  $\mathcal{A}$  and  $\mathcal{A}/\sim$  are  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO-equivalent.

**PROPOSITION 16.** *The class of problems  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO( $\mathcal{D}$ ) exhibits a dichotomy: if all relations of  $\mathcal{D}$  are trivial (either empty or contain all tuples), then the problem is in L; otherwise it is Pspace-complete.*

*Proof.* If all relations are trivial, then  $\sim$  has a single equivalence class and  $\mathcal{D}/\sim$  has a single element. It suffices to check whether an input  $\varphi$  in  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO holds in this one element structure; hence the problem is in L.

Otherwise, the equivalence relation  $\sim$  has at least  $n \geq 2$  equivalence classes since  $\mathcal{D}$  is nontrivial. We reduce from the Pspace-complete  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_n$ ) (see Example 11): the reduction consists in substituting every instance of  $E(x, y)$  by  $\varphi_{\sim}(x, y)$ .  $\square$

In our preliminary work [28, 27], the proof of the following is combinatorial and appeals to Hell and Nešetřil's dichotomy theorem for undirected graphs [16]. An alternative proof appeared later in [17, 2] when equality is present, and it uses the Galois connection  $\text{InV} - \text{End}$  due to Krasner [21]. We only sketch the proof in the absence of equality (a full proof is available on the webpages of the authors).

**PROPOSITION 17.** *The class of problems  $\{\exists, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) exhibits a dichotomy: if the core of  $\mathcal{D}$  has one element, then the problem is in L; otherwise it is NP-complete. As a corollary, the class of problems  $\{\exists, \wedge, \vee, =\}$ -FO( $\mathcal{D}$ ) exhibits the same dichotomy.*

*Proof sketch.* The first step consists in showing that the classical notion of a core is also the correct notion of  $\mathcal{L}$ -core when  $\mathcal{L}$  is  $\{\exists, \wedge, \vee\}$ -FO or  $\{\exists, \wedge, \vee, =\}$ -FO.

For a structure  $\mathcal{B}$ , let  $\text{hE}(\mathcal{B})$  be the set of hyper-endomorphisms of  $\mathcal{B}$ . The next step consists in proving a variant of Krasner's Galois connection when equality is absent, namely the Galois connection  $\text{InV} - \text{hE}$ . As a corollary, we obtain that when  $\mathcal{B}$  and  $\mathcal{B}'$  are two structures over the same domain  $B$ , if  $\text{hE}(\mathcal{B}) \subseteq \text{hE}(\mathcal{B}')$ , then  $\{\exists, \wedge, \vee\}$ -FO( $\mathcal{B}'$ )  $\leq_L$   $\{\exists, \wedge, \vee\}$ -FO( $\mathcal{B}$ ).

We assume w.l.o.g. that  $\mathcal{D}$  is a core. This means that every hyper-endomorphism of  $\mathcal{D}$  is in fact an automorphism—we identify hyper-endomorphisms whose range is a set of singletons with automorphisms—and thus  $\text{hE}(\mathcal{D})$  is a subset of  $S_n$  where

$n = |D|$ . If  $D$  has one element, then the problem is trivial. If  $D$  has two elements, then  $\text{hE}(\mathcal{D}) \subseteq \text{hE}(\mathcal{B}_{\text{NAE}}) = S_2$  and it follows that  $\{\exists, \wedge, \vee\}\text{-FO}(\mathcal{B}_{\text{NAE}}) \leq_L \{\exists, \wedge, \vee\}\text{-FO}(\mathcal{D})$ . Since the former is a generalization of the NP-complete  $\text{CSP}(\mathcal{B}_{\text{NAE}})$ , the latter is NP-complete. If  $\mathcal{D}$  has  $n \geq 2$  elements, then we proceed similarly with  $\mathcal{K}_n$ .

The same classification holds for the fragment with equality via the same argument as in the proof of Proposition 13.  $\square$

The proof of the following is similar to that of Proposition 13.

**PROPOSITION 18.** *The class of problems  $\{\exists, \wedge, \vee, \neq\}\text{-FO}(\mathcal{D})$  exhibits a dichotomy: if  $|D| = 1$ , then the problem is in L; otherwise it is NP-complete. Consequently, the fragment extended with  $=$  follows the same dichotomy.*

*Remark 19.* It is not very difficult to show that the notion of an  $\mathcal{L}$ -core coincides with the classical notion of a core for fragments with  $\{\exists, \wedge\}$  and possibly  $\vee$  or  $=$ . For  $\{\exists, \forall, \wedge, \vee, \neg\}\text{-FO}$ , we get the quotient by  $\sim$ , and for any fragment  $\mathcal{L}$  that contains  $\{\exists, \wedge, \neq\}\text{-FO}$  or its dual, every structure is its own  $\mathcal{L}$ -core.

Thus, all fragments of the second class follow a natural dichotomy.

**COROLLARY 20.** *For any syntactic fragment  $\mathcal{L}$  of FO in the second class, the model checking problem  $\mathcal{L}(\mathcal{D})$  is trivial (in L) when the  $\mathcal{L}$ -core of  $\mathcal{D}$  has one element and hard otherwise (NP-complete for existential fragments and Pspace-complete for fragments containing both quantifiers).*

### 3.3. Third class.

**PROPOSITION 21.** *The problem  $\{\exists, \wedge, \neq\}\text{-FO}(\mathcal{D})$  is in L if  $|D| = 1$  and in P if  $|D| = 2$ , and  $\mathcal{D}$  is bijunctive or affine, and NP-complete otherwise. The fragment extended with  $=$  follows the same dichotomy.*

*Proof.* We classify first the fragment extended with  $=$ . When  $|D| \geq 3$ , we may use  $\neq$  for a trivial reduction from  $\text{CSP}(\mathcal{K}_{|D|})$ , which is NP-complete. When  $|D| = 1$ , the problem is trivially in L by Proposition 12. We are left with the Boolean case. Let  $\mathcal{D}_{\neq}$  denote the expansion of  $\mathcal{D}$  with  $\neq^{\mathcal{D}}$ . Note that  $\{\exists, \wedge, \neq, =\}\text{-FO}(\mathcal{D})$  coincides with  $\{\exists, \wedge, =\}\text{-FO}(\mathcal{D}_{\neq})$ , which is the Boolean  $\text{CSP}(\mathcal{D}_{\neq})$ . We apply Schaefer's theorem. The relation  $\neq$  is neither Horn, nor dual-Horn, nor 0-valid, nor 1-valid, as it is not closed under any of the following Boolean operations:  $\wedge$ ,  $\vee$ ,  $c_0$ , or  $c_1$  (the constant functions 0 and 1). The relation  $\neq$  is both bijunctive and affine, as it is closed under both the Boolean majority and the minority operations (see Chen's survey for the definitions [10]). Consequently,  $\{\exists, \wedge, \neq, =\}\text{-FO}(\mathcal{D})$  is in P if  $\mathcal{D}$  is bijunctive or affine, and NP-complete otherwise.

It remains to classify the fragment without  $=$ . Note that above we have not used  $=$  in the hardness proof when  $|D| \geq 3$ . When  $|D| = 2$ , we appealed to Schaefer's theorem (Theorem 6). We can propagate out  $=$  in polynomial time by renaming variables. Thus, such cases remain hard also in the absence of  $=$ , and our claim follows for the fragment  $\{\exists, \wedge, \neq\}\text{-FO}$ .  $\square$

**PROPOSITION 22.** *The problem  $\{\exists, \forall, \wedge, \neq\}\text{-FO}(\mathcal{D})$  is in L if  $|D| = 1$  and in P if  $|D| = 2$ , and  $\mathcal{D}$  is bijunctive or affine, and Pspace-complete otherwise. The fragment extended with  $=$  follows the same dichotomy.*

*Proof.* We use the same notation as in the proof of Proposition 21 and proceed similarly. When  $|D| \geq 3$ , we may use  $\neq$  to trivially reduce the Pspace-complete  $\text{QCSP}(\mathcal{K}_{|D|})$  to  $\{\exists, \forall, \wedge, \neq\}\text{-FO}(\mathcal{D})$ . In the Boolean case, we apply Theorem 7 to  $\{\exists, \forall, \wedge\}\text{-FO}(\mathcal{D}_{\neq})$ , and the result follows. Again, equality may be propagated out and

hardness results extend to the fragment without  $=$ .  $\square$

The cases of  $\{\exists, \wedge\}$ -FO and  $\{\exists, \wedge, =\}$ -FO almost coincide, as equality may be propagated out by substitution, and every sentence of the latter is logically equivalent to a sentence of the former, with the exception of sentences using only  $=$  as an extensional predicate like  $\exists x x = x$  which are tautologies, as we only ever consider structures with at least one element. In the case of  $\{\exists, \forall, \wedge, =\}$ -FO, some equalities like  $\exists x \exists y x = y$  and  $\forall x \exists y x = y$  may also be propagated out by substitution. However, equalities like  $\exists x \forall y x = y$  and  $\forall x \forall y x = y$  cannot, but they hold only in structures with a single element. This technical issue does not really affect the complexity classification, and it would suffice to consider  $\{\exists, \wedge\}$ -FO and  $\{\exists, \forall, \wedge\}$ -FO. The complexity classifications for these two fragments remain open and correspond to the dichotomy conjecture for the CSP and the classification program of the QCSP. In practice, we like to pretend that equality is present, as it provides a better behaved algebraic framework, without affecting complexity.

**3.4. Fourth class.** The following—left as a conjecture at the end of [25, 32]—is the main contribution of this paper.

**THEOREM 23.** *Let  $\mathcal{D}$  be any structure:*

I. *If  $\mathcal{D}$  is preserved by both a singleton-surjective shop and a singleton-total shop, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is in L.*

II. *If  $\mathcal{D}$  is preserved by a singleton-surjective shop but is not preserved by any singleton-total shop, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is NP-complete.*

III. *If  $\mathcal{D}$  is preserved by a singleton-total shop but is not preserved by any singleton-surjective shop, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is co-NP-complete.*

IV. *If  $\mathcal{D}$  is preserved neither by a singleton-surjective shop nor by a singleton-total shop, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is Pspace-complete.*

The upper bounds (membership in L, NP, and co-NP) for Cases I, II, and III were known from [25]. The lower bounds are proved in section 5.

#### 4. Containment, relativization, and a core for $\{\exists, \forall, \wedge, \vee\}$ -FO.

**4.1. A theorem à la Chandra and Merlin for  $\{\exists, \forall, \wedge, \vee\}$ -FO.** It is well known that conjunctive query containment is characterized by the presence of homomorphism between the corresponding canonical databases.

