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PERIODICITY IN THE TRANSIENT REGIME OF EXHAUSTIVE POLLING SYSTEMS

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We consider an exhaustive polling system with three nodes in its transient regime under a switching rule of generalized greedy type. We show that, for the system with Poisson arrivals and service times with finite second moment, the sequence of nodes visited by the server is eventually periodic almost surely. To do this, we construct a dynamical system, the triangle process, which we show has eventually periodic trajectories for almost all sets of parameters and in this case we show that the stochastic trajectories follow the deterministic ones a.s. We also show there are infinitely many sets of parameters where the triangle process has aperiodic trajectories and in such cases trajectories of the stochastic model are aperiodic with positive probability.

1. Introduction. A polling system has N nodes where jobs arrive and queue and a single server which switches between the nodes to process the jobs. In exhaustive polling systems the server processes all jobs at its current node i , say, including any that arrive while jobs there are being processed, before switching to another node j , chosen by some rule. Conditions for transience/recurrence of polling systems using a *greedy* switching rule were given in [5, 6]. There is a critical case which has been investigated in [8] and [9]. In this paper we show that, for an exhaustive polling system with $N = 3$ nodes, with arrival streams and service times putting the system in its transient regime and switching according to a *threshold rule* (a type of generalized greedy rule), the sequence in which the server visits the nodes is eventually periodic for almost all choices of threshold parameters.

To show this, we consider in Section 2 the embedded Markov chain in \mathbf{Z}_+^N where we observe the polling system at service and switching time completions. We construct a dynamical system using the vector field of expected drifts of the chain for the server at each node. As the polling system is transient, this dynamical system exits any finite ball eventually, so we project it onto the unit simplex. With $N = 3$ nodes, we call this the *triangle process*, as it lives in a triangle—it has piecewise linear trajectories that change direction when they meet the triangle boundary so

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it looks something like a billiards model, but where “reflections” are caused by changes of dynamics due to the server switching to another node. We call any trajectory which returns to its start point after finitely many server switchings an *orbit*.

It is worth mentioning that this deterministic system closely resembles (and in some cases *is*) the so-called affine interval exchange transformation model. However, the affine interval exchange transformation here is contracting so it is not a bijection and it reverses orientation so it is not order-preserving. It is thus rather different from the usual interval exchange transformation model that has been much studied during the past 30 years. We comment more on this in Section 5.

We state our main results in Section 3. Theorem 3.1 states that for a.s. all choices of switching thresholds the triangle process has a finite number of orbits and every trajectory converges toward one of these orbits. We say the triangle process is stable in this case. That there are no more than four orbits for any triangle process parameters is the content of Theorem 3.2. In Theorem 3.3 we show that there are (infinitely many) choices of the switching thresholds that lead to the existence of aperiodic trajectories of the triangle process. These results are proved in Sections 4 and 5. Using these results for the deterministic triangle process, we show, in Theorem 3.4, that the projections of the trajectories of the stochastic queueing model a.s. converge onto the orbits of the triangle process, when it is stable. This implies the periodicity of the sequence of nodes receiving service for the polling system. Finally, in Theorem 3.5, we show that when the triangle process has nonperiodic trajectories then, in some situations, the sequence of nodes visited by the stochastic model is periodic with positive probability and aperiodic with positive probability. These results are proved in Sections 6 and 7.

2. System description. An exhaustive polling system has N nodes where jobs queue and a single server which switches to the next node j , chosen with some rule, after processing all jobs at the current node i , including any that arrive while jobs at i are being processed. In the general model switching takes a time which depends upon the pair i, j . We will assume the following at each node i : the arrival processes are independent Poisson streams with arrival rate λ_i ; the service times are i.i.d. and independent of the arrivals with finite mean μ_i^{-1} and variance $\sigma_i^2 < \infty$; the switching times are independent with finite means that depend upon the initial and final node while the server just waits for the next arrival at any node when it completes service at i and finds the system is empty.

For the results in our paper, we will assume that the switching times are zero. We believe our argument can be extended to a more general situation, but our main interest is in the transient case, where the system behavior is not sensitive to the switching time distributions, though of course it is affected by the switching rule.

The methods we know for showing that periodicity of the node sequence for the deterministic triangle process implies the same for the stochastic process are applicable only to Markov processes. As the sequence of nodes visited is determined

by the jump chain, we will consider the *discrete* time process

$$\Xi = \{(\xi(t); s(t))\}, \quad t = 0, 1, \dots,$$

where $\xi(t)$ describes the queue lengths and $s(t)$ the server location at the epochs of service time and switching time completions. Its state space is $\mathbb{Z}_+^N \times \{1, 2, \dots, N\}$. That Ξ is well defined, irreducible and aperiodic follows from the assumptions of Poisson arrivals and finite first moments of the service times. For later convergence arguments, we also require finite second moments of the service times and under these conditions, standard results imply this embedded chain is essentially equivalent to the continuous time process as regards transience/recurrence. The transition probabilities can be computed via the Laplace transforms of the service times, but we do not need them explicitly at any point. A standard conditioning argument readily produces the expected one-step mean drifts

$$(1) \quad \mathbf{E}(\xi_i(t+1) - \xi_i(t) | (\xi(t); s(t)) = (x; j)) = \lambda_i \mu_j^{-1} - I_{\{i=j\}},$$

$$i = 1, \dots, N,$$

for $x \in \mathbb{Z}_+^N$ with $x_j \geq 1$ and the server at node j and we make considerable use of these. A similar result holds for expected drifts during switching times, but we will not detail these as, in fact, we will assume switching to occur instantaneously. Other arrival and service processes for which there is an embedded Markov chain can be found, but their treatment needs no significant extension of our methods.

We will start by stating the known explicit conditions for recurrence/transience. For exhaustive polling models, the conditions for recurrence/transience depend only upon the *total loading* (or traffic intensity)

$$\rho = \sum_{i=1}^N \rho_i \quad \text{where } \rho_i = \lambda_i / \mu_i \text{ at each node } i$$

for a great many (and seemingly all *sensible*) switching rules under the assumption that switching times have finite first moment. We have the following:

THEOREM 2.1. *The process Ξ is positive recurrent if $\rho < 1$, transient if $\rho > 1$.*

REMARKS. Foss and Last [5] establish the result for a more general model. The method of proof is via Lyapunov functions as described in [1, 4] or [10]. The papers by MacPhee and Menshikov [8] and Menshikov and Zuyev [9] consider the critical case $\rho = 1$, where the behavior of the system depends strongly on the first two moments of the switching time distribution. Foss and Last [6] obtain the result of this theorem for nonexhaustive polling systems under a greedy switching policy.

