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# Closed-form equation for subsidence due to fluid production from a cylindrical confined aquifer

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## Abstract

Ground surface subsidence due to groundwater production is a significant problem. Many attempts have been made to develop analytical models to forecast subsidence rates as a consequence of groundwater production. Previous analytical solutions either make limiting assumptions about the stress regime (e.g., radially symmetric with uniaxial strain or radially symmetric with zero incremental vertical total stress) or assume that the pressure distribution within the aquifer is uniform. Imposing assumptions about the stress regime lead to an overestimate of subsidence. Imposing a uniform pressure assumption often leads to an underestimate of subsidence. In this article, the principle of superposition is applied to extend a previous analytical solution, for a cylindrical uniform pressure change, to allow for a non-uniform pressure distribution resulting from constant rate production of a viscous fluid from a cylindrical confined aquifer of finite permeability. Results from the analytical solution are verified by comparison with a set of fully coupled hydro-mechanical finite element simulations. The analytical solution for subsidence directly above the production well (or uplift above an injection well) can be written in closed-form and is straightforward to evaluate. The equation also shows that, for many practical purposes, ground surface subsidence is insensitive to production fluid viscosity and aquifer permeability when the aquifer radius is less

than the aquifer depth below the ground surface.

8 *Keywords:* Subsidence, Groundwater production, Confined aquifer, Analytical solution

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## 9 **1. Introduction**

10 Ground surface subsidence due to groundwater production has been a significant problem  
11 around the world for many decades (Gambolati and Teatini, 2015). When water is produced from  
12 an aquifer, the pressure within the aquifer is reduced, leading to a reduction in effective stress,  
13 which results in subsidence at the ground surface. Many attempts have been made to develop  
14 analytical models to forecast subsidence rates as a consequence of groundwater production.

15 Early models assumed radial symmetry around a groundwater production well. These models  
16 then either assumed that strain occurred only in the vertical direction (uniaxial strain) (Verruijt,  
17 1969; Bear and Corapcioglu, 1981a) or that incremental vertical total stress is zero (Verruijt, 1969;  
18 Bear and Corapcioglu, 1981b). Verruijt (1969) argues that the zero incremental vertical total stress  
19 model is analogous to assuming that the aquifer is overlain by a soft clay overburden, which offers  
20 negligible resistance to displacement. Both approaches lead to the elegant result that subsidence,  
21 at any point on the ground surface, is linearly proportional to the change in pressure in the aquifer  
22 immediately below.

23 However, the uniaxial strain model overestimates subsidence at the ground surface because  
24 it neglects the way the surrounding geological media distributes deformation laterally away from  
25 the aquifer of concern (Wu et al., 2018). The zero incremental vertical total stress model also

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26 overestimates subsidence at the ground surface because it neglects the vertical resistance of the  
27 overburden.

28 Geertsma (1973) developed an alternative analytical solution whereby the three-dimensional  
29 stress distribution is resolved without invoking uniaxial strain or zero incremental vertical total  
30 stress assumptions. Specifically, Geertsma (1973) considered the stress, strain and displacement  
31 around a cylindrical region of uniform pressure change. In particular, Geertsma (1973) derived a  
32 closed-form equation to calculate the ground surface subsidence (induced by the pressure change)  
33 immediately above the center of this cylindrical region.

34 Geertsma's closed-form equation can be related to the ground surface subsidence immediately  
35 above a production well at the center of a cylindrical confined aquifer. However, the assumption  
36 of uniform pressure leads to an underestimate in ground surface subsidence in this context. This  
37 is because the drawdown in pressure at the production well is much more significant than at the  
38 far-field of the aquifer (Wu et al., 2018).

39 Selvadurai and Kim (2015) sought to extend the analytical solution of Geertsma (1973) to  
40 allow for a non-uniform pressure distribution controlled by fluid production rate, fluid viscosity  
41 and aquifer permeability. However, the resulting equation for ground surface subsidence at the  
42 production well is significantly more complicated to evaluate, rendering it beyond application for  
43 most practical purposes.