**THEOREM 24** (see Chandra and Merlin [7]; see also [15, Chapter 6]). *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two structures. The following are equivalent:*

(i) *For every sentence  $\varphi$  in  $\{\exists, \wedge\}$ -FO, if  $\mathcal{A} \models \varphi$ , then  $\mathcal{B} \models \varphi$ .*

(ii) *There exists a homomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ .*

(iii)  *$\mathcal{B} \models \varphi_{\mathcal{A}}^{\{\exists, \wedge\}\text{-FO}}$ , where  $\varphi_{\mathcal{A}}^{\{\exists, \wedge\}\text{-FO}} := \exists v_1 \exists v_2 \dots v_{|\mathcal{A}|} \varphi_{\mathcal{A}}$ .*

Similarly, we characterize both algebraically and in terms of a canonical sentence the containment for  $\{\exists, \forall, \wedge, \vee\}$ -FO, a result that was not explicitly stated in [25] but follows easily from the ingredients needed to establish the Galois connection  $\text{Inv} - \text{shE}$  (see Theorem 4). It is crucial that we provide here some details, as we shall tweak some of these proofs in the next section.

**FACT 25.** *Any sentence of  $\{\exists, \forall, \wedge, \vee\}$ -FO is logically equivalent to a prenex sentence whose quantifier-free part is a disjunction of conjunction-of-positive-atoms  $\psi_i$ .*

Winning the  $(\mathcal{A}, \varphi)$ -game for a sentence  $\varphi$  may be recast as the existence of a surjective hyper-endomorphism from the canonical database  $\mathcal{D}_{\psi_i}$  of some disjunct  $\psi_i$  to  $\mathcal{A}$ . Thus we may prove in the same fashion the following analogue of Lemma 15.

LEMMA 26. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two structures such that there is a surjective hyper-morphism from  $\mathcal{A}$  to  $\mathcal{B}$ . Then, for every sentence  $\varphi$  in  $\{\exists, \forall, \wedge, \vee\}$ -FO, if  $\mathcal{A} \models \varphi$ , then  $\mathcal{B} \models \varphi$ .

We extend the notion of *canonical conjunctive query* of a structure  $\mathcal{A}$ . Given a tuple of (not necessarily distinct) elements  $\mathbf{r} := (r_1, \dots, r_l) \in A^l$ , define the quantifier-free formula  $\varphi_{\mathcal{A}(\mathbf{r})}(v_1, \dots, v_l)$  to be the conjunction of the positive facts of  $\mathbf{r}$ , where the variables  $v_1, \dots, v_l$  correspond to the elements  $r_1, \dots, r_l$ . That is,  $R(v_{\lambda_1}, \dots, v_{\lambda_i})$  appears as an atom in  $\varphi_{\mathcal{A}(\mathbf{r})}$  if and only if  $R(r_{\lambda_1}, \dots, r_{\lambda_i})$  holds in  $\mathcal{A}$ . When  $\mathbf{r}$  enumerates the elements of the structure  $\mathcal{A}$ , this definition coincides with the usual definition of canonical conjunctive query. Note also that there is a full homomorphism from the canonical database  $\mathcal{D}_{\varphi_{\mathcal{A}(\mathbf{r})}}$  to  $\mathcal{A}$  given by the map  $v_{\lambda_i} \mapsto r_i$ . Below, we write  $\mathcal{A}(\mathbf{r}, \mathbf{t})$  to stress that the tuple of elements of  $\mathcal{A}$  consists of two subtuples  $\mathbf{r}$  and  $\mathbf{t}$ .

DEFINITION 27 (see [25]). Let  $\mathcal{A}$  be a structure, and let  $m > 0$ . Let  $\mathbf{r}$  be an enumeration of the elements of  $\mathcal{A}$ :

$$\theta_{\mathcal{A}, m}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}} := \exists v_1, \dots, v_{|A|} \varphi_{\mathcal{A}(\mathbf{r})}(v_1, \dots, v_{|A|}) \wedge \forall w_1, \dots, w_m \bigvee_{\mathbf{t} \in A^m} \varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{v}, \mathbf{w}).$$

Observe that  $\mathcal{A} \models \theta_{\mathcal{A}, m}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$ . Indeed, we may take as a witness for the variables  $\mathbf{v}$  the corresponding enumeration  $\mathbf{r}$  of the elements of  $\mathcal{A}$ , and, for any assignment  $\mathbf{t} \in A^m$  to the universal variables  $\mathbf{w}$ , it is clear that  $\mathcal{A} \models \varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{r}, \mathbf{t})$  holds.

LEMMA 28. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two structures. If  $\mathcal{B} \models \theta_{\mathcal{A}, |\mathcal{B}|}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$ , then there is a surjective hyper-morphism from  $\mathcal{A}$  to  $\mathcal{B}$ .

*Proof.* Let  $\mathbf{b}' := b'_1, \dots, b'_{|A|}$  be witnesses for  $v_1, \dots, v_{|A|}$ . Assume that an enumeration  $\mathbf{b} := b_1, b_2, \dots, b_{|\mathcal{B}|}$  of the elements of  $\mathcal{B}$  is chosen for the universal variables  $w_1, \dots, w_{|\mathcal{B}|}$ . Let  $\mathbf{t} \in A^m$  such that  $\mathcal{B} \models \varphi_{\mathcal{A}(\mathbf{r})}(\mathbf{b}') \wedge \varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{b}', \mathbf{b})$ .

Let  $f$  be the map from the domain of  $\mathcal{A}$  to the power set of that of  $\mathcal{B}$  which is the union of the following two partial hyper-operations  $h$  and  $g$  (i.e.,  $f(a_i) := h(a_i) \cup g(a_i)$  for any element  $a_i$  of  $\mathcal{A}$ ), which guarantee totality and surjectivity, respectively. Let  $\mathbf{r}$  be the enumeration  $a_1, a_2, \dots, a_{|A|}$  of  $\mathcal{A}$ . We set

- $h(a_i) := b'_i$  (totality).

We set  $g$  to be the partial hyper-operation that is the smallest w.r.t. to inclusion (of hyper-operations) such that

- $b_i \in g(t_i)$  (surjectivity).

It remains to show that  $f$  is preserving. This follows from  $\mathcal{B} \models \varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{b}', \mathbf{b})$ .

Let  $R$  be an  $r$ -ary relational symbol such that  $R(a_{i_1}, \dots, a_{i_r})$  holds in  $\mathcal{A}$ . Let  $b''_{i_1} \in f(a_{i_1}), \dots, b''_{i_r} \in f(a_{i_r})$ . We will show that  $R(b''_{i_1}, \dots, b''_{i_r})$  holds in  $\mathcal{B}$ . Assume for clarity of the exposition and w.l.o.g. that from  $i_1$  to  $i_k$  the image is set according to  $h$  and from  $i_{k+1}$  to  $i_r$  according to  $g$ , i.e., for  $1 \leq j \leq k$ ,  $h(a_{i_j}) = b'_{i_j} = b''_{i_j}$ , and for  $k+1 \leq j \leq r$ , there is some  $l_j$  such that  $t_{l_j} = a_{i_j}$  and  $b''_{i_j} = b_{l_j} \in g(t_{l_j})$ . By the definition of  $\mathcal{A}(\mathbf{r}, \mathbf{t})$ , the atom  $R(v_{i_1}, \dots, v_{i_k}, w_{l_{k+1}}, \dots, w_r)$  appears in  $\varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{v}, \mathbf{w})$ . It follows from  $\mathcal{B} \models \varphi_{\mathcal{A}(\mathbf{r}, \mathbf{t})}(\mathbf{b}', \mathbf{b})$  that  $R(b''_{i_1}, \dots, b''_{i_r})$  holds in  $\mathcal{B}$ .  $\square$

THEOREM 29. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two structures. The following are equivalent:

- (i) For every sentence  $\varphi$  in  $\{\exists, \forall, \wedge, \vee\}$ -FO, if  $\mathcal{A} \models \varphi$ , then  $\mathcal{B} \models \varphi$ .
- (ii) There exists a surjective hyper-morphism from  $\mathcal{A}$  to  $\mathcal{B}$ .
- (iii)  $\mathcal{B} \models \theta_{\mathcal{A}, |\mathcal{B}|}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$ .

*Proof.* By construction,  $\mathcal{A} \models \theta_{\mathcal{A},|B|}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$ , so (i) implies (iii). By Lemma 26, (ii) implies (i). By Lemma 28, (iii) implies (ii).  $\square$

**4.2. Relativization.** Let  $\mathcal{A}$  be a finite structure over a set  $A$ , and let  $U, X$  be two subsets of  $A$ . Given a formula  $\varphi$ , we denote by  $\varphi_{[\forall u/\forall u \in U, \exists x/\exists x \in X]}$  the formula obtained from  $\varphi$  by relativizing simultaneously every universal quantifier to  $U$  and every existential quantifier to  $X$ . When we only relativize universal quantifiers to  $U$ , we write  $\varphi_{[\forall u/\forall u \in U]}$ , and when we only relativize existential quantifiers to  $X$ , we write  $\varphi_{[\exists x/\exists x \in X]}$ . Formally,  $U$  should also be understood as denoting a new unary symbol, not present in the signature of  $\mathcal{A}$ , whose interpretation is the subset  $U$  of  $A$ . However, for the sake of simplicity and readability, rather than writing  $\langle \mathcal{A}, U \rangle \models \varphi_{[\forall u/\forall u \in U]}$ , we will abuse this notation and write  $\mathcal{A} \models \varphi_{[\forall u/\forall u \in U]}$  instead. We proceed similarly with  $X$ .

**DEFINITION 30.** Let  $\mathcal{A}$  be a finite structure over a set  $A$ , and let  $U, X$  be two subsets of  $A$ . We say that  $\mathcal{A}$  has  $\forall U\text{-}\exists X$ -relativization if, for all sentences  $\varphi$  in  $\{\exists, \forall, \wedge, \vee\}$ -FO, the following are equivalent:

- (i)  $\mathcal{A} \models \varphi$ .
- (ii)  $\mathcal{A} \models \varphi_{[\forall u/\forall u \in U]}$ .
- (iii)  $\mathcal{A} \models \varphi_{[\exists x/\exists x \in X]}$ .
- (iv)  $\mathcal{A} \models \varphi_{[\forall u/\forall u \in U, \exists x/\exists x \in X]}$ .

**LEMMA 31.** Let  $\mathcal{A}$  be a finite structure over a set  $A$ , and let  $U, X$  be two subsets of  $A$ . If  $\mathcal{A}$  has a  $U$ -surjective  $X$ -total hyper-endomorphism, then  $\mathcal{A}$  has  $\forall U\text{-}\exists X$ -relativization.

