

The constitutive equation and flow dynamics of bubbly magmas

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[1] A generalized constitutive equation for bubbly liquids is presented which successfully reproduces the expected viscosity response for both steady flows with varying capillary number Ca (a measure of the bubble deformation) and unsteady flows with varying dynamic capillary number Cd (a measure of the steadiness of the flow) previously given in separate studies. The constitutive equation is given in terms of observable material and flow parameters and is valid at least up to $Ca \sim O(1000)$ and $Cd \sim O(10)$. Analytical solutions are presented for steady, simple-shearing flow (Ca variable) and oscillatory flow with small total-strains (Cd variable). The special case of steady flow in a circular pipe—analogue to magma flow in a volcanic conduit—is investigated. Velocity profiles across the conduit are found to be parabolic or plug-flow depending on a dimensionless number, the conduit capillary number Cc . Plug flow is predicted for $Cc \approx 4$ which is in the mid-range for volcanic-eruption conditions. INDEX TERMS: 8160 Tectonophysics: Evolution of the Earth: Rheology—general; 8429 Volcanology: Lava rheology and morphology; 8434 Volcanology: Magma migration. Citation: Llewellyn, E. W., H. M. Mader, and S. D. R. Wilson, The constitutive equation and flow dynamics of bubbly magmas, *Geophys. Res. Lett.*, 29(24), 2170, doi:10.1029/2002GL015697, 2002.

1. Introduction

[2] Magmas typically contain appreciable amounts of gas in the form of bubbles. Gas volume-fractions ϕ can vary over an enormous range from 0 to > 0.9 . The rheology of pure silicate melts is comparatively well-known and is Newtonian to a close approximation for a wide range of conditions. By contrast, the rheological effect of adding bubbles to such a liquid has remained controversial for many years with some authors observing an increase in viscosity with increasing ϕ [Sibree, 1934; Stein and Spera, 1992] and others a decrease [Sura and Panda, 1990; Bagdassarov and Dingwell, 1992, 1993; Lejeune et al., 1999]. A resolution to this apparent contradiction has recently been proposed by Rust and Manga [2002], Stein and Spera [2002] and Llewellyn et al. [2002].

[3] Rust and Manga [2002] observed the viscosity as a function of capillary number Ca in steady, simple-shearing flows. Ca is a measure of the equilibrium deformation of the bubbles and is given by $Ca = \eta_0 a \dot{\gamma} / \Gamma$ where η_0 is the

viscosity of the liquid phase, a is the radius of the relaxed, undeformed bubble, Γ is the surface tension, and $\dot{\gamma}$ is the strain rate. Both Rust and Manga [2002] and Stein and Spera [2002] identify two flow regimes: at low Ca , the viscosity is seen to increase with ϕ whereas, at high Ca , the viscosity decreases as ϕ increases. The two regimes can be explained as follows: Bubbles distort the flow lines in the surrounding liquid causing an increase in viscosity. Bubbles also provide free-slip surfaces within the suspension, decreasing its viscosity. At low Ca , the bubbles remain approximately spherical so the distortion of the flow lines is great and the free-slip surface-area is small, hence viscosity is an increasing function of ϕ . At high Ca , the bubbles are significantly elongate so the free-slip surface-area is large and the distortion of the flow lines is small, hence viscosity is a decreasing function of ϕ .

[4] The importance of distinguishing between steady and unsteady flows is highlighted by Llewellyn et al. [2002]. In a steady flow, shear conditions have remained constant for a time significantly longer than the relaxation time of the bubbles, $\lambda = k \eta_0 a / \Gamma$, where k is a dimensionless number that is a positive function of gas volume-fraction ($k \approx 1$ in the dilute limit). There are many magmatic flows which will never reach such a steady-state condition. This is especially true for explosive flows (below the fragmentation level) in which fluid particles experience enormous accelerations (in which case the Lagrangian $\dot{\gamma}$ is not steady) and for high-viscosity flows (e.g. for $\eta_0 = 10^9$ Pa s, $a = 1 \mu\text{m}$, $\Gamma = 0.3$ N m^{-1} and $k = 1$, $\lambda \approx 3000$ s). In Llewellyn et al. [2002] we investigated the effect of unsteadiness on the rheology of a bubbly flow. This necessarily includes a consideration of the viscoelastic properties of the two-phase mixture. We described the steadiness of the flow using a dimensionless number, the dynamic capillary number $Cd = \lambda \ddot{\gamma} / \dot{\gamma}$ which gives the ratio of the bubble relaxation time (λ) to the timescale over which the strain rate changes appreciably ($\ddot{\gamma} / \dot{\gamma}$). When $Cd \ll 1$ relaxation is rapid compared to the timescale of appreciable change in strain rate, hence the flow is approximately steady and the viscosity of the two-phase mixture can either increase or decrease with ϕ depending on the bubble shape as in the case of steady-flow conditions. By contrast, when $Cd \gg 1$, the bubbles cannot relax fast enough to reach their equilibrium deformation. They deform with the flow to a greater extent so flow past the bubble and, therefore, flow-line distortion, is reduced. The free-slip surfaces are more important, causing a reduction in the suspension, therefore, viscosity as ϕ increases.