44 More recently, Pujades et al. (2017) developed a numerical model to look at subsidence above a  
45 production well in an unconfined aquifer. They found that the zero incremental vertical total stress  
46 model was effective at estimating the subsidence far away from the production well. But close to  
47 the production well, the zero incremental vertical total stress model significantly overestimates the

48 subsidence. Pujades et al. (2017) then derived an empirical correction factor based on studying a  
49 sensitivity analysis of their numerical model. However, a limitation of their numerical model was  
50 that the model domain was restricted to the extent of the aquifer. Therefore their model was unable  
51 to properly account for how fluid production induced deformations propagate out into laterally and  
52 vertically extensive geological formations surrounding the aquifer region.

53 In this article, we build on the work of Geertsma (1973) to develop a closed-form equation  
54 for ground surface subsidence due to constant rate production of a viscous fluid from a cylindrical  
55 aquifer of finite permeability. This is achieved by application of the principle of superposition.  
56 Results from the new analytical solution are compared with equivalent results from a set of finite  
57 element simulations obtained using COMSOL Multiphysics v5.4.

## 58 **2. Mathematical model**

59 The mathematical model in this article is developed as follows. An analytical solution for  
60 the pressure distribution around a production well within a confined aquifer is presented. The  
61 original analytical solution of Geertsma (1973), for ground surface subsidence due to a cylindrical  
62 uniform pressure change, is presented. It is then shown how to incorporate non-uniform pressure  
63 distributions, resulting from constant rate production of a viscous fluid from a cylindrical aquifer  
64 of finite permeability, using the principle of superposition. A closed-form equation is then derived  
65 to calculate the ground surface subsidence directly above the production well.

### 66 *2.1. Pressure distribution in a confined aquifer*

67 Consider constant-rate single-phase fluid production from a vertically oriented and fully com-  
68 pleted production well, of infinitesimally small radius, located in the center of a homogenous,

69 isotropic, cylindrical and confined aquifer (see Fig. 1a). The pressure distribution,  $P$  [ $\text{ML}^{-1}\text{T}^{-2}$ ],  
70 within the aquifer can be found from (Theis, 1935; Dake, 1983; Mijic et al., 2013)

$$P(r, t) = \begin{cases} P_i - \frac{Q\mu}{4\pi kH} E_1\left(\frac{S\mu r^2}{4kt}\right), & 0 < t < t_c \\ P_i - \frac{Q\mu}{4\pi kH} \left[ \ln\left(\frac{R^2}{r^2}\right) + \frac{r^2}{R^2} - \frac{3}{2} + \frac{4kt}{S\mu R^2} \right] F(R-r), & t > t_c \end{cases} \quad (1)$$

71 where  $t$  [T] is time,  $P_i$  [ $\text{ML}^{-1}\text{T}^{-2}$ ] is the uniform initial pressure of the aquifer prior to com-  
72 mencement of fluid production,  $Q$  [ $\text{L}^3\text{T}^{-1}$ ] is the constant fluid production rate,  $\mu$  [ $\text{ML}^{-1}\text{T}^{-1}$ ] is the  
73 dynamic viscosity of the fluid,  $k$  [ $\text{L}^2$ ] is the permeability of the aquifer,  $H$  [L] is the thickness of  
74 the aquifer,  $r$  [L] is radial distance from the production well,  $S$  [ $\text{M}^{-1}\text{LT}^2$ ] is the specific storage  
75 coefficient of the aquifer,  $R$  [L] is the radial extent of the aquifer,  $F(x)$  denotes the Heaviside step  
76 function,  $E_1(x) = -\text{Ei}(-x)$  and  $\text{Ei}(x)$  is the exponential integral function and  $t_c$  [T] is the charac-  
77 teristic time at which the pressure front, caused by the initiation of fluid production, reaches the  
78 boundary of the confined aquifer at  $r = R$ .

79 Eq. (1) is exact for  $t \gg t_c$  and  $t \ll t_c$  but also works as an accurate approximation for  $t < t_c$   
80 and  $t > t_c$ . However, Eq. (1) is not valid in the immediate region around  $t_c$ . However, this is of  
81 little consequence for our subsequent results. The exact solution to this problem is provided by  
82 VanEverdingen (1949). However, their solution is provided as a Laplace transform, which requires  
83 numerical inversion, and is therefore not suitable for our subsequent analysis.