*Proof.* Note that in Definition 30, we have (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii) trivially. It suffices to prove that (ii)  $\Rightarrow$  (i) and (i)  $\Rightarrow$  (iii) to complete the proof. To do so, we will consider the  $(\mathcal{A}, \varphi)$ -game corresponding to case (i), called the *unrelativized game* hereafter, and we will consider the relativized games corresponding to the relativized formulae from cases (ii) and (iii). The relativized game is considered to be clear from context: we consider either partial Skolem functions that need only be defined when universal variables range over  $U$  or Skolem functions that must range over  $X$ .

Let  $h$  be a  $U$ -surjective  $X$ -total hyper-endomorphism of  $\mathcal{A}$ . We fix an arbitrary linear order over  $A$  and denote the minimum by  $\min_A$ .

((ii)  $\Rightarrow$  (i)). Let  $\{\sigma_x : \text{“}\exists x\text{”} \in \varphi\}$  be a winning strategy in the universally relativized game. We derive a winning strategy in the unrelativized game using  $h$  as follows.

Let  $\pi_A : Y_x \rightarrow A$  be an assignment to the universal variables  $Y_x$  preceding an existential variable  $x$  in  $\varphi$ , where  $\pi_A$  ranges over both  $U$  and  $A \setminus U$ . We set  $\sigma'_x(\pi_A) := \min_A h(\sigma_x(\pi_U))$ , where  $\pi_U : Y_x \rightarrow U$  is an assignment such that for any universal variable  $y$  preceding  $x$ , we have  $\pi_U(y) := \min_A (h^{-1}(\pi_A(y)) \cap U)$ . This is possible since  $h$  is  $U$ -surjective (which means that  $\pi_U$  is well defined) and  $A$ -total (which means that  $h(\sigma_x(\pi_U)) \neq \emptyset$ ). This does provide us with a winning strategy since  $h$  is a hyper-endomorphism of  $\mathcal{A}$  and will preserve the winning condition from the universally relativized game to the unrelativized game.

((i)  $\Rightarrow$  (iii)). This is dual to the above and relies on  $X$ -totality. For any  $\pi'_A : Y_x \rightarrow A$ , set  $\sigma'_x(\pi'_A) := \min_A (h(\sigma_x(\pi_A)) \cap X)$ , where  $\pi_A(y) := \min_A (h^{-1}(\pi'_A(y)))$ .  $\square$

**PROPOSITION 32.** The following are equivalent:

- (i)  $\mathcal{A}$  has  $\forall U\text{-}\exists X$ -relativization.
- (ii)  $\bar{\mathcal{A}}$  has  $\forall X\text{-}\exists U$ -relativization.

*Proof.* It suffices to prove one implication. We prove that (ii) implies (i). Let  $\varphi$  be a sentence of  $\{\exists, \forall, \wedge, \vee\}$ -FO. We use the duality principle and prove that  $\mathcal{A} \models \varphi \iff \mathcal{A} \models \varphi_{[\forall u/\forall u \in U]}$ . The other cases are similar and are omitted.

We follow the same notation as in the proof of Proposition 2. By assumption,  $\overline{\mathcal{A}} \not\models \overline{\varphi} \iff \overline{\mathcal{A}} \not\models \overline{\varphi}_{[\exists u/\exists u \in U]}$ . Using the chain of equivalences from this proof backward and propagating the relativization, we obtain the following chain of equivalences:

$$\begin{aligned} \overline{\mathcal{A}} \not\models \overline{\varphi}_{[\exists u/\exists u \in U]} &\iff \mathcal{A} \not\models \psi_{[\exists u/\exists u \in U]} \iff \mathcal{A} \models \neg(\psi_{[\exists u/\exists u \in U]}) \\ &\iff \mathcal{A} \models \neg(\neg\varphi_{[\forall u/\forall u \in U]}) \iff \mathcal{A} \models \varphi_{[\forall u/\forall u \in U]}. \quad \square \end{aligned}$$

**LEMMA 33.** *Let  $\mathcal{A}$  be a finite structure over a set  $A$ , and let  $U, X$  be two subsets of  $A$ . If  $\mathcal{A}$  has  $\forall U$ - $\exists X$ -relativization, then  $\mathcal{A}$  has a  $U$ -surjective  $X$ -total hyper-*endomorphism*.*

*Proof.* Using the fact that the identity (defined as  $i(x) := \{x\}$  for every  $x$  in  $\mathcal{A}$ ) is a surjective hyper-*endomorphism* of  $\mathcal{A}$  and applying Theorem 29, we derive that  $\mathcal{A} \models \theta$ , where  $\theta$  is short for  $\theta_{\mathcal{A}, |A|}^{\{\exists, \forall, \wedge, \vee\}\text{-FO}}$ . By assumption, we may equivalently relativize its existential quantifiers to  $X$  (Definition 30 (i)  $\Rightarrow$  (iii)) and  $\mathcal{A} \models \theta_{[\exists x/\exists x \in X]}$ . Proceeding as in the proof of Lemma 28 but over this relativized sentence, we derive the existence of an  $X$ -total surjective hyper-*operation*  $g$ . Using Proposition 32 and working over  $\overline{\mathcal{A}}$ , we derive similarly that  $\overline{\mathcal{A}}$  has a  $U$ -total surjective hyper-*operation*. Let  $f$  be the inverse of this hyper-*operation*. Observe that it is a  $U$ -surjective hyper-*operation*. By Lemma 3, the composition of these operations  $g \circ f$  is an  $X$ -total  $U$ -surjective hyper-*endomorphism*, as required.  $\square$

Together, the two previous lemmata establish an algebraic characterization of relativization.

**THEOREM 34.** *Let  $\mathcal{A}$  be a finite structure over a set  $A$ , and let  $U, X$  be two subsets of  $A$ . The following are equivalent:*

- (i) *The structure  $\mathcal{A}$  has  $\forall U$ - $\exists X$ -relativization.*
- (ii) *The structure  $\mathcal{A}$  has an  $X$ -total  $U$ -surjective hyper-*endomorphism*.*

**COROLLARY 35.** *Let  $\mathcal{A}$  be a finite structure that has a  $U$ -surjective  $X$ -total hyper-*endomorphism*. Let  $\tilde{\mathcal{A}}$  be the substructure of  $\mathcal{A}$  induced by  $U \cup X$ . The following hold:*

- (i)  *$\mathcal{A}$  and  $\tilde{\mathcal{A}}$  are  $\{\exists, \forall, \wedge, \vee\}$ -FO-equivalent.*
- (ii)  *$\tilde{\mathcal{A}}$  has  $\forall U$ - $\exists X$ -relativization.*

*Proof.* Let  $f$  be the  $U$ -surjective  $X$ -total hyper-*endomorphism* of  $\mathcal{A}$ . Its range restriction  $g$  to  $\tilde{A} = U \cup X$  is a surjective hyper-*morphism* from  $\mathcal{A}$  to  $\tilde{\mathcal{A}}$ . The inverse  $g^{-1}$  of  $g$  is a surjective hyper-*morphism* from  $\tilde{\mathcal{A}}$  to  $\mathcal{A}$  by  $X$ -totality of  $f$ . Appealing to Lemma 26 twice, once with  $g$  and once with  $g^{-1}$ , we obtain (i).

The restriction of  $g$  to  $\tilde{A}$  is a  $U$ -surjective  $X$ -total hyper-*endomorphism* of  $\tilde{\mathcal{A}}$ , and (ii) follows from Lemma 31.  $\square$

**4.3. The  $\forall U$ - $\exists X$ -core.** Unless otherwise stated, from now on minimality or maximality of sets is w.r.t. inclusion. Given a structure  $\mathcal{D}$ , we consider all minimal subsets  $X$  of  $D$  such that there is an  $X$ -total surjective hyper-*endomorphism*  $g$  of  $\mathcal{D}$  and all minimal subsets  $U$  such that there is a  $U$ -surjective hyper-*endomorphism*  $f$  of  $\mathcal{D}$ . Such sets always exist, as totality and surjectivity of surjective hyper-*endomorphisms* mean that in the worst case we may choose  $U = X = D$ . Recall that  $g \circ f$  is an  $X$ -total  $U$ -surjective hyper-*endomorphisms* of  $\mathcal{D}$  by Lemma 3. Thus, we may furthermore require that among all minimal sets satisfying the above, we choose a set  $U$  and

a set  $X$  in  $D$  with  $U \cap X$  maximal. Let  $\tilde{D}$  be the substructure of  $\mathcal{D}$  induced by  $U \cup X$ . We call  $\tilde{D}$  a  $\forall U\text{-}\exists X$ -core of  $\mathcal{D}$ .

**PROPOSITION 36.** *Given a structure  $\mathcal{D}$ , minimal subsets  $X$  (resp.,  $U$ ) of  $D$  such that there is an  $X$ -total (resp.,  $U$ -surjective) surjective hyper-endomorphism  $g$  of  $\mathcal{D}$  all have the same size.*

*Proof.* Assume that there is an  $X_1$ -total shop  $h_1$  and an  $X_2$ -total shop  $h_2$  that preserve  $\mathcal{D}$  such that  $|X_1| > |X_2|$ . We consider images of  $h_1 \circ h_2$ . For each element  $x_2$  in  $X_2$ , pick a single element  $x'_1$  of  $X_1$  in  $h_1(X_2)$  such that  $x'_1 \in h_1(x_2)$ . Let  $X'_1$  denote the set of picked elements. Since  $|X_1| > |X_2|$ , then  $h_1 \circ h_2$  is an  $X'_1$ -shop that preserves  $\mathcal{D}$  with  $|X'_1| \leq |X_2|$ . Diagrammatically, this can be written as

$$D \xrightarrow{h_2} X_2 \xrightarrow{h_1} X'_1 \subseteq h_1(X_2) \cap X_1 \subseteq X_1 \subseteq h_1 \circ h_2(D).$$

This completes the proof.  $\square$

*Remark 37.* This means that we may look for an  $X$ -total shop where the set  $X$  is minimal w.r.t. inclusion, or equivalently for a set with minimal size  $|X|$ . So, in order to find an  $X$ -total shop with a minimal set  $|X|$ , we may proceed greedily, removing elements from  $D$  while we have an  $X$ -total shop until we obtain a set  $X$  such that there is no  $X'$ -shop for  $X' \subsetneq X$ . The dual argument applies to  $U$ -surjective shops and consequently to  $U$ -surjective  $X$ -total shops.

This further explains why minimizing  $U$  and  $X$ , and then maximizing their intersection, necessarily leads to a minimal  $\tilde{D} := U \cup X$  also. This is because if there were  $U' \cup X'$  of smaller size, we might look within  $U'$  and  $X'$  for potentially smaller sets of cardinality  $|U|$  and  $|X|$ , thus contradicting minimality.

Note that the sets  $U$  and  $X$  are not necessarily unique. However, the  $\forall U\text{-}\exists X$ -core is unique up to isomorphism (see Theorem 45). Moreover, within  $\tilde{D}$ , the sets  $U$  and  $X$  are uniquely determined (see Theorem 49).