Generalized greedy switching rules. When the server has completed all the tasks at its current node, it chooses its next node using a switching rule which we will assume depends upon the queue lengths, that is, the rule is a function $\mathfrak{R}: \partial\mathbb{R}_+^N \rightarrow \{1, 2, \dots, N\}$, where $\partial\mathbb{R}_+^N = \bigcup_j \{y \in \mathbb{R}_+^N : y_j = 0\}$. Any \mathfrak{R} must satisfy $\mathfrak{R}(y) \neq j$ if $y_j = 0$ and a variety of such rules have been studied in the literature. We will study only a class of generalized greedy rules defined as follows. For each node j , there is a vector of positive weights $b_j = (b_{j1}, \dots, b_{jN})$ and at states $y \in \partial\mathbb{R}_+^N$ with $y_j = 0$, $\mathfrak{R}(y) = i$, where $b_{ji}y_i = \max_k (b_{jk}y_k)$ (for our results it is not important how ties are resolved). The simple greedy rule is the special case where all b_{jk} are equal.

A deterministic model. Our subsequent analysis of the transient case is based around the following deterministic model of the system. Consider a particle moving in $\mathbb{R}_+^N \times \{1, \dots, N\}$ with linear dynamics given by the one-step mean drifts of Ξ as calculated in (1). With its position denoted $y = (y_1, y_2, \dots, y_N)$ and the server at node j , we see that at (y, j) with $y_j > 0$ the particle has velocity

$$\mu_j^{-1} \sum_{i=1}^N \lambda_i e_i - e_j = \mu_j^{-1} \left(\sum_{i \neq j} \lambda_i e_i + (\lambda_j - \mu_j) e_j \right),$$

where the e_i denote the axial unit vectors in \mathbb{R}^N . If a point with $y_j = 0$ is reached, then a different set of dynamics (corresponding to the server switching to another node) is chosen instantaneously according to some generalized greedy switching rule \mathfrak{R} .

The trajectory $y(t)$ of our particle is constructed as follows. From start point $y(t_0) = \bar{y} \in \mathbb{R}_+^N$ with $\bar{y}_j > 0$ and the server at node j , the particle travels along the line

$$\ell_j(\bar{y}) = \left\{ y \in \mathbb{R}^N : y = \bar{y} + \mu_j^{-1}(t - t_0) \left(\sum_{i \neq j} \lambda_i e_i + (\lambda_j - \mu_j) e_j \right), t \in \mathbb{R} \right\}$$

through \bar{y} with the appropriate velocity. If $\rho_j \geq 1$, the particle never reaches $\partial\mathbb{R}_+^N$, while if $\rho_j < 1$, the particle reaches

$$(2) \quad y(t_1) = \sum_{i \neq j} \left(\bar{y}_i + \frac{\lambda_i \bar{y}_j}{\mu_j - \lambda_j} \right) e_i \in \partial\mathbb{R}_+^N$$

at time $t = t_0 + \bar{y}_j / (1 - \rho_j) =: t_1$. Now the server switches to node $\mathfrak{R}(y(t_1))$ and the next and subsequent pieces of the trajectory are computed as above.

The node process. In the transient case the trajectory exits from any finite ball eventually, so, in order to study whether there is any periodicity in the order the

server visits the queues, we project onto the unit simplex. Project the lines $\ell_j(\bar{y})$ onto the hyperplane $\mathcal{S}_1 = \{y \in \mathbb{R}^N : \sum_i y_i = 1\}$ using

$$\Lambda : \mathbb{R}^N \setminus \{0\} \rightarrow \mathcal{S}_1 \quad \text{where } \Lambda(y) = \frac{y}{\sum_i y_i},$$

which maps each line $\ell_j(\bar{y})$ onto the intersection of \mathcal{S}_1 with the plane containing $\ell_j(\bar{y})$ and the origin. The feasible positions for the particle will be mapped onto points in the unit simplex $\mathcal{S}_1^+ = \{y \in \mathbb{R}_+^N : \sum_i y_i = 1\}$. For starting points \bar{y} and $\alpha\bar{y}$ for any $\alpha > 0$, the lines $\ell_j(\bar{y})$ and $\ell_j(\alpha\bar{y})$ have exactly the same image in \mathcal{S}_1 , so we can restrict our attention to reference points $\bar{y} \in \partial\mathcal{S}_1^+$, the boundary of \mathcal{S}_1^+ .

Under the condition $\rho_j > 1$, the server in the stochastic process can remain serving at queue j indefinitely, so we will only consider the cases where $\rho_j < 1$ at all queues j . Under the condition $\rho_j < 1$, we see from equation (2) that the trajectory $y(t)$ leaving $\bar{y} \in \partial\mathcal{S}_1^+$ along $\ell_j(\bar{y})$ next reaches $\partial\mathbb{R}_+^N$ in finite time at the point

$$\sum_{i \neq j} \left(\bar{y}_i + \frac{\lambda_i \bar{y}_j}{\mu_j - \lambda_j} \right) e_i \quad \text{with projection } \sum_{i \neq j} \frac{(\mu_j - \lambda_j) \bar{y}_i + \lambda_i \bar{y}_j}{(\mu_j - \lambda_j) + \mu_j \theta_j} e_i \in A_j^0,$$

where $A_j^0 = \{y \in \mathbb{R}^N : \sum_i y_i = 1, y_j = 0, y_i \geq 0 \text{ for } i \neq j\}$ for $j = 1, 2, \dots, N$ are regions of the switching boundary $\partial\mathcal{S}_1^+$ and

$$(3) \quad \theta_j := \mu_j^{-1} \sum_i \lambda_i - 1, \quad j = 1, 2, \dots, N.$$

There is a very nice geometric description of the projected process. For any fixed j , the lines $\ell_j(\bar{y})$ through different points \bar{y} are parallel so they are concurrent after projection, that is, their image lines on \mathcal{S}_1 are either parallel or share a common focus, v_j , say. The line $\ell_j(\bar{y})$ with $\bar{y} = \sum_{i \neq j} \lambda_i e_i + (\lambda_j - \mu_j) e_j$ passes through 0 so v_j is the point where $\ell_j(\bar{y})$ meets \mathcal{S}_1 if this happens. This is the case whenever $\theta_j \neq 0$, in which case

$$(4) \quad v_j = \frac{1}{\mu_j \theta_j} \left(\sum_{i \neq j} \lambda_i e_i + (\lambda_j - \mu_j) e_j \right).$$

When $\theta_j = 0$, the $\ell_j(\bar{y})$ with $\bar{y} \in \mathcal{S}_1$ all lie in \mathcal{S}_1 and the projected lines through different \bar{y} are parallel.