[5] If the results of Rust and Manga [2002] and Llewellyn et al. [2002] are combined then we conclude that the viscosity will decrease as ϕ increases except when both $Ca < O(1)$ and $Cd < O(1)$.

[6] Below we present a generalized equation that is valid for varying Ca and Cd so that no assumptions need be made

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a priori about the steadiness of the flow or the bubble shape.

2. General Constitutive Equation

[7] A constitutive equation for a bubbly liquid valid up to at least $Cd = 10$, $Ca \ll 1$ and $\phi \lesssim 0.5$ is given in *Llewellyn et al.* [2002]:

$$\tau_{ij} + \frac{6\lambda}{5} \dot{\tau}_{ij} = \eta_0(1 + b\phi) \dot{\gamma}_{ij} + \eta_0 \frac{6\lambda}{5} \left(1 - \frac{5\phi}{3}\right) \dot{\gamma}_{ij} \quad (1)$$

where τ_{ij} is the deviatoric-stress tensor ($T_{ij} = -P\delta_{ij} + \tau_{ij}$ where T_{ij} is the total-stress tensor, P is the pressure and δ_{ij} is the Kronecker delta), η_0 is the viscosity of the liquid phase, ϕ is the gas volume-fraction and b is an empirically-determined parameter. This model is based on the physical analysis of *Frankel and Acrivos* [1970] which was simplified by assuming small total-strains ($Ca \ll 1$). *Llewellyn et al.* [2002] parameterize the resultant model according to data collected from oscillatory rheometric measurements of aerated golden syrup and find $b = 9$.

[8] In order to generalize this constitutive model to include arbitrarily large strains, the simple time derivatives in equation 1 have to be changed back to the *co-rotational* or *Jaumann* derivatives of the unsimplified *Frankel and Acrivos* [1970] model. The Jaumann derivative for an arbitrary tensor A_{ij} is given by:

$$\dot{A}_{ij} = \frac{\partial A_{ij}}{\partial t} + u_k \frac{\partial A_{ij}}{\partial x_k} + \varpi_{ik} A_{kj} - A_{ik} \varpi_{kj} \quad (2)$$

where u_i is the velocity vector and ϖ_{ij} is the vorticity tensor [*Barthés-Biesel and Acrivos*, 1973]. Thus, a Jaumann derivative includes advective (2nd term on r.h.s.) and vorticity terms (3th and 4th terms on r.h.s.).

[9] The Jaumann derivative appears commonly in analytical studies that derive general constitutive (i.e. stress-strain) relations for complex materials from first principles [see, e.g. *Rivlin and Erickson*, 1955; *Oldroyd*, 1950; *Barthés-Biesel and Acrivos*, 1973]. The only assumption that enters into these studies is that the length scale of the motion is much larger than the dimensions of the particles. The suspension can then be treated as a continuum with bulk properties that are ensemble averages of the corresponding local quantities. The detailed flow-field around each particle is obtained analytically and is used to derive, without any further assumptions, an exact analytical rheological equation of state, which contains no empirically adjustable parameters and in which the functional relation between the stress and all the relevant physical quantities is shown explicitly.

[10] If the Jaumann derivatives and the non-linear terms are reinstated in our constitutive equation 1, then it becomes [*Rivlin and Erickson*, 1955; *Frankel and Acrivos*, 1970; *Barthés-Biesel and Acrivos*, 1973]:

$$\tau_{ij} + \lambda_1 \dot{\tau}_{ij} = 2\eta_0(\alpha e_{ij} + \lambda_2 \dot{e}_{ij} + \lambda_3 \ell d[e_{ik}e_{kj}]) \quad (3)$$

where the Jaumann derivative is denoted by the ring operator, e_{ij} is the rate-of-strain tensor ($e_{ij} = \dot{\gamma}_{ij}/2$) and:

$$\lambda_1 = \frac{6\lambda}{5} \quad \alpha = 1 + b\phi \quad (4)$$

$$\lambda_2 = \left(1 - \frac{5\phi}{3}\right) \lambda_1 \quad \lambda_3 = \frac{4\lambda_1\phi}{7} \quad (5)$$

and the operator ℓd denotes the symmetric, traceless part (e.g., $\ell d[A_{ij}] = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}A_{ii}\delta_{ij}$). Note that e_{ij} and ϖ_{ij} are the symmetric and anti-symmetric components of the velocity gradient tensor $\partial u_j/\partial x_i$.