**4.4. Uniqueness of the  $\forall U\text{-}\exists X$ -core.** Throughout this section, let  $\mathcal{D}$  be a finite structure and  $\mathcal{M}$  its associated DSM; i.e.,  $\mathcal{M}$  is the set of surjective hyper-endomorphisms of  $\mathcal{D}$ . Let  $U$  and  $X$  be subsets of  $D$  such that the substructure  $\tilde{D}$  of  $\mathcal{D}$  induced by  $\tilde{D} = U \cup X$  is a  $\forall U\text{-}\exists X$ -core of  $\mathcal{D}$ . We will progress through various lemmata and eventually derive the existence of a canonical  $\forall U\text{-}\exists X$ -shop in  $\mathcal{M}$  which will be used to prove that the  $\forall U\text{-}\exists X$ -core is unique up to isomorphism. Uniqueness of the  $\forall U\text{-}\exists X$ -core has no real bearing on our classification program, but the canonical shop will allow us to characterize all other shops in  $\mathcal{M}$ , which will be instrumental in the hardness proofs for  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ).

**LEMMA 38.** *Let  $f$  be a shop in  $\mathcal{M}$  (note that  $f$  is an arbitrary shop). For any element  $z$  in  $D$ ,  $f(z)$  contains at most one element of the set  $U$ , that is,  $|f(z) \cap U| \leq 1$ .*

*Proof.* Assume for contradiction that there are some  $z$  and some distinct elements  $u_1$  and  $u_2$  of  $U$  such that  $f(z) \supseteq \{u_1, u_2\}$ . Let  $z_3, z_4, \dots$  be any choice of antecedents under  $f$  of the remaining elements  $u_3, u_4, \dots$  of  $U$  (recall that  $f$  is surjective). By assumption, the monoid  $\mathcal{M}$  contains a  $U$ -surjective shop  $g$ . Hence,  $g \circ f$  would be a  $U'$ -surjective shop with  $U' = \{z, z_3, z_4, \dots\}$  since  $f(U') \subseteq U$  and  $g(U) = D$ . We get a contradiction, as  $|U'| < |U|$ .  $\square$

**LEMMA 39.** *Let  $f$  be a  $U$ -surjective shop in  $\mathcal{M}$ . There exists a permutation  $\alpha$  of  $U$  such that for any  $u$  in  $U$ ,*

$$(i) \quad f(u) \cap U = \{\alpha(u)\} \text{ and}$$

(ii)  $f^{-1}(u) \cap U = \{\alpha^{-1}(u)\}$ .

*Proof.* It follows from Lemma 38 that for any  $u$  in  $U$ ,  $|f(u) \cap U| \leq 1$ . Since  $f$  is a  $U$ -surjective shop, every element in  $D$  has an antecedent in  $U$  under  $f$ , and thus in particular for any  $u$  in  $U$ ,  $|f^{-1}(u) \cap U| \geq 1$ . Note that if some element of  $U$  had no image in  $U$ , then as  $U$  is finite, we would have an element of  $U$  with two distinct images in  $U$ . Hence, for any  $u$  in  $U$ ,  $|f(u) \cap U| = 1$  and the result follows.  $\square$

The dual statements concerning  $X$  and  $X$ -total shops hold.

LEMMA 40. *Let  $f$  be a shop in  $\mathcal{M}$  (note that  $f$  is an arbitrary shop). For any element  $z$  in  $D$ ,  $f^{-1}(z)$  contains at most one element of the set  $X$ , that is,  $|f^{-1}(z) \cap X| \leq 1$ .*

*Proof.* This is by duality from Lemma 38.  $\square$

LEMMA 41. *Let  $f$  be an  $X$ -total shop in  $\mathcal{M}$ . There exists a permutation  $\beta$  of  $X$  such that for any  $x$  in  $X$ ,*

- (i)  $f(x) \cap X = \{\beta(x)\}$  and
- (ii)  $f^{-1}(x) \cap X = \{\beta^{-1}(x)\}$ .

*Proof.* This is by duality from Lemma 39.  $\square$

LEMMA 42. *If  $f$  is a  $U$ -surjective  $X$ -total shop in  $\mathcal{M}$ , then  $f(X) \cap (U \setminus X) = \emptyset$ .*

*Proof.* Assume for contradiction that for some  $x_1 \in X$  and some  $u_1 \in U \setminus X$ , we have  $u_1 \in f(x_1)$ . Since  $f$  is an  $X$ -total shop, every element is an antecedent under  $f$  of some element in  $X$ ; in particular, every element  $x_2, x_3, \dots \in X$  (different from  $x_1$ ) has a unique image  $x'_2, x'_3, \dots \in X$  (see Lemma 41). Some element of  $X$ , say  $x_i$ , does not occur in these images. Necessarily,  $x_1$  reaches  $x_i$ . Note that  $x_i$  cannot also belong to  $U$ , as otherwise  $x_i$  and  $u_1$ , two distinct elements of  $U$ , would be reached by  $x_1$ , contradicting Lemma 38. Thus, we must have that  $x_i$  belongs to  $X \setminus U$ . Let  $U' := U$  and  $X' := X \setminus \{x_i\} \cup \{u_1\}$ . Note that  $f^2 := f \circ f$ , the second iterate of  $f$ , is a  $U'$ -surjective  $X'$ -total shop with  $|U'| = |U|$ ,  $|X'| = |X|$ , and  $|U' \cap X'| > |U \cap X|$ . This contradicts our hypothesis on  $U$  and  $X$ .  $\square$

PROPOSITION 43. *Let  $\mathcal{M}$  be a DSM over a set  $D$  and  $U$ , and let  $X$  be minimal subsets of  $D$  such that there is a  $U$ -surjective shop in  $\mathcal{M}$ , there is an  $X$ -total shop in  $\mathcal{M}$ , and  $U \cup X$  is minimal. Then there is a  $U$ -surjective  $X$ -total shop  $h$  in  $\mathcal{M}$  that has the following properties:*

- (i) for any  $z$  in  $X$ ,  $h(z) \cap (U \cup X) = \{z\}$ ;
- (ii) for any  $u$  in  $U \setminus X$ ,  $h(u) \cap (U \cup X) = \{u\} \cup X_u$ , where  $X_u \subseteq X \setminus U$ ; and
- (iii)  $h(U \setminus X) \cap X = \bigcup_{u \in U \setminus X} X_u = X \setminus U$ .

*Proof.* By assumption,  $\mathcal{M}$  contains a  $U$ -surjective  $X$ -total shop  $f$ . Let  $\alpha$  and  $\beta$  be permutations of  $U$  and  $X$ , respectively, as in Lemmas 39 and 41. Let  $r$  be the least common multiple of the order of the permutations  $\beta$  and  $\alpha$ . We set  $h$  to be the  $r$ th iterate of  $f$ , and we now know that  $z \in h(z)$  for any element  $z$  in  $U \cup X$  and that  $h$  is a  $U$ -surjective  $X$ -total shop by Lemma 3. Letting  $y$  in  $U \cap X$ , we know that  $y \in h(y)$ . We cannot have another element from  $U \cup X$  in  $h(y)$  by Lemmas 38 and 41. This proves (i) in part; the other case follows. Letting  $x$  in  $X \setminus U$ , we know that  $x \in h(x)$ . We cannot have an element from  $X$  distinct from  $x$  in  $h(x)$  by Lemma 41, and we cannot have an element from  $U \setminus X$  in  $h(x)$  by Lemma 42. This proves (i). Letting  $u$  in  $U \setminus X$ , we know that  $u \in h(u)$ . We cannot have an element from  $U$  distinct from  $u$  in  $h(u)$  by Lemma 39. We may have, however, some elements from  $X \setminus U$  in  $h(u)$ . Thus, there is a set  $\emptyset \subseteq X_u \subseteq X \setminus U$  such that  $h(u) \cap (U \cup X) = \{u\} \cup X_u$ . This

proves (ii). By construction,  $h$  is a  $U$ -surjective shop and every element must have an antecedent in  $U$  under  $h$ . Since by the first two points elements from  $X \setminus U$  can only be reached from elements of  $U$  that are in  $U \setminus X$ , the last point (iii) follows.  $\square$

*Remark 44.* Given  $h_1$  and  $h_2$  of the form in Proposition 43,  $h_1 \circ h_2$  is also of the required form, and further satisfies  $h_1 \circ h_2(z) = h_1(z) \cup h_2(z)$ , for all  $z \in U \setminus X$ . Hence, in general, given suitable  $U$  and  $X$  and taking the composition of all such shops results in a *canonical shop*  $h$  of this form with  $|h(z)|$  maximal for each  $z \in U \setminus X$ .

**THEOREM 45.** *The  $\forall U$ - $\exists X$ -core is unique up to isomorphism.*

*Proof.* Let  $h_1$  be a  $U_1$ - $X_1$ -shop with minimal  $|U_1|$ ,  $|X_1|$ , and  $|U_1 \cup X_1|$ , and let  $h_2$  be a  $U_2$ - $X_2$ -shop with minimal  $|U_2|$ ,  $|X_2|$ , and  $|U_2 \cup X_2|$ . Hence,  $h_1 \circ h_2$  is an  $h_1(X_2) \cap X_1$ -shop with  $|h_1(X_2)| \leq |X_1|$ . By minimality of  $X_1$ ,  $|h_1(X_2)| = |X_1|$ , and the restriction of  $h_1$  to domain  $X_2$  and codomain  $X_1$  induces a surjective homomorphism from the substructure induced by  $X_2$  to the substructure induced by  $X_1$ . Similarly,  $h_2$  induces a surjective homomorphism in the other direction. As we work with finite structures,  $h_1$  induces an isomorphism  $i$  from the substructure induced by  $X_1$  to the substructure induced by  $X_2$ . By duality, we also get that  $h_1$  induces an isomorphism  $i'$  from the substructure induced by  $U_1$  to the substructure induced by  $U_2$ . By construction,  $i$  and  $i'$  agree on  $U_1 \cap X_1$  (necessarily to  $U_2 \cap X_2$ ) and the result follows.  $\square$

**5. Proving hardness to establish the tetrachotomy.** Our aim is to derive the lower bounds of Theorem 23 to conclude its proof. It follows from Proposition 43 and Corollary 35 that the complexity of a structure  $\mathcal{D}$  is the same as the complexity of its  $\forall U$ - $\exists X$ -core. Hence, in this section, we assume *w.l.o.g.* that  $U \cup X = D$ . We will say in this case that the DSM  $\mathcal{M}$  is *reduced*.