The focus point v_j will be outside \mathcal{S}_1^+ when $\theta_j < 0$ or when $\theta_j > 0$ and $\rho_j = \lambda_j / \mu_j < 1$, conditions which correspond to the next switching event taking place after some finite time.

As our aim is to study the switching sequence, we assume from now on that

$$(5) \quad \rho_i < 1, \quad i = 1, \dots, N \quad \text{and} \quad \rho = \sum_{i=1}^N \rho_i > 1$$

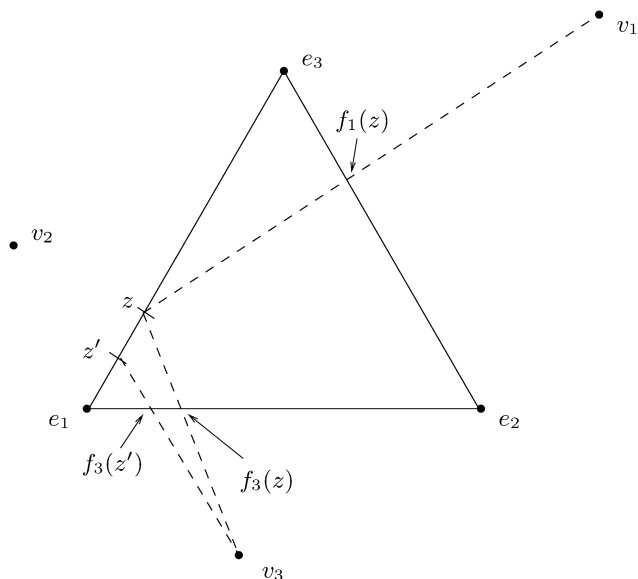


FIG. 1. The node process when $N = 3$.

(which means that the whole system is transient, yet the server does not get stuck at any individual node). We can now define the discrete dynamical system $Z = \{z(t)\}, t = 0, 1, \dots$, living on $A^0 \equiv \bigcup_i A_i^0 = \partial \mathcal{S}_1^+$. Define mappings

$$(6) \quad f_j(z) = \sum_{i \neq j} \frac{(\mu_j - \lambda_j)z_i + \lambda_i z_j}{(\mu_j - \lambda_j) + \mu_j \theta_j z_j} e_i \in A_j^0, \quad z \in A^0 \setminus A_j^0, \quad j = 1, \dots, N.$$

The image under f_j of a point $z \in A_i^0$ lies at the intersection of the line through z and v_j and the boundary region A_j^0 , as shown for the case $N = 3$ in Figure 1. For given $z(0) \in \partial \mathcal{S}_1^+$ let

$$(7) \quad z(t + 1) = \varphi(z(t)), \quad t = 0, 1, \dots,$$

$$\text{where } \varphi(z) = \sum_{j=1}^N I_{\{\mathfrak{R}(z)=j\}} f_j(z), \quad z \in A^0.$$

We will call Z the *node process*, as it records information about the projection of the dynamical system $y(t)$ only at switching epochs.

3. The three node case. From this point our analysis is restricted to the case where the system has three nodes and the switching decision is made using a generalized greedy rule. This enables us to give clear statements of our results and methods, but is complex enough to be very interesting in our opinion.

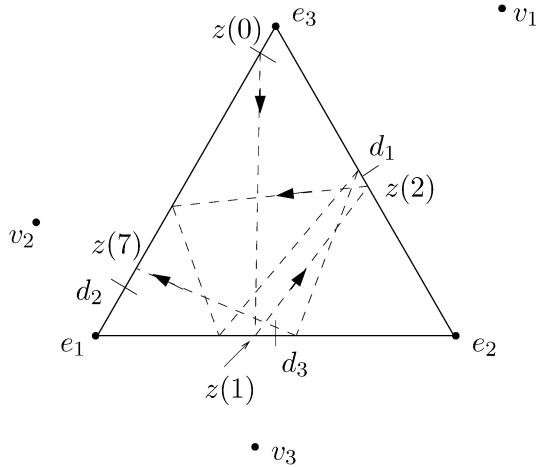


FIG. 2. The triangle process trajectory.

Recall that when the server has completed all jobs at queue i , the switching rule is defined by parameters $b_{ij} > 0$ and selects the next node k when $b_{ij}x_j < b_{ik}x_k$ for i, j, k any permutation of 1, 2, 3. Projection by Λ onto \mathcal{A}_+^1 reduces the switching boundaries $\{x : b_{ij}x_j = b_{ik}x_k, x_i = 0\}$ to decision points, one on each side of the triangle A^0 as shown in Figure 2. This rule can be applied to the stochastic process Ξ and the dynamical process $y(t)$ as stated with the following resolution of boundary cases. For the stochastic process Ξ , a randomized rule may be used whenever $b_{ij}x_j = b_{ik}x_k$, so we will consider the consequences for the node process Z of both possible decisions at such points. Specifically, we will consider trajectories of the node process which branch when they exactly hit the decision points. From now on, we will call the node process Z the *triangle process* and specialize our notation. The construction in Section 2 is somewhat abstract, but the system which we have defined and wish to study is really very simple to describe. Figure 2 shows a trajectory starting from $z(0) = z$ near e_3 . This is mapped to $z(1)$ along the line from z to v_3 and then to $z(2)$ just below d_1 . As $z(2)$ is toward e_2 from d_1 , the next point $z(3)$ is on the line from $z(2)$ to v_2 . From here the trajectory continues toward v_3 , then v_1 , then back toward v_3 and from there toward v_2 and so on.