84 Note that the above set of equations represents a flow model, which has been uncoupled from  
85 the associated geomechanical processes. However, a good approximation for the pressure distribu-

86 tion, from a fully coupled flow model, can be obtained using a specific storage coefficient derived  
 87 assuming zero lateral strain (Gambolati et al., 2000). A recent demonstration was provided by  
 88 (Andersen et al., 2017). Analogous to Eq. (7.90) of Jaeger et al. (2009, p. 189) and Eq. (6a) of  
 89 Gambolati et al. (2000), such an expression takes the form

$$S = \frac{\phi}{K_f} + \frac{(1 - \alpha)(\alpha - \phi)}{K} + \alpha^2 C_m \quad (2)$$

90 where  $\phi$  [-] is the porosity,  $K_f$  [ $\text{ML}^{-1}\text{T}^{-2}$ ] is the bulk modulus of the fluid,  $\alpha$  [-] is the Biot coeffi-  
 91 cient,  $K$  [ $\text{ML}^{-1}\text{T}^{-2}$ ] is the bulk modulus of the rock and  $C_m$  [ $\text{M}^{-1}\text{LT}^2$ ] is the vertical (oedometric)  
 92 bulk compressibility as measured in an oedometer with lateral expansion precluded, found from  
 93 (Fjær et al., 2008, p.394)

$$C_m = \frac{1}{3K} \left( \frac{1 + \nu}{1 - \nu} \right) \quad (3)$$

94 where  $\nu$  [-] is Poisson's ratio.

95 The drawdown of the piezometric surface within the aquifer,  $s$  [L], can be found from

$$s = \frac{P_i - P}{\rho g} \quad (4)$$

96 The characteristic time,  $t_c$ , can be thought of as the time at which  $P = P_i$  at  $r = R$  for the  $t > t_c$   
 97 expression given in Eq. (1). It follows that

$$t_c = \frac{S\mu R^2}{8k} \quad (5)$$

98 2.2. *Ground surface subsidence due to a cylindrical uniform pressure change*

99 The geological material surrounding the aquifer is assumed to be homogenous, isotropic, im-  
100 permeable and semi-infinite. Furthermore, the elastic properties of the surrounding material are  
101 assumed to be the same as those of the confined aquifer.

102 When the change in fluid pressure within the aquifer can be assumed uniform, Eq. (1) reduces  
103 to

$$P = P_i - \frac{Qt}{\pi H S R^2}, \quad 0 \leq r \leq R \quad (6)$$

104 and the subsidence at the surface directly above the production well,  $w$  [L], can be found from  
105 (Geertsma, 1973; Fjær et al., 2008, p. 405)

$$w = 2C_m H \alpha (P_i - P) (1 - \nu) \left( 1 - \frac{D}{\sqrt{D^2 + R^2}} \right) \quad (7)$$

106 where  $D$  [L] is the depth of the center of the aquifer from the ground surface.

107 Substituting Eq. (6) into Eq. (7) leads to

$$w = \frac{2C_m \alpha (1 - \nu) Q t}{\pi S R^2} \left( 1 - \frac{D}{\sqrt{D^2 + R^2}} \right) \quad (8)$$

108 Geertsma (1973) also derived analytical solutions for displacement in the radial and vertical  
109 directions,  $u_r(r, z)$  [L] and  $u_z(r, z)$  [L], respectively, normal total stress in the radial, angular and  
110 vertical directions,  $\sigma_r(r, z)$  [ $\text{ML}^{-1}\text{T}^{-2}$ ],  $\sigma_\theta(r, z)$  [ $\text{ML}^{-1}\text{T}^{-2}$ ] and  $\sigma_z(r, z)$  [ $\text{ML}^{-1}\text{T}^{-2}$ ], respectively,  
111 and the stress,  $\tau_{rz}(r, z)$  [ $\text{ML}^{-1}\text{T}^{-2}$ ] for this case. Note that  $z$  [L] is depth from the ground surface  
112 and  $r$  [L] is, again, the horizontal distance from the center of the well. In this way it can be



113 understood that  $w = -u_z(0, 0)$  (see Fig. 1b). These analytical solutions are substantially more  
 114 complicated to evaluate as compared to Eq. (7) because they involve numerical approximations of  
 115 several integral expressions. Nevertheless, all the mathematical expressions needed to determine  
 116 these analytical solutions are presented in Appendix D5 of Fjær et al. (2008).