The lower bounds can be rephrased as follows:

- II. *If  $U$  is of size one and  $X$  of size at least two, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is NP-hard.*
- III. *If  $X$  is of size one and  $U$  of size at least two, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is co-NP-hard.*
- IV. *If both  $U$  and  $X$  have at least two elements, then  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is Pspace-hard.*

In subsection 5.1, we study extensively reduced monoids. Cases II and III are proved in subsection 5.3. Finally, case IV is proved in subsection 5.4.

In the following, we will describe a DSM  $\mathcal{M}$  as being (NP, co-NP, Pspace)-hard in the case that  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) is hard for some structure  $\mathcal{D}$ , with the same domain as  $\mathcal{M}$  and relations from  $\text{Inv}(\mathcal{M})$  (we shall write  $\mathcal{D} \in \text{Inv}(\mathcal{M})$  to denote this hereafter).

In order to facilitate the hardness proof, we will show the hardness of a monoid  $\widehat{\mathcal{M}}$  with a very simple structure of which  $\mathcal{M}$  is in fact a sub-DSM (as in general  $\widehat{\mathcal{M}}$  preserves fewer relations than  $\mathcal{M}$ , the hardness of  $\mathcal{M}$  would follow). We ensure that the structure of  $\widehat{\mathcal{M}}$  is sufficiently simple to allow us to build canonically some relatively simple gadgets for our hardness proof.

**5.1. Characterizing reduced DSMs.** Any  $U$ -surjective  $X$ -total shop in  $\mathcal{M}$  will be shown to be in the following special form, reminiscent of the form of the canonical shop.

**DEFINITION 46.** *We say that a shop  $f$  is in the 3-permuted form if there are a permutation  $\zeta$  of  $X \cap U$ , a permutation  $\chi$  of  $X \setminus U$ , and a permutation  $\nu$  of  $U \setminus X$  such that  $f$  satisfies the following:*

- for any  $y$  in  $U \cap X$ ,  $f(y) = \{\zeta(y)\}$ ;
- for any  $x$  in  $X \setminus U$ ,  $f(x) = \{\chi(x)\}$ ; and,
- for any  $u$  in  $U \setminus X$ ,  $f(u) = \{v(u)\} \cup X_u$ , where  $X_u \subseteq X \setminus U$ .

LEMMA 47. *If a shop  $f$  satisfies  $f(X) \cap (U \setminus X) = \emptyset$ , then  $f$  is in the 3-permuted form (note that  $f$  is an arbitrary shop).*

We have to be careful to prove the above, as we do not assume  $f$  to be a  $U$ -surjective  $X$ -total shop: in particular, we may not use the “permutation Lemmata” 39 and 41. We have to use the weaker but more general Lemmas 38 and 40.

*Proof.* The hypothesis forces an element of  $X$  to reach an element of  $X$ , and Lemma 40 forces two elements of  $X$  to have different images. Since  $X$  is finite, there exists a permutation  $\beta$  of  $X$  such that for every  $x$  in  $X$ ,  $f(x) = \{\beta(x)\}$ . Since Lemma 38 forces in particular an element of  $U$  to have at most one element of  $U$  in its image and since  $U$  is finite, it follows that there exists a permutation  $\alpha$  of  $U$  such that for every  $u$  in  $U$ ,  $f(u) \cap U = \{\alpha(u)\}$  and  $f^{-1}(u) \cap U = \{\alpha^{-1}(u)\}$ .

It follows that there exists a permutation  $\zeta$  of  $U \cap X$  such that for any  $y$  in  $U \cap X$ ,  $f(y) = \{\zeta(y)\}$ .

The existence of a permutation  $\chi$  of  $X \setminus U$  such that  $\beta$  is the disjoint union of  $\chi$ , and  $\zeta$  follows. Hence, for any  $x$  in  $X \setminus U$ ,  $f(x) = \{\chi(x)\}$ .

Similarly, there must also be a permutation  $v$  of  $U \setminus X$  such that  $\alpha$  is the disjoint union of  $v$  and  $\zeta$ . Hence, for any  $u$  in  $U \setminus X$ ,  $f(u) \cap U = \{v(u)\}$ . Elements of  $U \setminus X$  may, however, have some images in  $X \setminus U$ . So we get finally that for any  $u$  in  $U \setminus X$ , there is some  $\emptyset \subseteq X_u \subseteq X \setminus U$  such that  $f(u) = \{v(u)\} \cup X_u$ . This proves that  $f$  is in the 3-permuted form, and we are done.  $\square$

THEOREM 48. *Let  $\mathcal{M}$  be a reduced DSM. Every shop in  $\mathcal{M}$  is in the 3-permuted form. Moreover, every  $U$ -surjective  $X$ -total shop in  $\mathcal{M}$  follows the additional requirement that the elements of  $U \setminus X$  cover the set  $X \setminus U$ , more formally that*

$$f(U \setminus X) \cap X = \bigcup_{u \in U \setminus X} X_u = X \setminus U.$$

*Proof.* We can now deduce easily from Lemmas 42 and 47 that  $U$ -surjective  $X$ -total shops in  $\mathcal{M}$  must take the 3-permuted form. It remains to prove that an arbitrary shop  $f$  in  $\mathcal{M}$  is in the 3-permuted form. Let  $h$  be the canonical shop of  $\mathcal{M}$ . It follows from Lemma 3 that  $f' := h \circ f \circ h$  is a  $U$ -surjective  $X$ -total shop. Hence,  $f'$  is in the 3-permuted form. Let  $z$  in  $X$  and  $u$  in  $U \setminus X$ . If  $u \in f(z)$ , then  $u \in f'(z)$  and  $f'$  would not be in the 3-permuted form. It follows that  $f(X) \cap (U \setminus X) = \emptyset$  and appealing to Lemma 47 that  $f$  is in the 3-permuted form.  $\square$

We do not need the following result in order to prove our main result. But surprisingly in a reduced DSM,  $U$  and  $X$  are unique. This means that we may speak of *the canonical shop of  $\mathcal{M}$*  instead of some canonical  $U$ -surjective  $X$ -total shop. It also means that we can define the  $\forall U$ - $\exists X$ -core of a structure  $\mathcal{D}$  *without explicitly referring to  $U$  or  $X$*  as the minimal substructure of  $\mathcal{D}$  which satisfies the same  $\{\exists, \forall, \wedge, \vee\}$ -FO sentences.

THEOREM 49. *Let  $\mathcal{D}$  be a structure that is both a  $\forall U$ - $\exists X$ -core and a  $\forall U'$ - $\exists X'$ -core; then it follows that  $U = U'$  and  $X = X'$ .*

*Proof.* We prove this by contradiction. Let  $h$  and  $h'$  be the canonical  $U$ -surjective  $X$ -total shop and  $U'$ -surjective  $X'$ -total shop, respectively. Assume  $U' \neq U$ , and let  $x$  in  $U' \setminus U$ . Note that since  $D = U \cup X$ ,  $x$  does belong to  $X \setminus U$ . Thus, there exists some  $u$  in  $U \setminus X$  such that  $h(u) \supseteq \{u, x\}$  (and necessarily  $u \neq x$ ).

By Theorem 48,  $h$  has to be in the 3-permuted form w.r.t.  $U'$  and  $X'$ , which means that  $h$  can send an element to at most one element of  $U'$ . Since  $x$  belongs to  $U'$ , it follows that  $u$  belongs to  $D \setminus U' = X' \setminus U'$ . But the 3-permuted form prohibits an element of  $X'$  to reach an element of  $U'$ , a contradiction.

The dual argument yields  $X = X'$ .  $\square$

Recall that the  $\{\exists, \forall, \wedge, \vee\}$ -FO-core  $\mathcal{D}'$  of  $\mathcal{D}$  is the smallest (w.r.t. domain size) structure that is  $\{\exists, \forall, \wedge, \vee\}$ -FO-equivalent to  $\mathcal{D}$ .

**PROPOSITION 50.** *A structure is said to be a  $\forall U$ - $\exists X$ -core if and only if it is a  $\{\exists, \forall, \wedge, \vee\}$ -FO-core. Note also that the subsets  $U$  and  $X$  are uniquely determined in a core.*

*Proof.* Let  $\mathcal{D}$  be a structure that is a  $\forall U$ - $\exists X$ -core with (unique) subsets  $U$  and  $X$ . Let  $c$  be the canonical shop of  $\mathcal{D}$ .

Let  $\mathcal{D}'$  be a  $\{\exists, \forall, \wedge, \vee\}$ -FO-core of  $\mathcal{D}$ , that is, a smallest (w.r.t. domain size) structure that is  $\{\exists, \forall, \wedge, \vee\}$ -FO-equivalent to  $\mathcal{D}$ . Let  $U'$  and  $X'$  be subsets of  $\mathcal{D}'$  witnessing that  $\mathcal{D}'$  is a  $U'$ - $X'$  core. Note that  $U' \cup X' = \mathcal{D}'$  by minimality of  $\mathcal{D}'$  (and, consequently,  $U'$  and  $X'$  are uniquely determined by Theorem 49). Let  $c'$  be the canonical shop of  $\mathcal{D}'$ .

By Theorem 29, since  $\mathcal{D}$  and  $\mathcal{D}'$  are  $\{\exists, \forall, \wedge, \vee\}$ -FO-equivalent, there exist two surjective hyper-morphisms  $g$  from  $\mathcal{D}$  to  $\mathcal{D}'$  and  $f$  from  $\mathcal{D}'$  to  $\mathcal{D}$ .

Let  $U''$  be a minimal subset of  $(g)^{-1}(U')$  such that  $g(U'') = U'$ . Note that  $f \circ c' \circ g$  is a  $U''$ -surjective shop of  $\mathcal{D}$ . By minimality of  $U$ , it follows that  $|U| \leq |U''| \leq |U'|$ . A similar argument over  $\mathcal{D}'$  gives  $|U'| \leq |U|$ , and, consequently,  $|U| = |U'|$ . Moreover, since  $c \circ (f \circ c' \circ g)$  is a  $U''$ -surjective  $X$ -total surjective hyperendomorphism of  $\mathcal{D}$ , by Theorem 49, it follows that  $U = U''$ .

This means that there is a bijection  $\alpha'$  from  $U'$  to  $U$  such that, for any  $u'$  in  $U'$ ,  $g^{-1}(u') = \{\alpha'(u')\}$ .

By duality, we obtain similarly that  $|X| = |X'|$  and that there is a bijection  $\beta$  from  $X$  to  $X'$  such that, for any  $x$  in  $X$ ,  $g(x) = \{\beta(x)\}$ .

Thus,  $g$  acts necessarily as a bijection from  $U \cap X$  to  $U' \cap X'$ .

The map  $\tilde{g}$  from  $\mathcal{D}$  to  $\mathcal{D}'$  defined for any  $u$  in  $U$  as  $\tilde{g}(u) := \alpha'^{-1}(u)$  and  $\tilde{g}(x) := \beta(x)$  is a homomorphism from  $\mathcal{D}$  to  $\mathcal{D}'$  that is both injective and surjective.