We note that the mapping which determines the trajectories is not continuous at the d_i . However, during extensive numerical investigation, we found that, for all the configurations of rates and decision points we tried, the trajectories we examined converged toward periodic orbits.

Triangle process notation. Let $\hat{i}, \hat{j}, \hat{k}$ denote the values 1, 2, 3 or either of its cyclic permutations 2, 3, 1 or 3, 1, 2 and let i, j, k denote any permutation of 1, 2, 3. For $z = (1 - x)e_j + xe_k \in A_i^0$ with $x \in [0, 1]$, we define $\text{Side}(z) := \hat{i}, \pi(z) := x$

and write $z = (x, \hat{i})$ (with standard right-hand axes, increasing x corresponds to going around the triangle anticlockwise).

For the triangle process Z , any generalized greedy rule \mathfrak{R} has the form

$$\mathfrak{R}(x, \hat{i}) = \begin{cases} \hat{j}, & x \leq d_{\hat{i}}, \\ \hat{k}, & x \geq d_{\hat{i}}, \end{cases}$$

for values $d_{\hat{i}} = (1 + b_{\hat{i}\hat{k}}/b_{\hat{i}\hat{j}})^{-1} \in (0, 1)$ for $\hat{i} = 1, 2, 3$. The *decision points* $(d_{\hat{i}}, \hat{i})$ will usually be written simply $d_{\hat{i}}$. We will usually refer to rules of this type as *threshold* rules.

For $z = (x, i), z' = (x', j)$, we will use distance $|z - z'|_1 := \|z - z'\|/\sqrt{2}$, the Euclidean distance scaled so that when $i = j, |z - z'|_1 = |\pi(z) - \pi(z')| = |x - x'|$, that is, it is length along the side of the triangle.

The forward mapping φ defined in (7) is 1–1 except at the $d_{\hat{i}}$ which have two images, while the inverse of φ is 1–1 where it exists. Call z a *pre-image* of z' when $z' = \varphi^{(t)}(z)$ for some $t \geq 1$.

DEFINITION. A sequence $z(t), t = 0, 1, \dots$, satisfying (7) is a *trajectory* of Z . If $z(0)$ is a pre-image of a decision point $d_{\hat{i}}$, then there are at least two trajectories starting from $z(0)$.

We will study closely the sets $A^t = \varphi(A^{t-1}), t = 1, 2, \dots$, which are such that A^t contains the possible locations for $z(t)$ from all initial points $z(0) \in A^0$. Alternatively, $A^0 \setminus A^t$ is the set of points with fewer than t pre-images, a description which we will use later. The set $A^1 = \varphi(A^0)$ is a strict subset of A^0 for any decision points since, for example, $e_i \notin A^1, i = 1, 2, 3$. Hence, $A^0 \supset A^1 \supset \dots \supset A^t$ for $t = 2, 3, \dots$. Further, A^1 is a union of three disjoint closed intervals, one in each A_i^0 . We introduce now the notation $A_i^t \equiv A^t \cap A_i^0$ for later use— A_1^1 is depicted in Figure 3. As only intervals containing a decision point will be split by φ , it follows that A^t is a union of at most $3t$ disjoint closed intervals.

Periodicity definitions. We call a trajectory $z(t), t = 0, 1, \dots$, *eventually- m -periodic* if there is $N > 0$ such that, for all $n \geq N$,

$$\text{Side}(z(n + m)) = \text{Side}(z(n)).$$

A trajectory which is not eventually- m -periodic for any m is called *nonperiodic*.

We say that Z has a *periodic orbit* if there is a finite sequence of points u_1, u_2, \dots, u_m with $\phi(u_i) = u_{i+1}$ for $i = 1, 2, \dots, m - 1$ and $\phi(u_m) = u_1$. The sequence $\text{Side}(u_1), \text{Side}(u_2), \dots, \text{Side}(u_m)$ is the *node-cycle* of the orbit. We will consider cyclic permutations of orbits/node-cycles to be equivalent to the original orbit/node-cycle. All orbits are disjoint since, for any point u on an orbit, $\varphi^{(-t)}(u)$ is uniquely defined for all $t \geq 0$ so there is no point at which two orbits could join.

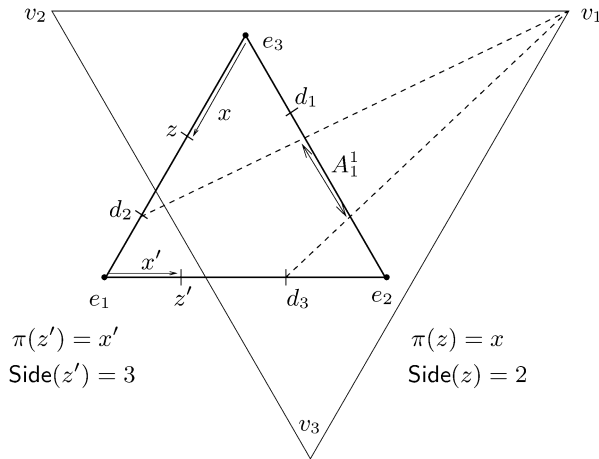


FIG. 3. The triangle process.

A trajectory converges onto an orbit u_1, \dots, u_m when there exists $n_0 \geq 0$ such that

$$|z(mt + n + n_0) - u_n|_1 \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ for each } n = 1, 2, \dots, m,$$

that is, for each n , the subsequence of the trajectory corresponding to phase n of the node-cycle converges to u_n . For any point $z = (x, i) \in A_i^0$ and $0 < \varepsilon < x$, let

$$(8) \quad \mathcal{N}_\varepsilon(z) = \{z' \in A_i^0 : |\pi(z') - x|_1 < \varepsilon\}$$

denote the ε -neighborhood of u , $\mathcal{N}_\varepsilon^+(z) = \{z' \in A_i^0 : 0 \leq \pi(z') - x < \varepsilon\}$ and $\mathcal{N}_\varepsilon^-(z) = \{z' \in A_i^0 : 0 \leq x - \pi(z') < \varepsilon\}$ the one-sided ε -neighborhoods of z . We will say an orbit is *stable* when, for each u_n , there exists $\varepsilon > 0$ such that trajectories starting from any $z \in \mathcal{N}_\varepsilon(u_n)$ converge onto the orbit. When trajectories started from $z \in \mathcal{N}_\varepsilon^+(u_n)$ but not from $\mathcal{N}_\varepsilon^-(u_n)$ (or vice versa) converge onto the orbit, we say the orbit is *stable on one side* and otherwise we say the orbit is *unstable*. An orbit containing a decision point must be unstable if it is of odd length m and may be stable on one side if m is even.