117 Because the problem being solved is a linear elastic problem, all the analytical solutions pre-  
 118 sented in Appendix D5 are linearly proportional to  $P - P_i$ . It is therefore useful to define the  
 119 following auxiliary terms:

$$\tilde{w}(R) = \frac{w}{P - P_i}, \quad \tilde{u}_j(r, z, R) = \frac{u_j(r, z, R)}{P - P_i}, \quad \tilde{\sigma}_j(r, z, R) = \frac{\sigma_j(r, z)}{P - P_i}, \quad \tilde{\tau}_{rz}(r, z, R) = \frac{\tau_{rz}(r, z)}{P - P_i} \quad (9)$$

120 where  $j$  is  $r$  for radial direction and  $z$  for vertical direction and the  $w$ ,  $u_j$ ,  $\sigma_j$  and  $\tau_{rz}$  terms in Eq.  
 121 (9) hereafter specifically relate to the expressions presented in Appendix D5 of Fjær et al. (2008).  
 122 Note that we are also identifying these expressions are functions of the radius of the uniform  
 123 pressure cylinder,  $R$ , which corresponds to the radius of the confined aquifer in this case. For  
 124 example, from Eq. (7),

$$\tilde{w}(R) = -2C_m H \alpha (1 - \nu) \left( 1 - \frac{D}{\sqrt{D^2 + R^2}} \right) \quad (10)$$

### 125 2.3. Ground surface subsidence due to production of a viscous fluid

126 The analytical solutions presented by Geertsma (1973) explicitly assumes that the pressure  
 127 within the aquifer is uniform. However, it is possible to derive approximate solutions to allow  
 128 for non-uniform pressures by discretising the pressure distribution and applying the principle of  
 129 superposition as follows:

130 Let  $r \in [0, R]$  be discretized into  $N$ , not necessarily equally spaced, points located at  $r_k$  where  
 131  $k = 1, 2, 3, \dots, N$  (see Fig. 1c). In this way it can be said that:

$$w \approx \sum_{k=2}^N \tilde{w}(r_{k-1/2})(P_{k-1} - P_k) \quad (11)$$

132

$$u_j(r, z) \approx \sum_{k=2}^N \tilde{u}_j(r, z, r_{k-1/2})(P_{k-1} - P_k) \quad (12)$$

133

$$\sigma_j(r, z) \approx \sum_{k=2}^N \tilde{\sigma}_j(r, z, r_{k-1/2})(P_{k-1} - P_k) \quad (13)$$

134

$$\tau_{rz}(r, z) \approx \sum_{k=2}^N \tilde{\tau}_{rz}(r, z, r_{k-1/2})(P_{k-1} - P_k) \quad (14)$$

135 where

$$r_{k-1/2} = \frac{r_k + r_{k-1}}{2} \quad (15)$$

136 *2.4. Closed-form equation for subsidence above the production well*

137 The series expansion of the  $E_1(x)$  function takes the form (Cooper and Jacob, 1946)

$$E_1\left(\frac{S\mu r^2}{4kt}\right) = -\gamma - \ln\left(\frac{S\mu r^2}{4kt}\right) + O\left(\frac{S\mu r^2}{4kt}\right) \quad (16)$$

138 where  $\gamma = 0.5772$  is known as the Euler-Mascheroni constant.

139

It follows that Eq. (1) can be written as (considering Cooper and Jacob, 1946)

$$P(r, t) = \begin{cases} P_i - \frac{Q\mu}{4\pi kH} \ln\left(\frac{r_e^2}{r^2}\right) F(r_e - r) + O\left(\frac{S\mu r^2}{4kt}\right), & 0 < t < t_c \\ P_i - \frac{Q\mu}{4\pi kH} \left[ \ln\left(\frac{R^2}{r^2}\right) + \frac{r^2}{R^2} - \frac{3}{2} + \frac{4kt}{S\mu R^2} \right] F(R - r), & t > t_c \end{cases} \quad (17)$$

140

where  $r_e$  [L] can be thought of as the radius of influence of the production well, found from

$$r_e = \sqrt{\frac{4kte^{-\gamma}}{S\mu}} \quad (18)$$

141

Because of the simple forms of Eqs. (17) and (7), an exact solution for  $w$  can be obtained by

142 considering

$$w = \int_0^R \tilde{w}(r) \frac{dP}{dr} dr \quad (19)$$

143

Differentiating Eq. (17) with respect to  $r$  leads to

$$\frac{dP}{dr} = \frac{Q\mu}{2\pi kH} \begin{cases} \frac{1}{r} F(r_e - r) + O\left(\frac{S\mu r}{4kt}\right), & 0 < t < t_c \\ \left(\frac{1}{r} - \frac{r}{R^2}\right) F(R - r) + \left(\frac{2kt}{S\mu R^2} - \frac{1}{4}\right) \delta(R - r), & t > t_c \end{cases} \quad (20)$$

144

where  $\delta(x)$  is the Dirac delta function.