A symmetric argument yields a map  $\tilde{f}$  that is a bijective homomorphism from  $\mathcal{D}'$  to  $\mathcal{D}$ . Isomorphism of  $\mathcal{D}'$  and  $\mathcal{D}$  follows.  $\square$

*Remark 51.* To simplify the presentation, we defined the  $\mathcal{L}$ -core as a minimal structure w.r.t. domain size. Considering minimal structures w.r.t. inclusion, we would get the same notion for  $\{\exists, \forall, \wedge, \vee\}$ -FO. This remains true for the CSP and the similarly robust notion of a (classical) core, but it is not the case for the logic  $\{\exists, \forall, \wedge\}$ -FO, which corresponds to the QCSP [26].

**LEMMA 52.** *Let  $\mathcal{M}$  be a reduced DSM with associated sets  $U$  and  $X$ . One of the following three cases holds:*

1.  $U \cap X \neq \emptyset$ ,  $U \setminus X \neq \emptyset$ , and  $X \setminus U \neq \emptyset$ .
2.  $U = X$ .
3.  $U \cap X = \emptyset$ .

*Proof.* We prove that  $U \subsetneq X$  is not possible. Otherwise, let  $x$  in  $X \setminus U$ , and let  $h$  be the canonical shop. There exists some  $u$  in  $U \subsetneq X$  such that  $x \in h(u)$  by the  $U$ -surjectivity of  $h$ . Since  $u$  does not occur in the image of any other element than  $u$  under the canonical shop, this would mean that  $h$  is  $X \setminus \{u\}$ -total, contradicting the minimality of  $X$ .

By duality,  $X \subsetneq U$  is not possible either and the result follows. □

**5.2. The hard DSM above  $\mathcal{M}$ .** Let  $\mathcal{M}$  be a reduced DSM with associated sets  $U$  and  $X$ . Define the completion  $\widehat{\mathcal{M}}$  of  $\mathcal{M}$  to be the DSM generated by *all* shops in the 3-permuted form of  $\mathcal{M}$ . More precisely, the canonical shop of  $\widehat{\mathcal{M}}$  is the shop  $\hat{h}$  where every set  $X_u$  is the whole set  $X \setminus U$ , and, for every permutation  $\zeta$  of  $X \cap U$ ,  $\chi$  of  $X \setminus U$ , and  $v$  of  $U \setminus X$ , any shop in the 3-permuted form with these permutations is in  $\widehat{\mathcal{M}}$ . Note that by construction,  $\mathcal{M}$  is a sub-DSM of  $\widehat{\mathcal{M}}$ . Note also that the minimality of  $U$  and  $X$  still holds in  $\widehat{\mathcal{M}}$ . We will establish hardness for  $\widehat{\mathcal{M}}$ , whereupon hardness of  $\mathcal{M}$  follows from Corollary 5.

**5.3. Cases II and III: NP-hardness and co-NP-hardness.** We begin with case II. We note first that  $U = \{u\}$  and  $|X| \geq 2$  implies  $U \cap X = \emptyset$  by Lemma 52. The structure  $\mathcal{K}_{|X|} \uplus \mathcal{K}_1$ , the disjoint union of a clique of size  $|X|$  with an isolated vertex  $u$ , has associated DSM  $\widehat{\mathcal{M}}$ . The problem  $\{\exists, \wedge, \vee\}$ -FO( $\mathcal{K}_{|X|} \uplus \mathcal{K}_1$ ) is NP-hard by Proposition 17 since the (classical) core of  $\mathcal{K}_{|X|} \uplus \mathcal{K}_1$  is  $\mathcal{K}_{|X|}$ .

For case III, we may assume similarly to above that  $X = \{x\}$ ,  $|U| \geq 2$ , and  $U \cap X = \emptyset$  by Lemma 52. We use the duality principle, which corresponds to taking the inverse of shops. Since the inverse of an  $\{x\}$ -total  $U$ -surjective shop with  $U \geq 2$  is a  $\{U\}$ -total  $\{x\}$ -surjective shop, we may use the structure  $\overline{\mathcal{K}_{|U|}} \uplus \mathcal{K}_1$ , which is  $\{\forall, \vee, \wedge\}$ -FO-equivalent to  $\overline{\mathcal{K}_{|U|}}$  (and  $\{\forall, \vee, \wedge\}$ -FO( $\overline{\mathcal{K}_{|U|}}$ ) is co-NP-hard).

**5.4. Case IV: Pspace-hardness.** We assume that  $|U| \geq 2$  and  $|X| \geq 2$  and consider the three possible cases given by Lemma 52.

*Case 1: when  $U \cap X \neq \emptyset$ ,  $U \setminus X \neq \emptyset$ , and  $X \setminus U \neq \emptyset$ .* We write  $U \Delta X$  as an abbreviation for  $(X \setminus U) \cup (U \setminus X)$ . We consider the DSM  $\mathcal{M}'$ , which is generated by a single shop  $g'$  defined as follows:

- for every  $y$  in  $X \cap U$ ,  $g'(y) := X \Delta U$ , and
- for every  $z$  in  $X \Delta U$ ,  $g'(z) := X \cap U$ .

Note that  $\mathcal{M}$  is a sub-DSM of  $\mathcal{M}'$  since any shop  $g$  in  $\mathcal{M}$  is in the 3-permuted form and is therefore a subshop of  $g' \circ g'$ . Note also that  $g'$  is a shop of the complete bipartite graph  $\mathcal{K}_{X \Delta U, X \cap U}$ . Moreover, any surjective hyper-endomorphism of  $\mathcal{K}_{X \Delta U, X \cap U}$  is easily seen to be either a subshop of  $g'$  or a subshop of  $g' \circ g'$ . So we have proved that  $\mathcal{M}'$  is a super-DSM of  $\mathcal{M}$  and that  $\mathcal{K}_{X \Delta U, X \cap U}$  admits  $\mathcal{M}'$  as a DSM.

Observing that there is a full surjective homomorphism from  $\mathcal{K}_{X \Delta U, X \cap U}$  to  $\mathcal{K}_2$ , thus by Lemma 15 the two structures agree on all sentences of  $\{\exists, \forall, \wedge, \vee, \neg\}$ -FO and so also on all sentences of  $\{\exists, \forall, \wedge, \vee\}$ -FO. We know that  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_2$ ) is Pspace-complete (see Example 11).

*Case 2: when  $U = X$ .* Clique  $\mathcal{K}_{|U|}$  has DSM  $\widehat{\mathcal{M}}$ . We know  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{K}_{|U|}$ ) is Pspace-complete.

*Case 3: when  $U \cap X = \emptyset$ .* The remainder of this section is devoted to a generic hardness proof. Assume that  $|U| = j \geq 2$  and  $|X| = k \geq 2$ , and w.l.o.g. let  $U = \{1, 2, \dots, j\}$  and  $X = \{j + 1, j + 2, \dots, j + k\}$ . Recalling that the symmetric group is generated by a transposition and a cyclic permutation, the DSM  $\widehat{\mathcal{M}}$  can be generated

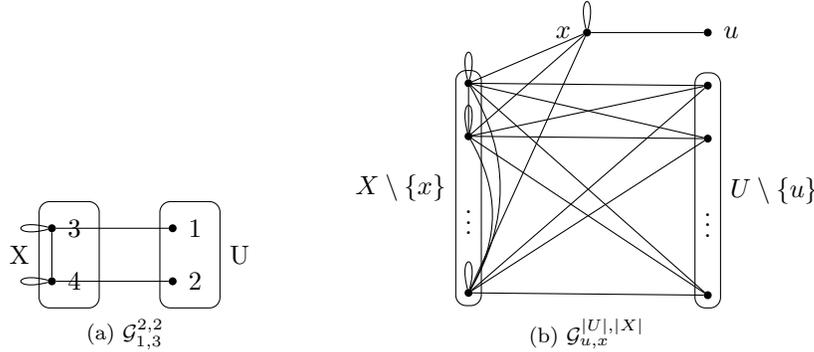


FIG. 2. Main gadget.

as follows:

$$\left\langle \begin{array}{c|c} 1 & 2, j+1, \dots, j+k \\ \hline 2 & 1, j+1, \dots, j+k \\ \hline 3 & 3, j+1, \dots, j+k \\ \hline \vdots & \vdots \\ \hline j & j, j+1, \dots, j+k \\ \hline j+1 & j+1 \\ \hline j+2 & j+2 \\ \hline j+3 & j+3 \\ \hline \vdots & \vdots \\ \hline j+k & j+k \end{array}, \begin{array}{c|c} 1 & 2, j+1, \dots, j+k \\ \hline 2 & 3, j+1, \dots, j+k \\ \hline 3 & 4, j+1, \dots, j+k \\ \hline \vdots & \vdots \\ \hline j & 1, j+1, \dots, j+k \\ \hline j+1 & j+1 \\ \hline j+2 & j+2 \\ \hline j+3 & j+3 \\ \hline \vdots & \vdots \\ \hline j+k & j+k \end{array}, \begin{array}{c|c} 1 & 1, j+1, \dots, j+k \\ \hline 2 & 2, j+1, \dots, j+k \\ \hline 3 & 3, j+1, \dots, j+k \\ \hline \vdots & \vdots \\ \hline j & j, j+1, \dots, j+k \\ \hline j+1 & j+2 \\ \hline j+2 & j+1 \\ \hline j+3 & j+3 \\ \hline \vdots & \vdots \\ \hline j+k & j+k \end{array}, \begin{array}{c|c} 1 & 1, j+1, \dots, j+k \\ \hline 2 & 2, j+1, \dots, j+k \\ \hline 3 & 3, j+1, \dots, j+k \\ \hline \vdots & \vdots \\ \hline j & j, j+1, \dots, j+k \\ \hline j+1 & j+2 \\ \hline j+2 & j+3 \\ \hline j+3 & j+4 \\ \hline \vdots & \vdots \\ \hline j+k & j+1 \end{array} \right\rangle.$$

We will give a structure  $\widehat{\mathcal{D}}$  such that  $\text{shE}(\widehat{\mathcal{D}}) = \widehat{\mathcal{M}}$ . First, though, given some fixed  $u$  in  $U$  and  $x$  in  $X$ , let  $\mathcal{G}_{u,x}^{[U],|X|}$  be the symmetric graph with self-loops with domain  $D = U \cup X$  such that the following hold:

- $u$  and  $x$  are adjacent.
- The graph induced by  $X$  is a reflexive clique  $\mathcal{K}_X^{\text{ref}}$ .
- $U \setminus \{u\}$  and  $X \setminus \{x\}$  are related via a complete bipartite graph  $\mathcal{K}_{|X \setminus \{x\}|, |U \setminus \{u\}|}$ .