We are now in position to state our main results for the triangle process Z . These hold for any set of parameters $\lambda_i, \mu_i, i = 1, 2, 3$, satisfying conditions (5).

THEOREM 3.1. *For almost all decision points $d_i \in A^0, i = 1, 2, 3$, the triangle process Z has finitely many periodic orbits. For such sets of decision points, all trajectories $z(t)$ are eventually periodic and each converges onto one of these orbits as $t \rightarrow \infty$.*

The quantifier “almost all” is used in the sense of Lebesgue measure \times counting measure on $[0, 1] \times \{1, 2, 3\}$. Due to the convergence behavior of all trajectories,

we will call this the *stable case*. Typically, in this case all the orbits are stable, but unstable orbits are possible when an orbit includes a decision point.

THEOREM 3.2. *For any set of decision points $d_i, i = 1, 2, 3$, there are at most four periodic orbits.*

The next result shows that not all choices of decision points result in all trajectories being eventually periodic.

THEOREM 3.3. *There is an uncountable set of decision points for which Z has nonperiodic trajectories.*

These results have important consequences for the behavior of the underlying stochastic process $\Xi = \{\xi(t)\}_t$. We now introduce the random times $\tau_t, t = 1, 2, \dots$, at which the server changes queues. Under the assumptions in (5) on the parameters, the τ_t are all a.s. finite. The next result concerns the stochastic triangle process

$$(9) \quad \zeta(t) = \Lambda(\xi(\tau_t)), \quad t = 1, 2, \dots$$

This process has trajectories living on A^0 , so we can use the same definition of convergence for it as for the triangle process.

THEOREM 3.4. *Suppose the service times have variances $\sigma_i^2 < \infty$. For any set of decision points (d_1, d_2, d_3) such that Z is stable, the stochastic process ζ is also stable in the sense that a.s. each trajectory of ζ converges onto one of the periodic orbits of Z .*

THEOREM 3.5 (No zero–one law). *There exist configurations of the decision points such that the stochastic process ζ has nonperiodic trajectories with positive probability.*

We construct an example with trajectories that can converge to a periodic orbit of Z with positive probability and also can be nonperiodic with positive probability.

Some open problems.

- It seems from numerical computations that there are, in fact, at most three orbits. Is this correct or are there examples with four orbits?
- Is it also true that, for any given set of decision points d_i , the sets of parameters λ, μ where there are nonperiodic orbits has measure zero?
- Which of the results proved here also hold when there are four or more nodes?
- Can the stochastic process ζ converge with positive probability to an orbit of even length which contains a decision point and is only stable on one side?

4. Proofs for the triangle process. The conditions for transience or recurrence of the stochastic process Ξ are in terms of the traffic intensities ρ_i in accord with the intuition that considering weighted work at nodes rather than just the numbers of queued jobs should not affect such properties. Such a re-weighting also helps simplify the treatment of the triangle process. To describe its effect, we introduce the fractional linear functions $F_\alpha : [0, 1] \rightarrow [0, 1]$, where, for any $\alpha > 0$, $F_\alpha(x) = x/(\alpha + (1 - \alpha)x)$. These have a key composition property $F_\alpha \circ F_\beta = F_{\alpha\beta}$.

LEMMA 4.1 (Re-weighting). *Consider the triangle process Z with parameters λ_i, μ_i, d_i for $i = 1, 2, 3$ and define $T : A^0 \rightarrow A^0$ by $T(z) = F_{\mu_{\hat{i}}/\mu_j}(z), z \in A^0, \hat{i} = 1, 2, 3$. Z is isomorphic to the triangle process Z' with parameters $\mu'_i = 1, \lambda'_i = \lambda_i/\mu_i = \rho_i$ and $d'_i = T(d_i), i = 1, 2, 3$. Specifically, for any given $z(0) \in A^0$, if Z' is started from $T(z(0))$, then $z'(t) = T(z(t))$ for $t = 1, 2, \dots$*

PROOF. We start by assuming that only parameters $\lambda_{\hat{i}}, \mu_{\hat{i}}$ for some \hat{i} are transformed by multiplication by $\alpha > 0$ and briefly describe an isomorphism between trajectories of the dynamical processes y and y' . Define $\Gamma_{\hat{i}} : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$ by $\Gamma_{\hat{i}}(y) = y + (\alpha - 1)y_{\hat{i}}e_{\hat{i}}$, so $\Gamma_{\hat{i}}$ rescales \mathbb{R}_+^3 by α in the $e_{\hat{i}}$ direction. Let $\ell_j(\bar{y})$ and $\ell'_j(\bar{y})$ denote the lines through \bar{y} parallel to the trajectories of $y(t)$ and $y'(t)$ respectively when the server is at node j . Then $\Gamma_{\hat{i}}(\ell_j(\bar{y})) = \ell'_j(\Gamma_{\hat{i}}(\bar{y}))$. This space rescaling provides an isomorphism between entire trajectories whenever the switching decisions are identical, or, equivalently, $r'(\Gamma_{\hat{i}}(\bar{y})) = r(\bar{y})$ for all $\bar{y} \in \partial\mathbb{R}_+^3$.