145 It follows that

$$w_D = \begin{cases} 4 \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 + \frac{\epsilon e^{-\gamma} t_D}{2}} \right) \right], & 0 < t_D < 1 \\ \left( 1 - \frac{1}{\sqrt{1 + \epsilon}} \right) (t_{0D} + t_D), & t_D > 1 \end{cases} \quad (21)$$

146 where

$$t_{0D} = \left( 1 - \frac{1}{\sqrt{1 + \epsilon}} \right)^{-1} \left[ 4 \ln \left( \frac{1 + \sqrt{1 + \epsilon}}{2} \right) + \frac{4 + 5\epsilon}{\epsilon \sqrt{1 + \epsilon}} - \frac{4}{\epsilon} - 3 \right] \quad (22)$$

147 and

$$w_D = \frac{4\pi k w}{Q\mu C_m \alpha (1 - \nu)}, \quad t_D = \frac{8kt}{S\mu R^2}, \quad \epsilon = \frac{R^2}{D^2} \quad (23)$$

148 It can be seen that the deviation of Eq. (21) from the original solution for a uniform pressure  
 149 distribution, Eq. (8), is controlled by the value of  $t_D$ . When  $t_D \gg t_{0D}$ , Eq. (21) reduces to  
 150 Eq. (8). High  $t_D$  values imply high permeability, long production duration, low compressibility,  
 151 low viscosity and/or small aquifer radius. From Eq. (22), it can be shown that  $t_{0D} < 1$  when  
 152  $\epsilon < 3.453$ . It follows that if  $t_D > 1$ , ground surface subsidence can be calculated to a reasonable  
 153 accuracy using a uniform pressure distribution providing the radius of the aquifer is a lot less  
 154 than 1.858 times the depth of the aquifer below the ground surface. This further implies that, for  
 155 many practical purposes, ground surface subsidence is insensitive to production fluid viscosity and  
 156 aquifer permeability when the aquifer radius is less than the aquifer depth.

### 157 3. Finite element modeling

158 Results from the analytical solution were compared with results from four equivalent finite  
 159 element (FE) simulations, described by the parameter values given in Table 1. These simulations

160 were obtained using COMSOL Multiphysics v5.4.

161 Cases 1 and 3 in Table 1 are relatively shallow scenarios with the aquifers situated at a depth of  
162 200 m. In contrast, Cases 2 and 4 are deeper scenarios with the aquifers situated at a depth of 1000  
163 m. Cases 1 and 2 are based on the Berea sandstone properties presented in Table 7.2 of Jaeger et  
164 al. (2009). Cases 3 and 4 are based on a softer rock with a Bulk modulus an order of magnitude  
165 less than that for the Berea sandstone.

166 The FE simulations involved full hydro-mechanical coupling such that changes in fluid pres-  
167 sure result in changes in volume of the porous material and deformation whilst concomitant  
168 changes in stress results in a change in fluid pressure. Fluid production is specified as an out-  
169 ward mass flux on a vertical well segment along the radial symmetry axis. Since the formation  
170 surrounding the aquifer is assumed to be impervious, the aquifer has no-flow boundary condi-  
171 tions on all other boundaries. To simulate an infinitely large domain outside of the aquifer, the  
172 lateral and lower sides of the formation surrounding the aquifer is padded with infinite element  
173 domains. These domains have a geometrical scaling corresponding to an extent of several hundred  
174 kilometers, enough for the stress perturbation (caused by fluid production) not to reach the outer  
175 boundary of the computational model. The associated boundaries are treated as zero deformation  
176 boundaries. In contrast, the free surface upper boundary is treated as a zero traction boundary.

177 Pressure dissipation is fast in nearly incompressible fluids and formations. Since the aquifer  
178 is confined, there are no particularly large gradients in the solution for the fluid pressure or the  
179 displacement that require a particularly fine computational grid. The mesh used therefore consists  
180 of a fairly uniform grid with a maximum grid size of 125 meters, mainly to ensure a high resolution  
181 in the output for presentation of the results.