The structure  $\mathcal{G}_{u,x}^{[U],|X|}$  and the more specific  $\mathcal{G}_{1,3}^{2,2}$  are drawn in Figure 2. Denote by  $E_{u,x}^{[U],|X|}$  the binary relation of  $\mathcal{G}_{u,x}^{[U],|X|}$ , and let  $\widehat{\mathcal{D}}$  be the structure with a single 4-ary relation  $R^{\widehat{\mathcal{D}}}$  with domain  $\widehat{D} = U \cup X$  specified as follows:

$$R^{\widehat{\mathcal{D}}} := \bigcup_{u \in U} \left( \left( \bigcup_{x \in X} (u, x) \times E_{u,x}^{[U],|X|} \right) \cup \left( \bigcup_{x_1, x_2, x_3 \in X} (x_1, x_2) \times E_{u, x_3}^{[U],|X|} \right) \right).$$

Essentially, when the first argument in a quadruple is from  $U$ , the rest of the structure allows for the unique recovery of some  $\mathcal{G}_{u,x}^{[U],|X|}$ , but if the first argument is from  $X$ , then all possibilities from  $X$  for the remaining arguments are allowed. In particular, we note from the last big union that  $(x_1, x_2, x_3, x_4)$  is a tuple of  $R^{\widehat{\mathcal{D}}}$  for all quadruples  $x_1, x_2, x_3, x_4$  in  $X$ .

LEMMA 53.  $\text{shE}(\widehat{\mathcal{D}}) = \widehat{\mathcal{M}}$ .

*Proof.* Recall that, according to Theorem 48 and our assumption on  $U$ ,  $X$ , and  $\widehat{\mathcal{M}}$ , a maximal (w.r.t. subshop inclusion) shop  $f$  is of the following form:

- for any  $x$  in  $X \setminus U = X$ ,  $f(x) = \{\chi(x)\}$ ; and
- for any  $u$  in  $U \setminus X = U$ ,  $f(u) = \{v(u)\} \cup X$ ,

where  $\chi$  and  $v$  are permutations of  $X$  and  $U$ , respectively.

(Backwards;  $\widehat{\mathcal{M}} \subseteq \text{shE}(\widehat{\mathcal{D}})$ .) It suffices to check that a maximal shop  $f$  in  $\widehat{\mathcal{M}}$  preserves  $\widehat{\mathcal{D}}$ . This holds by construction and can be easily verified:

- We consider first tuples from  $(x_1, x_2) \times E_{u,x_3}^{|U|,|X|}$ . Each such tuple is mapped to possibly several tuples from  $(\chi(x_1), \chi(x_2)) \times E_{v(u),\chi(x_3)}^{|U|,|X|}$  under  $f$ .
- We consider now tuples from  $(u, x) \times E_{u,x}^{|U|,|X|}$ . If the first coordinate  $u$  is mapped to  $v(u) \in U \setminus X$ , then the tuple is mapped to possibly several tuples from  $(v(u), \chi(x)) \times E_{v(u),\chi(x)}^{|U|,|X|}$ . Otherwise, the first coordinate  $u$  is mapped to an element  $x_1$  from  $X$ , and the second coordinate to  $\chi(x) \in X \setminus U$ , where  $\chi(x)$  and  $x_1$  may be equal. In this case, a tuple is mapped to possibly several tuples which appear in  $(x_1, \chi(x)) \times E_{v(u),\chi(x)}^{|U|,|X|}$ .

(Forwards;  $\text{shE}(\widehat{\mathcal{D}}) \subseteq \widehat{\mathcal{M}}$ .) We proceed by contraposition, demonstrating that  $R^{\widehat{\mathcal{D}}}$  is violated by any  $f \notin \widehat{\mathcal{M}}$ . We consider the different ways that  $f$  might not be in  $\widehat{\mathcal{M}}$ .

- If  $f$  is such that  $u \in f(x)$  for  $x \in X$  and  $u \in U$ , then we, e.g., take  $(u, x, x, x) \in R^{\widehat{\mathcal{D}}}$  but  $(z, u, u, u) \notin R^{\widehat{\mathcal{D}}}$  (for any  $z \in f(u)$ ), and we are done. It follows that  $f(X) = X$ .
- Assume now that  $f$  is such that  $\{x'_1, x'_2\} \subseteq f(x)$  for  $x'_1 \neq x'_2$  and  $x, x'_1, x'_2 \in X$ . Let  $u, u' \in U$  be such that  $u' \in f(u)$ . Take  $(u, x, u, x) \in R^{\widehat{\mathcal{D}}}$  but  $(u', x'_1, u', x'_2) \notin R^{\widehat{\mathcal{D}}}$ , and we are done. It follows that  $f$  is a permutation  $\chi$  on  $X$ .
- Assume now that  $f$  is such that  $\{u'_1, u'_2\} \subseteq f(u)$  for  $u'_1 \neq u'_2$  and  $u, u'_1, u'_2 \in U$ . Let  $x, x' \in X$  be such that  $x' \in f(x)$ . Take  $(u, x, u, x) \in R^{\widehat{\mathcal{D}}}$  but  $(u'_1, x', u'_2, x') \notin R^{\widehat{\mathcal{D}}}$ , and we are done. It follows that  $f$  restricted to  $U$  is a permutation  $v$  on  $U$ .

Hence,  $f$  is a subshop of a maximal shop  $f'$  from the DSM  $\widehat{\mathcal{M}}$ , and  $f$  belongs to  $\widehat{\mathcal{M}}$  (recall that a DSM is closed under subshops). The result follows.  $\square$

PROPOSITION 54.  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{G}_{u,x}^{|U|,|X|}$ ) is Pspace-complete.

*Proof.* This follows easily from the Pspace-completeness of  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{G}_{1,3}^{2,2}$ ), the simplest gadget which is depicted in Figure 2a. These gadgets  $\mathcal{G}_{u,x}^{|U|,|X|}$  agree on all equality-free sentences—even ones involving negation—by Lemma 15, as there is a full surjective homomorphism from  $\mathcal{G}_{u,x}^{|U|,|X|}$  to  $\mathcal{G}_{1,3}^{2,2}$ .

We will prove that  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{G}_{1,3}^{2,2}$ ) is Pspace-hard by reduction from the Pspace-complete problem QCSP( $\mathcal{B}_{\text{NAE}}$ ). Recall that we may assume w.l.o.g. that universal variables are relativized to  $U = \{1, 2\}$  and that existential variables are relativized to  $X = \{3, 4\}$  by Theorem 34. Let  $\varphi$  be an instance of QCSP( $\mathcal{B}_{\text{NAE}}$ ) that is a sentence in prenex form whose quantifier-free part is a conjunction of atoms of the form  $R(\alpha, \beta, \gamma)$ , where  $\alpha, \beta$ , and  $\gamma$  are (positive occurrences of) variables. We reduce  $\varphi$  to a (relativized) instance  $\psi$  of  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{G}_{1,3}^{2,2}$ ) as follows.

The first part of the prefix of  $\psi$  is obtained from the prefix of  $\varphi$  by replacing each quantifier as follows:

- $\exists x$  in the prefix of  $\varphi$  is replaced by  $\exists v_x \in X$  in the prefix of  $\psi$ ;
- $\forall u$  in the prefix of  $\varphi$  is replaced by  $\forall u \in U \exists v_u \in X$  in the prefix of  $\psi$ .

The second part of the prefix of  $\psi$  takes clauses into account. For every clause  $C_i := R(\alpha, \beta, \gamma)$  in  $\varphi$ , we extend the prefix of  $\psi$  with  $\forall c_i \in U$ .

The quantifier-free part is a conjunction of

- $E(u, v_u)$  for every universal variable  $u$  in  $\varphi$  and

- $E(c_i, v_\alpha) \vee E(c_i, v_\beta) \vee E(c_i, v_\gamma)$  for every clause  $C_i := R(\alpha, \beta, \gamma)$  in  $\varphi$ .

We claim that there is a family of Skolem functions  $\{\sigma_x : “\exists x” \in \varphi\}$  witnessing that  $\mathcal{B}_{\text{NAE}} \models \varphi$  if and only if there is a family of Skolem functions  $\{\sigma'_{v_x} : “\exists v_x” \in \psi\}$  in the relativized game witnessing that  $\mathcal{G}_{1,3}^{2,2} \models \psi$ . We will establish this claim by providing a tight correspondence between these families.

Note first that the edges present in  $\mathcal{G}_{1,3}^{2,2}$ , the atoms of the type  $E(u, v_u)$  for a universal variable, and the prefix  $\forall u \in U \exists v_u \in X$  mean that for  $\psi$  to hold, the Skolem function  $\sigma'_{v_u}$  takes value 3 (resp., 4) whenever  $u$  takes value 1 (resp., 2).

We fix arbitrarily that 1 and 3 correspond to true and 2 and 4 to false. Given  $\pi$  as a truth assignment of the universal variables of  $\varphi$ , we denote by  $\pi'$  the corresponding assignment to 1 and 2. The correspondence between the family of Skolem functions is given by  $\sigma_x(\pi|_{Y_x}) = \text{true}$  (resp., false) if and only if  $\sigma'_{v_x}(\pi'|_{Y_{v_x}}) = 3$  (resp., 4).

By construction, the winning conditions agree for both families of Skolem functions, because for each clause  $C_i := R(\alpha, \beta, \gamma)$  in  $\varphi$ , the subsentence of  $\psi$  of the form  $\exists v_\alpha \exists v_\beta \exists v_\gamma \forall c_i E(c_i, v_\alpha) \vee E(c_i, v_\beta) \vee E(c_i, v_\gamma)$  enforces that one of  $v_\alpha, v_\beta, v_\gamma$  is equal to 3 and one is equal to 4 (since we may select  $c_i$  to be equal to either 1 or 2 after the choice of values for  $v_\alpha, v_\beta, v_\gamma$ ).  $\square$

PROPOSITION 55.  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\widehat{\mathcal{D}}$ ) is Pspace-complete.

*Proof.* As in the previous proposition, we reduce from QCSP( $\mathcal{B}_{\text{NAE}}$ ). We adapt a little bit the previous reduction as follows. We substitute  $R(u_0, x_0, u, v)$  for each instance of  $E(u, v)$  in the previous proof. We quantify outermost the formula so produced with the prefix  $\forall u_0 \in U \exists x_0 \in X$ , obtaining a sentence  $\psi'$ .