The class of threshold policies is closed under space rescalings $\Gamma_{\hat{i}}$ with the parameters $b_{\hat{j}\hat{i}}$ being mapped to $\alpha b_{\hat{j}\hat{i}}$ and $b_{\hat{k}\hat{i}}$ to $\alpha b_{\hat{k}\hat{i}}$ with consequent changes to $d_{\hat{j}}$ and $d_{\hat{k}}$. Now for any $\bar{y} \in \partial\mathbb{R}_+^3$ and $z = \Lambda(\bar{y}) = (x, i)$, we find that $z' = \Lambda(\Gamma_{\hat{i}}(z))$ satisfies

$$z' = \begin{cases} (x, i), & i = \hat{i}, \\ (F_{1/\alpha}(x), i), & i = \hat{j}, \\ (F_\alpha(x), i), & i = \hat{k}, \end{cases}$$

where $(\hat{i}, \hat{j}, \hat{k})$ is a cyclic permutation of $(1, 2, 3)$. Applying successive rescalings by $1/\mu_i$ in the e_i direction for $i = 1, 2, 3$ and using the composition property of F_α gives the result. Note that F_α satisfies $F_\alpha(0) = 0, F_\alpha(1) = 1$ and $F'_\alpha(x) > 0, x \in (0, 1)$, so the mapping T fixes the corners e_i of A^0 and smoothly rescales the sides. \square

The implication of Lemma 4.1 is that we need only study the triangle process with parameters $\mu_1 = \mu_2 = \mu_3 = 1$ and general λ_i and d_i to understand the possible behavior of the general case. We will work with the rates $\mu_k = 1$ from this point. We now find, from (3), that $\theta_k = \sum_n \rho_n - 1 =: \theta > 0$ for each k as we are in the transient case. It follows from standard projective geometry results that the

foci v_k are the vertices of an equilateral triangle V with sides parallel to the A_i^0 , as shown in Figure 3. It can also be seen directly as, from equation (4) for each k ,

$$v_k = \frac{1}{\theta}(\rho_i e_i + \rho_j e_j + (\rho_k - 1)e_k) \quad \text{and, hence,} \quad v_i - v_j = \frac{1}{\theta}(e_j - e_i).$$

Under conditions (5), the v_k are outside \mathcal{R}_1^+ , as remarked after equation (4). Let ${}^V A$ denote those points of A^0 which are internal to V and ${}^V A_i = A_i^0 \cap {}^V A$. In Figure 3, ${}^V A$ contains all of A^0 except the neighborhood of e_1 .

When $\rho_k < \theta$ or, equivalently, $\rho_i + \rho_j > 1$, the line from v_i to v_j meets A_i^0 at the point $(1 - \alpha)e_j + \alpha e_k$ with $\alpha = \rho_k/\theta \in (0, 1)$, while if $\rho_i + \rho_j < 1$, then $e_k \in {}^V A$. If $e_{\hat{k}} \notin {}^V A$, let $J_{\hat{k}} = \{z \in A_i^0 : x > \rho_{\hat{k}}/\theta\} \cup \{z \in A_j^0 : x < 1 - \rho_{\hat{k}}/\theta\}$ denote the corner of A^0 containing $e_{\hat{k}}$ and lying outside ${}^V A$. Let $J_{\hat{k}} = \emptyset$ when $e_{\hat{k}} \in {}^V A$ (so $J_2 = J_3 = \emptyset$, but $J_1 \neq \emptyset$ in Figure 3).

The next two lemmas state key properties of the node process mappings but ignore the effect of the decision points. Translating definition (6) into the triangle process notation, we find that $f_j : A_i^0 \cup A_k^0 \rightarrow A_j^0$ satisfies

$$(10) \quad \begin{aligned} f_j(x, \hat{i}) &= \left(\frac{\rho_i(1-x)}{1-\rho_j+\theta(1-x)}, \hat{j} \right) \quad \text{and} \\ f_j(x, \hat{k}) &= \left(\frac{\rho_i x + (1-\rho_j)(1-x)}{1-\rho_j+\theta x}, \hat{j} \right) \end{aligned}$$

and $f_j(e_j) = f_j(0, \hat{i}) = f_j(1, \hat{k}) = \rho_i/(\rho_i + \rho_{\hat{k}})$, which demonstrates continuity of the f_j at the corners e_j of A^0 .

LEMMA 4.2 (Contraction property). *The function f_j is monotone on $A_i^0 \cup A_k^0$ and there exists $\gamma \in (0, 1)$ and $C_j(\gamma) \subset A_i^0 \cup A_k^0$ such that for $z, z' \in C_j(\gamma) \cap A_n^0$, $n = \hat{i}, \hat{k}$, we have $|f_j(z) - f_j(z')|_1 < \gamma|z - z'|_1$ and further,*

$$C_j(\gamma) \supset \{(x, \hat{i}) : 0 \leq x \leq \min(1, \rho_{\hat{k}}/\theta)\} \cup \{(x, \hat{k}) : \max(0, 1 - (\rho_i/\theta)) \leq x \leq 1\},$$

so $C_j(\gamma) \supset {}^V A_i \cup {}^V A_{\hat{k}}$. The containment is strict if $J_i \cup J_{\hat{k}} \neq \emptyset$.

PROOF. On A_i^0 , let $g(x) = \pi(f_j(x, \hat{i})) = \rho_i(1-x)/(1-\rho_j+\theta(1-x))$. As $g'(x) = -\rho_i(1-\rho_j)/(\rho_i+\rho_{\hat{k}}-\theta x)^2 < 0$, g is monotone decreasing in x . Choose γ_i such that if $\rho_{\hat{k}} > \theta$, then $|g'(1)| = \rho_i/(1-\rho_j) \leq \gamma_i < 1$, while if $\rho_{\hat{k}} < \theta$, $|g'(\rho_{\hat{k}}/\theta)| = (1-\rho_j)/\rho_i \leq \gamma_i < 1$. The same argument applies for selecting γ_k on A_k^0 using $g(x) = (\rho_i x + (1-\rho_j)(1-x))/(1-\rho_j+\theta x)$. Finally, choose $\gamma \geq \max(\gamma_i, \gamma_k)$ and set $C_j(\gamma) = \{z \in A_i^0 \cup A_k^0 : |g'(x)| \leq \gamma\}$. \square

The mapping f_j may not be contracting near e_i or e_k if they are not in ${}^V A$. This implies that φ may not be contracting around, say, $(x, 1)$ if $\rho_3 < \theta$, d_1 is close to e_3 and $\rho_3/\theta < x < d_1$.