182 The FE models were constructed using COMSOL's core functionality and did not require the  
183 use of any additional application packages. The relevant equations used are described in Sections 3  
184 and 4 of Bjørnarå (2018). Spatial discretisation was achieved using default quadratic Lagrange el-  
185 ements. Solution was achieved using COMSOL's direct solver, MUMPS (MULTifrontal Massively  
186 Parallel sparse direct Solver).

#### 187 **4. Results**

188 Fig. 2 shows plots of drawdown and ground surface subsidence as a function of radial distance  
189 from the production well for different times. The results from the finite element simulations are  
190 shown as circular dots. The results from the analytical solution are shown as solid lines. Draw-  
191 down was calculated using Eq. (1) and subsidence was calculated using Eq. (12). To perform  
192 the superposition,  $r \in [R \times 10^{-3}, R]$  was discretised into 100 logarithmically spaced points. Log-  
193 arithmic spacing is required to properly capture the steep pressure gradients that occur close to  
194 the production well. Also shown, as circular markers, are values of subsidence directly above the  
195 production well, calculated using the closed-form equation given by Eq. (21).

196 The results from the fully coupled hydro-mechanical finite element simulations and the an-  
197 alytical solution are very similar, confirming that the uniaxial strain assumption involved in the  
198 definition of storativity,  $S$ , in Eq. (2) is appropriate in this context, as previously shown by Gam-  
199 bolati et al. (2000). The results from the closed-form equation, given by Eq. (21), correspond  
200 increasingly well with Eq. (12) with increasing time. This is to be expected because the associ-  
201 ated approximation of the pressure profile, given by Eq. (17), assumes that  $t_D \gg 1$ . Despite this  
202 shortcoming, Eq. (21) provides very close estimates of the subsidence calculated by Eq. (12). The

203 advantage of Eq. (21) is that it is significantly more straightforward to evaluate, as compared to  
204 Eq. (12).

205 Looking at Fig. 2a it can be seen that the radius of influence moves out from the well until  
206 just after 30 days, when it reaches the aquifer boundary, at a radial distance of 3000 m. After  
207 this point, pressure across the aquifer increases in a relatively uniform fashion. After 300 days  
208 of water production, the drawdown in the aquifer ranges from 8 to 12 m. For the shallow case  
209 (i.e., Fig. 2b), the subsidence above the well reaches a maximum value of just over 0.6 mm. This  
210 appears relatively uniform throughout the confined aquifer. The subsidence then decreases to zero  
211 at 1000 m from the edge of the aquifer. For the deeper case, the maximum subsidence is reduced  
212 but subsidence persists much further away from the aquifer boundary (see Fig. 2c).

213 The softer rock scenarios, Cases 3 and 4, lead to less drawdown in the aquifer (see Fig. 2d).  
214 However, this is compensated for by a greater level of subsidence at the ground surface (compare  
215 Figs. 2b and e and 2c and f). It is also noted that the radius of influence takes longer to reach the  
216 aquifer boundary. This is due to the reduction in  $t_c$  caused by the reduction in bulk modulus (recall  
217 Eq. (5)). The non-uniform pressure profile in the aquifer is clearly pronounced in the surface  
218 subsidence profile for the shallow scenario depicted in Fig. 2e. However, the subsidence profile is  
219 much smoother at 1000 m depth (see Fig. 2f).

## 220 **5. Conclusions**

221 Geertsma (1973) provided an analytical solution, which can be used to calculate the ground  
222 surface subsidence due to a cylindrical uniform pressure change. In this article, the principle of  
223 superposition was used to build on the work of Geertsma (1973) to develop an analytical solution

224 for ground surface subsidence due to constant rate production of a viscous fluid from a cylindrical  
225 aquifer of finite permeability. Results from the analytical solution were verified by comparison  
226 with a set of fully coupled hydro-mechanical finite element simulations.

227 The analytical solution based on the principle of superposition requires a priori discretisation  
228 of the pressure distribution. However, using Geertsma's closed-form equation to describe ground  
229 surface subsidence directly above the center of the cylindrical uniform pressure change, it was also  
230 possible to derive a simple closed-form equation to describe ground surface subsidence directly  
231 above the production well (or uplift directly above an injection well) within the aforementioned  
232 aquifer. The resulting equation relates a dimensionless subsidence to a dimensionless time, with  
233 just one free dimensionless parameter, which represents the ratio of the aquifer radial extent to the  
234 aquifer depth. Furthermore, the equation shows that, for many practical purposes, ground surface  
235 subsidence is insensitive to production fluid viscosity and aquifer permeability when the aquifer  
236 radius is less than the aquifer depth below the ground surface.