The correctness of this new reduction is an easy consequence of the previous result and the construction of  $\widehat{\mathcal{D}}$ . Indeed, once  $u_0$  and  $x_0$  are chosen, everything proceeds as above but in the “copy”  $\mathcal{G}_{u_0, x_0}^{|U|, |X|}$ , and the result follows.  $\square$

**6. The complexity of the meta-problem.** The  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\sigma$ ) meta-problem takes as input a finite  $\sigma$ -structure  $\mathcal{D}$  and answers L, NP-complete, co-NP-complete, or Pspace-complete, according to the complexity of  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ). The principle result of this section is that this problem is NP-hard even for some fixed and finite signature  $\sigma_0$ , which consists of two binary and three unary predicates (the unaries are for convenience, but it is not clear whether a single binary suffices).

Note that one may determine whether a given shop  $f$  is a surjective hyper-endomorphism of a structure  $\mathcal{D}$  in, say, quadratic time in  $|D|$ . Since we are not interested here in distinguishing levels within P, we will henceforth consider such a test to be a basic operation. We begin with the most straightforward case.

PROPOSITION 56. On input  $\mathcal{D}$ , the question “is  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) in L?” is in P.

*Proof.* By Theorem 23, we need to check whether there is both a singleton-surjective shop and a singleton-total shop in  $\text{shE}(\mathcal{D})$ . In this special case, it suffices to test for each  $u, x$  in  $D$  if the following  $\{u\}$ -surjective  $\{x\}$ -total shop  $f$  preserves  $\mathcal{D}$ :  $f(u) := D$  and  $x \in f(z)$  for any element  $z$  in  $D$  (that is, such that  $f^{-1}(x) := D$ ).  $\square$

PROPOSITION 57. For some fixed and finite signature  $\sigma_0$ , on input of a  $\sigma_0$ -structure  $\mathcal{D}$ , the question “is  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) in NP (respectively, in NP-complete, in co-NP, and in co-NP-complete)?” is NP-complete.

*Proof.* The four variants are each in NP. For the first, one guesses and verifies that  $\mathcal{D}$  has a singleton-surjective shop; for the second, one further checks that there is no  $\{u\}$ -surjective  $\{x\}$ -total shop (see the proof of Proposition 56). Similarly, for the

third, one guesses and verifies that  $\mathcal{D}$  has a singleton-total shop; for the fourth, one further checks that there is no  $\{u\}$ -surjective  $\{x\}$ -total shop. The result then follows from Theorem 23.

For NP-hardness we will address the first problem only. The same proof will work for the second (for the third and fourth, recall that a structure  $\mathcal{D}$  has a singleton-surjective shop if and only if its complement  $\overline{\mathcal{D}}$  has a singleton-total shop). We reduce from *graph 3-colorability*. Let  $\mathcal{G}$  be an undirected graph with vertices  $V := \{v_1, v_2, \dots, v_s\}$ . We will build a structure  $\mathcal{S}_{\mathcal{G}}$  over the domain  $D$  which consists of the disjoint union of “three colors”  $\{0, 1, 2\}$ ,  $u$ , and the vertices from  $V$ .

The key observation is that there is a structure  $\mathcal{G}_V$  whose class of surjective hyper-endomorphisms  $\text{shE}(\mathcal{G}_V)$  is generated by the following singleton-surjective shop:

$$f_V := \begin{array}{c|c} 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline u & 0, 1, 2, u, v_1, \dots, v_s \\ \hline v_1 & 0, 1, 2 \\ \hline v_2 & 0, 1, 2 \\ \hline \vdots & \vdots \\ \hline v_s & 0, 1, 2 \end{array}$$

The existence of such a  $\mathcal{G}_V$  is in fact guaranteed by the Galois connection, fully given in [29], but that may require relations of unbounded arity, and we wish to establish our result for a fixed signature. So we will appeal to Lemma 58, below, for a  $\sigma_V$ -structure  $\mathcal{G}_V$  with the desired class of surjective hyper-endomorphisms, where the signature  $\sigma_V$  consists of one binary relation and three monadic predicates. The signature  $\sigma_0$  is  $\sigma_V$  together with a binary relational symbol  $E$ .

The structure  $\mathcal{S}_{\mathcal{G}}$  is defined as in  $\mathcal{G}_V$  for symbols in  $\sigma_V$ , and for the additional binary symbol  $E$ , as the edge relation of the instance  $\mathcal{G}$  of 3-colorability together with a clique  $\mathcal{K}_3$  for the colors  $\{0, 1, 2\}$ . By construction, the following hold:

- Any surjective hyper-endomorphism  $g$  of  $\mathcal{S}_{\mathcal{G}}$  will be a subshop of  $f_V$ .
- Restricting such a shop  $g$  to  $V$  provides a set of mutually consistent 3-colorings; i.e., we may pick arbitrarily a color from  $g(v_i)$  to get a 3-coloring  $\tilde{g}$ . If there is an edge between  $v_i$  and  $v_j$  in  $\mathcal{G}$ , then  $E(v_i, v_j)$  holds in  $\mathcal{S}_{\mathcal{G}}$ . Since  $g$  is a shop, for any pair of colors  $c_i, c_j$ , where  $c_i \in g(v_i)$  and  $c_j \in g(v_j)$ , we must have that  $E(c_i, c_j)$  holds in  $\mathcal{S}_{\mathcal{G}}$ . The relation  $E$  is defined as  $\mathcal{K}_3$  over the colors. Hence,  $c_i \neq c_j$  and we are done.
- Conversely, a 3-coloring  $\tilde{g}$  induces a subshop  $g$  of  $f_V$ : set  $g$  as  $f_V$  over elements from  $\{0, 1, 2, u\}$  and as  $\tilde{g}$  over  $V$ . The detailed argument is similar to the above.

This proves graph 3-colorability reduces to the meta-question “is  $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ ) in NP?” □

Note that it follows from the given proof that the meta-problem itself is NP-hard. To see this, we take the structure  $\mathcal{S}_{\mathcal{G}}$  from the proof of Proposition 57 and ask to which of the four classes L, NP-complete, co-NP-complete, or Pspace-complete the corresponding problem belongs. If the answer is NP-complete, then  $\mathcal{G}$  was 3-colorable; otherwise the answer is Pspace-complete and  $\mathcal{G}$  was not 3-colorable.

In the statement below,  $f_V$  denotes the same shop as in the proof of Proposition 57, which this lemma completes.

LEMMA 58. *Let  $\sigma_V$  be a signature involving one binary relation  $E'$  and three*

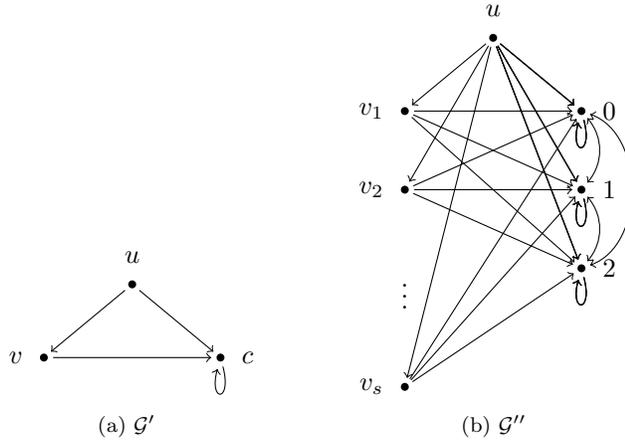


FIG. 3. Building a structure with  $\text{shE}(\mathcal{G}_V) = \langle f_V \rangle$ .

monadic predicates Zero, One, and Two. There is a  $\sigma_V$ -structure  $\mathcal{G}_V$  such that

$$\text{shE}(\mathcal{G}_V) = \langle f_V \rangle.$$

*Proof.* We begin with the graph  $\mathcal{G}'$  on signature  $\langle E' \rangle$ , depicted in Figure 3a. Note that

$$\text{shE}(\mathcal{G}') := \left\langle \begin{array}{c|c} c & c \\ \hline u & c, u, v \\ \hline v & c \end{array} \right\rangle.$$

We now replace  $c$  by  $\{0, 1, 2\}$  and  $v$  by  $V$  to obtain a graph  $\mathcal{G}''$ . Formally, this graph is the unique graph  $\mathcal{G}''$  with domain  $\{0, 1, 2, u\} \cup V$  such that the mapping which maps  $\{0, 1, 2\}$  to  $c$ , fixes  $u$ , and maps  $V$  to  $v$  is a strong surjective homomorphism. By construction,

$$\text{shE}(\mathcal{G}'') := \left\langle \begin{array}{c|c} 0 & 0, 1, 2 \\ \hline 1 & 0, 1, 2 \\ \hline 2 & 0, 1, 2 \\ \hline u & 0, 1, 2, u, v_1, \dots, v_s \\ \hline v_1 & 0, 1, 2 \\ \hline \vdots & \vdots \\ \hline v_s & 0, 1, 2 \end{array} \right\rangle.$$

We now build  $\mathcal{G}_V$  as the structure with binary relation  $E'$ , which is the edge relation from  $\mathcal{G}''$ , and by setting the unary predicates as follows: Zero holds only over 0, One holds only over 1, and Two holds only over 2. This effectively fixes surjective hyper-endomorphisms to act as the identity over the colors  $\{0, 1, 2\}$ , as required.  $\square$

**7. Conclusion.** We have classified the complexity of the model checking problem for all fragments of FO but those corresponding to the CSP and the QCSP. Our results are summarized in Figure 4. The inclusion of fragments is denoted by dashed edges, a larger fragment being above. Each fragment is classified in two fashions. First, we have indicated in the figure the notion of core used to classify fragments by regrouping them in the same box. Second, we have organized the fragments in four classes according to the nature of the complexity classification they follow (as outlined in the introduction of this paper).



Moreover, we are able to give the delineation of our tetrachotomy by two equivalent means. First, we do so by the presence or absence of *singleton-surjective shops* and *singleton-total shops*. Second, we do so by the existence or not of trivial sets for the relativization of universal and existential quantifiers (see Table 2).

TABLE 2

*Reformulations of the tetrachotomy ( $U$  and  $X$  denote the subsets of the domain to which universal and existential variables relativize, respectively; the relativization into a weaker logical fragment allows up to two constants).*

Tetrachotomy for $\{\exists, \forall, \wedge, \vee\}$ -FO( $\mathcal{D}$ )						
Case	Complexity	Singleton universal shop	Singleton-total shop	$\forall U$ - $\exists X$ -core	Relativizes into	Dual
I	L	yes	yes	$ U  = 1,  X  = 1$	$\{\wedge, \vee\}$ -FO	I
II	NP-complete	yes	no	$ U  = 1,  X  \geq 2$	$\{\exists, \wedge, \vee\}$ -FO	III
III	co-NP-complete	no	yes	$ U  \geq 2,  X  = 1$	$\{\forall, \vee, \wedge\}$ -FO	II
IV	Pspace-complete	no	no	$ U  \geq 2,  X  \geq 2$	$\{\exists, \forall, \wedge, \vee\}$ -FO	IV

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