Starting from $z \in A_i^0$, we can apply either f_j or f_k to z , then again apply either f_i or f_k to $f_j(z) \in A_j^0$ and so on. Let $\sigma \in \{1, 2, 3\}^{\mathbb{N}}$ denote a sequence with the properties $\sigma_1 \neq i$, $\sigma_t \neq \sigma_{t-1}$, $t = 1, 2, \dots$ (such a sequence will be called *allowed sequence*). For $z \in A_i^0$, let $f_\sigma^{(t)}(z) = f_{\sigma_t}(f_{\sigma_{t-1}}(\dots(f_{\sigma_1}(z)\dots))$. As the mappings f_i are monotone, they are invertible and we will need, a little later, to consider mappings $f_\sigma^{(-t)}(z) = f_{\sigma_t}^{(-1)}(f_{\sigma_{t-1}}^{(-1)}(\dots(f_{\sigma_1}^{(-1)}(z)\dots))$.

LEMMA 4.3. *There exists a constant $\kappa > 0$ such that, for any short enough interval $[u, w] \subset {}^V A_i$, $i \in \{1, 2, 3\}$, and any allowed sequence σ , with the constant $\gamma \in (0, 1)$ from Lemma 4.2:*

- (i) $|f_\sigma^{(t)}(w) - f_\sigma^{(t)}(u)|_1 \leq \gamma^t |w - u|_1 \quad \text{for } t = 1, 2, \dots,$
- (ii)
$$e^{-\kappa|w-u|_1} \frac{|v - u|_1}{|w - u|_1} \leq \frac{|f_\sigma^{(t)}(v) - f_\sigma^{(t)}(u)|_1}{|f_\sigma^{(t)}(w) - f_\sigma^{(t)}(u)|_1} \leq e^{\kappa|w-u|_1} \frac{|v - u|_1}{|w - u|_1}, \quad t = 1, 2, \dots,$$

for any $v \in (u, w)$.

PROOF. Inequalities (i) follow immediately from Lemma 4.2 since we have $f_j({}^V A_i) \subset {}^V A_j$ for all pairs $i \neq j$. For part (ii), consider any monotone function g on a short interval $[a, b]$ with $|g'(x)| > \alpha_1 > 0$ and $|g''(x)| < \alpha_2$. Expanding g to second order around a with Taylor’s theorem, we find that, for any $c \in (a, b)$,

$$\frac{g(c) - g(a)}{g(b) - g(a)} = \frac{c - a}{b - a}(1 + \eta) \quad \text{where } \eta = O(b - a)$$

and is nonzero by monotonicity of g . Composing this result t times, we obtain

$$\frac{|f_\sigma^{(t)}(v) - f_\sigma^{(t)}(u)|_1}{|f_\sigma^{(t)}(w) - f_\sigma^{(t)}(u)|_1} = \frac{|v - u|_1}{|w - u|_1} \prod_{n=1}^t (1 + \eta_n),$$

where $|\eta_n| < \kappa_1 |w - u|_1 \gamma^n$ for some $\kappa_1 > 0$ independent of u, v, w and σ . This establishes the second set of inequalities. Intuitively, the idea here is that lines from points $v \in [u, w]$ to vertex v_j will be almost parallel when $|w - u|_1$ is small, so relative lengths of subintervals will be about the same after mapping by f_j . \square

From Lemmas 4.2 and 4.3, it follows that there is $0 < \gamma < 1$ such that mapping φ is uniformly contracting on $C(\gamma) := C_i(\gamma) \cup C_j(\gamma) \cup C_k(\gamma)$. Fix this γ from now on.

LEMMA 4.4. *All points z on a periodic orbit lie in ${}^V A$.*

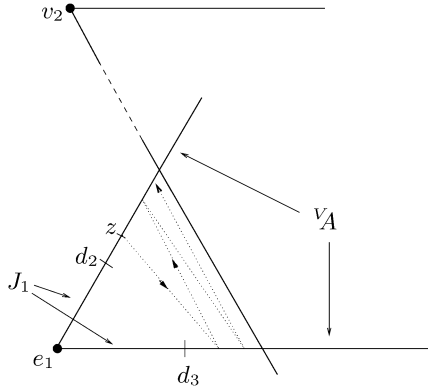


FIG. 4. A case in the proof of Lemma 4.4.

PROOF. We show instead that points outside V_A , that is, $z \in J_{\hat{k}}$ for some \hat{k} , cannot lie on a periodic orbit because they have terminating sequences of pre-images. Suppose $J_{\hat{k}} \neq \emptyset$.

If neither d_i or $d_j \in J_{\hat{k}}$, then $\varphi(J_{\hat{k}}) \subset A_{\hat{k}}^0 \cap V_A$ and no $z \in J_{\hat{k}}$ has a pre-image. If $d_j \in J_{\hat{k}}$ but $d_i \notin J_{\hat{k}}$, then, recalling the notation $\pi(z) = x$ for $z = (x, i)$, those $z \in A_i^0 \cap J_{\hat{k}}$ with $\pi(z) < \pi(f_i(d_j))$ have a single pre-image, while the other $z \in J_{\hat{k}}$ have none. The case with $d_i \in J_{\hat{k}}$ but $d_j \notin J_{\hat{k}}$ is similar.

If both $d_i, d_j \in J_{\hat{k}}$, as shown in Figure 4, then $f_j(d_i), f_i(d_j) \in J_{\hat{k}}$, but no $z \in \{z \in A_i^0 : \pi(z) > \pi(f_i(d_j))\} \cup \{z \in A_j^0 : \pi(z) < \pi(f_j(d_i))\}$ has a pre-image under φ . Let $\sigma = \{\hat{i}, \hat{j}, \hat{i}, \hat{j}, \dots\}$ and observe that $f_{\sigma}^{(-2t)}(z) \rightarrow e_{\hat{k}}$ as $t \rightarrow \infty$ for any $z \in A_j^0 \cap J_{\hat{k}}$. Hence, for $z \in A_j^0$ with $\pi(z) > \pi(f_j(d_i))$, we see that $\pi(f_{\sigma}^{(-2t)}(z)) < \pi(f_j(d_i))$ for some finite t and these z have only finitely many pre-images under φ . Any point in $A_j^0 \cap J_{\hat{k}}$ can be written as $f_j(z)$ for some $z \in A_i^0 \cap J_{\hat{k}}$, so they too have finitely many pre-images in this case. We note that a forward trajectory from any $z \in J_{\hat{k}}$ either enters V_A or converges onto the period two orbit on the points $(\rho_{\hat{k}}/\theta, \hat{i})$ and $(1 - \rho_{\hat{k}}/\theta, \hat{j})$. \square

LEMMA 4.5. For any starting point $z(0)$, there is $t_0 > 0$ such that $z(t) \in C(\gamma)$ for all $t \geq t_0$.