[11] Equation 3 is based on the analysis of *Frankel and Acrivos* [1970] and so, strictly speaking, is only valid for large material strains as long as the bubble deformation is small (bubbles remain approximately spherical) and the flow is only weakly time-dependent. However, *Llewellyn et al.* [2002] find that the model provides a good fit to their data, even in the unsteady limit where $Cd \approx O(10)$ and the model is consistent with the data presented in *Stein and Spera* [2002] up to at least $Ca \approx 1000$.

2.1. Simple Shearing Flow

[12] In certain simple cases, equation 3 can be solved analytically. One such case is that of simple shearing flow for which the velocity is given by $u = (u_1(x_2), 0, 0)$.

[13] In this case, if $S = du_1/dx_2$ then the relationship between deviatoric stress and strain is given by:

$$\begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{12} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} + \lambda_1 S \begin{pmatrix} -\tau_{12} & \frac{1}{2}(\tau_{11} - \tau_{22}) & 0 \\ \frac{1}{2}(\tau_{11} - \tau_{22}) & \tau_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ = 2\eta_0 \left[\alpha \begin{pmatrix} 0 & \frac{S}{2} & 0 \\ \frac{S}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\frac{S^2}{2} & 0 & 0 \\ 0 & \frac{S^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} \frac{S^2}{12} & 0 & 0 \\ 0 & \frac{S^2}{12} & 0 \\ 0 & 0 & -\frac{S^2}{6} \end{pmatrix} \right] \quad (6)$$

Comparing terms leads to the following set of equations:

$$\tau_{11} - \lambda_1 S \tau_{12} = \eta_0 S^2 \left(\frac{\lambda_3}{6} - \lambda_2 \right) \quad (7)$$

$$\tau_{22} + \lambda_1 S \tau_{12} = \eta_0 S^2 \left(\lambda_2 + \frac{\lambda_3}{6} \right) \quad (8)$$

$$\tau_{33} = \frac{-\eta_0 \lambda_3 S^2}{3} \quad (9)$$

$$\tau_{12} + \frac{\lambda_1 S}{2} (\tau_{11} - \tau_{22}) = \eta_0 \alpha S \quad (10)$$

From equations 7, 8, and 10 and using the result that the viscosity is given by $\eta = \tau_{12}/S$, the relative viscosity $\eta_r = \eta/\eta_0$ is given by:

$$\eta_r = \frac{\alpha + \lambda_1 \lambda_2 S^2}{1 + \lambda_1^2 S^2} = 1 - \frac{5\phi}{3} + \frac{(75b + 125)\phi}{75 + 108Ca^2} \quad (11)$$

where $Ca = \lambda S$, $b = 9$. Hence the material is shear thinning:

$$\eta_r = \begin{cases} 1 + b\phi & : Ca \ll 1 \\ 1 - 5\phi/3 & : Ca \gg 1. \end{cases} \quad (12)$$

The solid line in Figure 1 shows how η_r varies with Ca for a bubbly liquid. The normal-stress differences $N_1 = \tau_{11} - \tau_{22}$ and $N_2 = \tau_{22} - \tau_{33}$ can be calculated readily from equations 7–10. They are generally non-zero and $N_1 > \tau_{12}$ for intermediate Ca . *Spera et al.* [1988] present a value of the first normal-stress coefficient (which is given by $\psi_1 =$

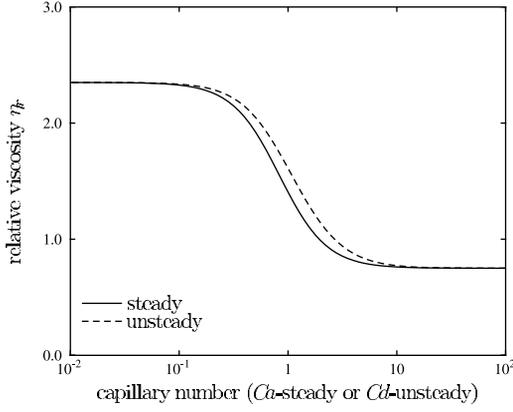


Figure 1. Relative viscosity against capillary number (Ca : steady, Cd : unsteady) for $\phi = 0.15$.