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244 grated aquifer subsidence equations based on vertical displacement only. *Water Resources Research*, 17, 937–946.



Table 1: Parameter values used to obtain the results presented in Fig. 2.

Parameter	Case 1	Case 2	Case 3	Case 4
Depth of aquifer, $D$ (m)	200	1000	200	1000
Radius of aquifer, $R$ (m)	3000	3000	3000	3000
Aquifer thickness, $H$ (m)	100	100	100	100
Production rate, $Q$ ( $\text{m}^3\text{day}^{-1}$ )	100	100	100	100
Bulk modulus, $K$ (GPa)	8.0	8.0	0.8	0.8
Poisson's ratio, $\nu$ (-)	0.2	0.2	0.2	0.2
Biot coefficient, $\alpha$ (-)	0.8	0.8	0.8	0.8
Porosity, $\phi$ (-)	0.19	0.19	0.19	0.19
Permeability, $k$ ( $\text{m}^2$ )	$190 \times 10^{-15}$	$190 \times 10^{-15}$	$190 \times 10^{-15}$	$190 \times 10^{-15}$
Fluid density, $\rho$ ( $\text{kg m}^{-3}$ )	1000	1000	1000	1000
Dynamic viscosity, $\mu$ (Pa s)	$10^{-3}$	$10^{-3}$	$10^{-3}$	$10^{-3}$
Fluid modulus, $K_f$ (GPa)	2.1	2.1	2.1	2.1
Aspect ratio, $\epsilon = R^2/D^2$ (-)	225	9	225	9
Value of $t_D$ at 300 days (-)	29.30	29.30	6.872	6.872

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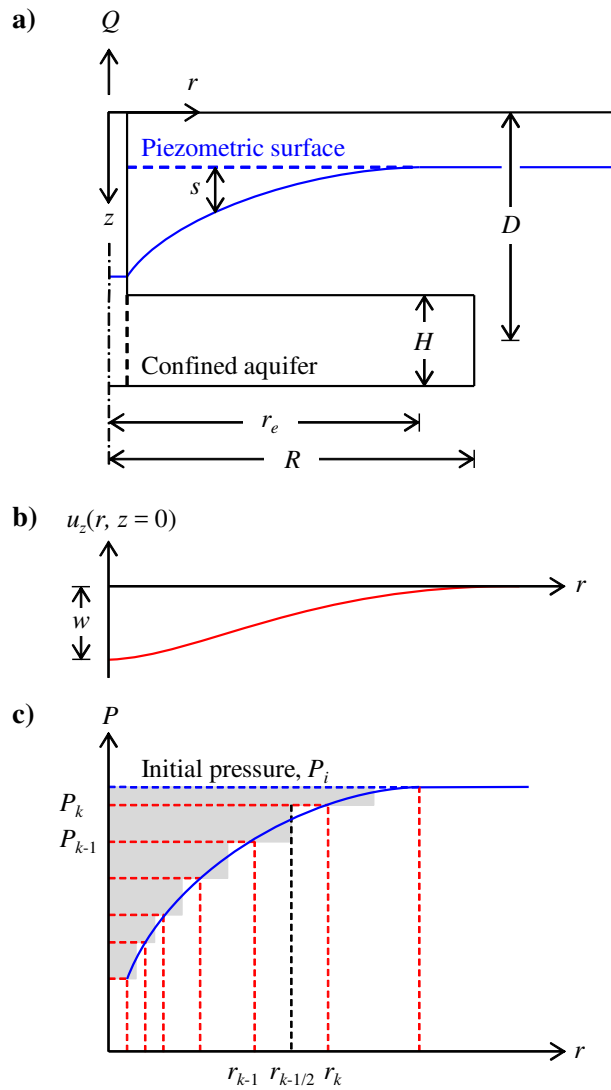


Figure 1: Schematic diagrams showing: a) The production well and its relation to the confined aquifer and surrounding semi-infinite geological formation. b) The maximum subsidence above the production well and the vertical displacement,  $u_z(r, z)$ , at the ground surface (i.e.,  $z = 0$ ). c) How the pressure is discretised to apply the principle of superposition for Eqs. (11) to (14).