$-N_1/\dot{\gamma}^2$) of $\psi_1 = -2.53 \times 10^3 \text{ Pa s}^2$ based on rod-climbing experiments with bubbly rhyolites. For the appropriate material properties and strain rate, our analysis predicts $\psi_1 = -2.49 \times 10^3 \text{ Pa s}^2$, which is in excellent agreement.

2.2. Unsteady Flow

[14] Equation 3 is also soluble for the case of oscillatory flow, e.g. in a concentric cylinder viscometer where the imposed torque varies sinusoidally. Small strains are assumed, equivalent to, for example, start-up flow in a volcanic conduit where Cd is, necessarily, large (since $\dot{\gamma} = 0$ initially) and strains are small.

[15] For a viscoelastic material undergoing oscillatory flow with angular frequency ω , the viscosity has a viscous component, η' and an elastic component η'' [see *Llewellyn et al.*, 2002 - Appendix A]:

$$\eta' = \eta_0 \frac{\alpha + \lambda_1 \lambda_2 \omega^2}{1 + \lambda_1^2 \omega^2} = \eta_0 \left(1 - \frac{5\phi}{3} + \frac{(75b + 125)\phi}{75 + 108Cd^2} \right) \quad (13)$$

$$\eta'' = \eta_0 \frac{(\lambda_1 \alpha - \lambda_2) \omega}{1 + \lambda_1^2 \omega^2} = \eta_0 \frac{(30b + 50)\phi Cd}{25 + 36Cd^2} \quad (14)$$

where $Cd = \lambda \omega$, $b = 9$. The complex viscosity $\eta^* = \eta' - i\eta''$ hence the relative viscosity is given by:

$$\begin{aligned} \eta_r &= \frac{|\eta^*|}{\eta_0} \\ &= \frac{1}{3} \sqrt{9 - 30\phi + 25\phi^2 + \frac{(450b + 750)\phi + (225b^2 - 625)\phi^2}{25 + 36Cd^2}} \end{aligned} \quad (15)$$

Hence the viscosity decreases as the flow becomes increasingly unsteady:

$$\eta_r = \begin{cases} 1 + b\phi & : Ca \ll 1 \\ 1 - 5\phi/3 & : Ca \gg 1. \end{cases} \quad (16)$$

This is the same as equation 12 for steady flow with Cd substituted for Ca . The dashed line in Figure 1 shows how η_r varies with Cd for an unsteady flow. The relationship is almost identical to that of $\eta_r = f(Ca)$ for steady-flow conditions (solid line in Figure 1), with only a little deviation

when $Ca = Cd \approx 1$. If Ca and Cd are small, $\eta_r = 1 + b\phi$. If either Ca or Cd are large, $\eta_r = 1 - 5\phi/3$.

3. Flow Along A Circular Pipe

[16] The special case of flow along a circular pipe is of interest to volcanologists as a model for flow of magma in a volcanic conduit. The analysis for Poiseuille flow in a circular pipe is identical to that for simple shearing flow described in section 2.1 with directions 1, 2 and 3 replaced by cylindrical coordinate components z , r and θ respectively ($u = (u_z(r), 0, 0)$) and the velocity profile and flow rate of a bubbly fluid along a pipe can be calculated for given physical parameters. The z -component of the equation of motion for a fluid undergoing simple shearing flow along a pipe is:

$$0 = -P' - \frac{1}{r} \frac{d}{dr} (r\tau_{rz}) \quad (17)$$

where P' is the pressure gradient above hydrostatic (so, in a volcanic conduit, $P' = d(P_{\text{lithostatic}} - P_{\text{magnastatic}})/dz$). Integration of equation 17 and substitution for τ_{rz} (from equations 7, 8, and 10) gives)

$$\eta_0 S \frac{\alpha + \lambda_1 \lambda_2 S^2}{1 + \lambda_1^2 S^2} = \frac{P'r}{2} \quad (18)$$

This allows $S(r)$ to be calculated as the root of a cubic equation. Since $S = du_z/dr$:

$$u_z(r) = \int_R^0 S(r) dr \quad (19)$$

where R is the radius of the pipe and subject to the no-slip boundary-condition $u_z(R) = 0$. Numerical solution of equation 19 allows the velocity profile $u_z(r)$ to be determined for any given values of η_0 , P' , R and λ .