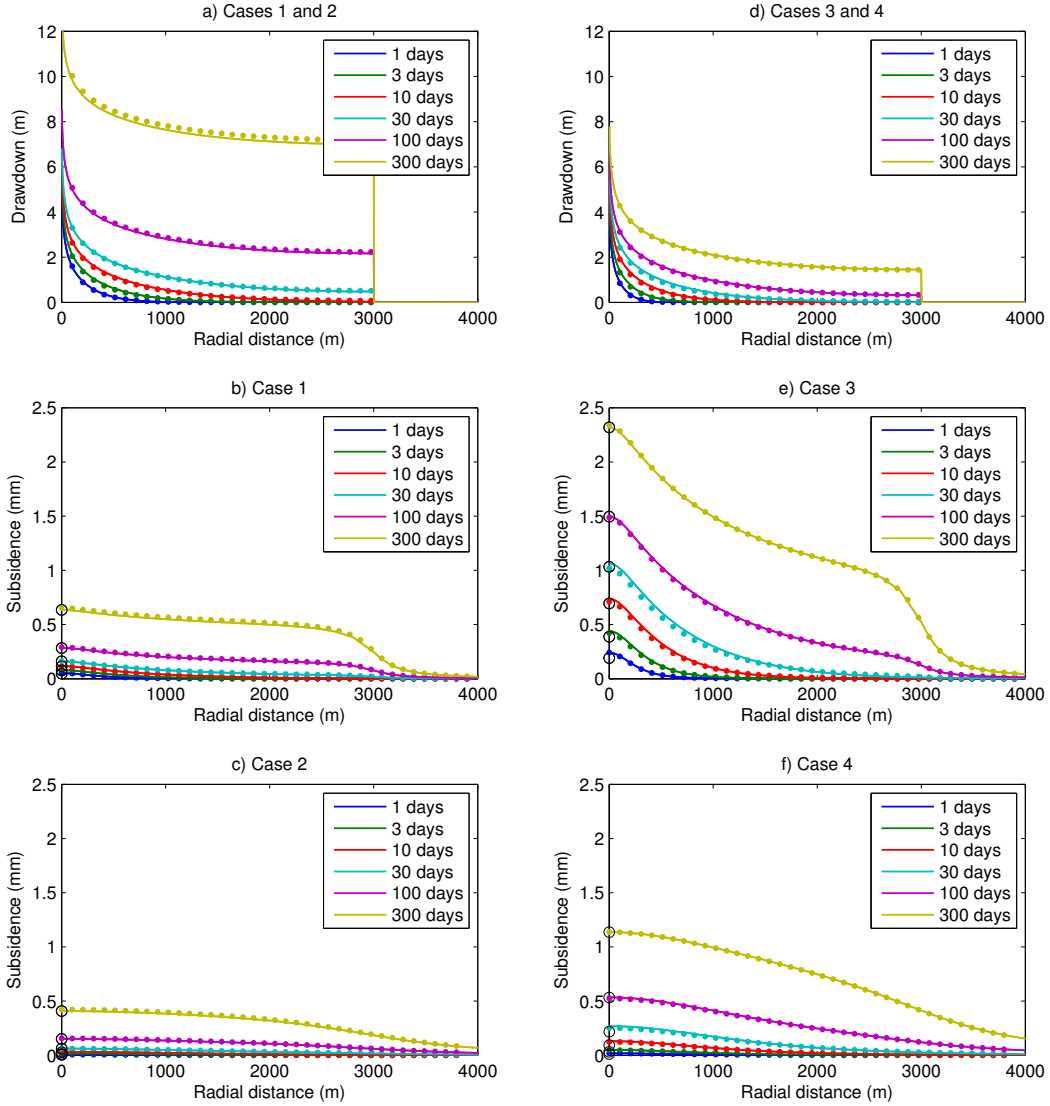


Figure 2: Plots of drawdown ( $s$ ) and subsidence ( $-u_z(r, 0)$ ) for Cases 1 to 4 as indicated by the subtitles. The solid lines were determined using Eq. (12). The circular dots were determined using the finite element simulations. The subsidence values directly above the production well ( $w$ ), as calculated using Eq. (21), are presented as black circular markers.