[17] Using Poiseuille's equation a relative effective Newtonian viscosity can be defined for a bubbly flow.

$$\eta_{\text{eff}} = \frac{\pi P' R^4}{8Q\eta_0} \quad (20)$$

This is the viscosity of the Newtonian fluid which has the same volume flow-rate as the bubbly fluid for the given

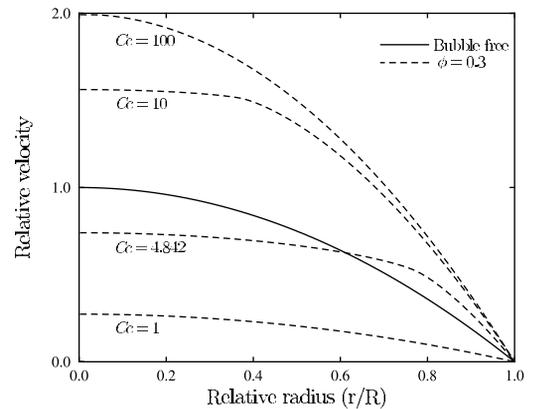


Figure 2. Velocity profile for a Newtonian material (solid line) and a bubbly fluid (dashed line). Profiles for bubbly fluid are shown at low, intermediate and high values of Cc .

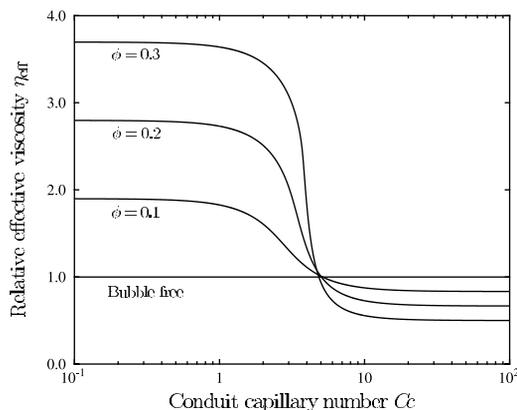


Figure 3. Relative effective viscosity (defined in section pipe) for flows with varying gas volume-fraction and Cc .

flow parameters normalized with respect to η_0 . Q is the volume flow-rate and is given by:

$$Q = 2\pi \int_R^0 r u_z(r) dr \quad (21)$$

[18] Figure 2 shows the velocity profile across a conduit for a fluid with $\phi = 0.3$. All velocities are normalized to the axial velocity for the bubble-free flow (which has a Newtonian rheology and a parabolic velocity-profile which is shown for comparison). The velocity profile for the bubbly flow is not constant, but varies as the physical parameters P' , R , λ and η_0 are varied. We define a dimensionless combination of these parameters, the conduit capillary number Cc , with which we can describe the flow.

$$Cc = \frac{P'R\lambda}{\eta_0} \quad (22)$$

At low Cc the material has an almost parabolic velocity profile indicating Newtonian rheology. The velocity of the flow is considerably lower than the bubble-free flow indicating that its effective viscosity is higher. For low Cc , $\eta_{\text{eff}} = 1 + b\phi$. At high Cc the velocity profile is again parabolic and $\eta_{\text{eff}} = 1 - 5\phi/3$.

[19] At intermediate values of Cc the fluid shows two regimes of flow: an inner plug where strain rates are low ($\eta_{\text{eff}} = 1 + b\phi$) and an outer sleeve where strain rates are high ($\eta_{\text{eff}} = 1 - 5\phi/3$). The volume flow-rate Q for $Cc = 4.842$ is the same as for the bubble-free flow and the corresponding velocity profile is shown for comparison in Figure 2.

[20] Figure 3 shows how the relative viscosity η_{eff} of the material varies with Cc for bubbly flows with a range of gas volume-fractions. The transition from $\eta_{\text{eff}} = 1 + b\phi$ to $\eta_{\text{eff}} = 1 - 5\phi/3$ occurs at $Cc \approx 4$ which is in the mid-range for volcanic eruptions.

[21] The presence of a relatively-undeformed plug of material surrounded by a rapidly-deforming sleeve has

many implications for physical volcanology. A study of emulsions in pipe flow by Grizzuti and Bifulco [1997] suggests that low strain-rates, such as those found in the plug-flow region, promote the coalescence of bubbles which may promote magma fragmentation in the plug. 'Tube' pumices with highly-elongate vesicles frequently appear together with pumices containing spherical vesicles [Marti et al., 1999]. This would be expected if fragmentation occurred in an intermediate- Cc flow where a magma plug bearing spherical bubbles coexists with a region of highly-elongate bubbles.

[22] Calculating Cc for a volcanic eruption in which steady-flow conditions have been reached provides a simple way to calculate the effect of bubbles on the velocity profile across the conduit and the effective viscosity of the magma.

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