PROOF. We see that V_A is closed under φ , so once a trajectory enters it never leaves. Any trajectory that never enters V_A must remain in one of the J_i and so, by Lemma 4.4, converges to the two cycle on the endpoints of that J_i . In either case Lemma 4.2 implies that it enters the contracting region and remains there. \square

LEMMA 4.6. For any eventually- m -periodic trajectory $z(t), t = 0, 1, 2, \dots$:

- (i) *there exists an orbit u_1, \dots, u_m of period m onto which the trajectory $z(t)$ converges;*
- (ii) *trajectories $z'(t)$ with the same node-cycle converge onto the same orbit;*
- (iii) *there can be at most one orbit having a given node-cycle.*

PROOF. (i) Note that, as $z(t)$ is eventually- m -periodic, there exists $t_0 > 0$ such that $\text{Side}(z(t + m)) = \text{Side}(z(t))$ for $t \geq t_0$. Now Lemma 4.5 implies there will be $t_1 > 0$ such that $z(t) \in C(\gamma)$ for all $t \geq t_1$. Since $\text{Side}(z(tm + i))$ is the same for all $t \geq t_0$, Lemma 4.2 implies

$$|z((t + 1)m + i) - z(tm + i)|_1 \leq \gamma^m |z(tm + i) - z((t - 1)m + i)|_1$$

and, hence, $u_i = \lim_{t \rightarrow \infty} z(tm + i)$ exists. For parts (ii) and (iii), Lemmas 4.4 and 4.2 imply that, for some $r \leq m$, $|z(t) - z'(t + r)|_1 \rightarrow 0$ exponentially quickly as $t \rightarrow \infty$, so the trajectory $z'(t)$ converges onto the orbit u_1, \dots, u_m . The existence of another orbit v_1, \dots, v_m with the same node-cycle is impossible because we can choose $z'(0) = v_1$. \square

Recall the notation $A^{t+1} = \varphi(A^t) = \dots = \varphi^{(t+1)}(A^0)$. The set A^t contains all points that have at least t pre-images under φ . Let $\mathcal{P} = \{\varphi^{(-t)}(d_i) : i = 1, 2, 3; t = 0, 1, 2, \dots\}$ be the set of pre-images of the decision points and $\overline{\mathcal{P}}$ be its closure. Let $|z - \mathcal{P}|_1 = \inf_{z' \in \mathcal{P}} |z - z'|_1$ denote the distance from z to \mathcal{P} —of course, $|z - \mathcal{P}|_1 = |z - \overline{\mathcal{P}}|_1$.

REMARKS. If $\{d_1, d_2, d_3\} \cap A^t = \emptyset$ for some finite t , then \mathcal{P} is finite. If $\{d_1, d_2, d_3\} \cap A^t \neq \emptyset$ for any finite t , then \mathcal{P} can still be finite when there is a finite orbit containing one or more of the decision points. The only other possibility is that at least one decision point has infinitely many distinct pre-images and so \mathcal{P} is infinite. An orbit containing a decision point must be unstable if it is of odd length m and may be stable on one side if m is even.

LEMMA 4.7. (i) *If \mathcal{P} is finite, then the triangle process has a finite number of periodic orbits and each trajectory $z(t)$, from any starting point $z(0)$, converges onto one of these orbits as $t \rightarrow \infty$.*

(ii) *Whether \mathcal{P} is finite or not, if $z \notin \overline{\mathcal{P}}$ and the trajectory $z(t)$ with $z(0) = z$ converges onto the orbit u_1, \dots, u_m , then, for some $\varepsilon > 0$, there is a unique trajectory starting from each $z' \in \mathcal{N}_\varepsilon(z)$ and it converges onto this orbit.*

(iii) *Any orbit u_1, \dots, u_m with $u_n \notin \overline{\mathcal{P}}$ for each $n = 1, \dots, m$ is stable.*

PROOF. (i) As \mathcal{P} is finite, we can partition A^0 into $|\mathcal{P}|$ open intervals O_n (indexed anticlockwise from the interval containing e_1 , say) with points of \mathcal{P} as endpoints. These O_n are never split by mapping with φ as they and their images never contain decision points, so each O_n is mapped into another by φ as it is

continuous and monotone except at the d_i . Thus, φ induces a mapping h of the indices $\{1, 2, \dots, |\mathcal{P}|\}$ into itself and, hence, h has a core $K \subseteq \{1, 2, \dots, |\mathcal{P}|\}$ such that $h(K) = K$. As h has no fixed points, it permutes K and so factors into a product of disjoint nontrivial cycles and, hence, all trajectories starting from $z \notin \mathcal{P}$ are periodic with one of a finite number of node-cycles. Each node-cycle supports only a single orbit by Lemma 4.6 and these trajectories converge onto one of these orbits.

Now consider trajectories with $z(0) = z \in \mathcal{P}$. If, for any $t \geq 1$, we have $z(t) \in O_n$ for some n , then the trajectory is periodic and converges onto a finite orbit by the above argument. This only leaves trajectories with $z(t) \in \mathcal{P}$ for all $t \geq 0$. As was remarked when orbits were defined, all orbits are disjoint, so a trajectory can only remain in \mathcal{P} by following a single orbit.

(ii) As $z \notin \overline{\mathcal{P}}$, the trajectory started from z never hits a decision point and so never branches. As $\overline{\mathcal{P}}$ is closed, there exists $\varepsilon > 0$ such that $|z - \overline{\mathcal{P}}|_1 > \varepsilon$, so no trajectory started from any $z' \in \mathcal{N}_\varepsilon(z)$ ever branches. As $\mathcal{N}_\varepsilon(z)$ contains no pre-images of decision points, the same switching decisions are made for each of these trajectories at all times t , so they have the same node-cycle and by Lemma 4.6, they converge to the orbit u_1, \dots, u_m . Part (iii) follows immediately by considering $z(0) = u_1$. \square

We are now ready to establish Theorems 3.1 and 3.2. Let μ denote Lebesgue measure on A^0 in the following proofs. This measure is consistent with the distance $|\cdot|_1$ we are using and any null sets in A^0 will remain so under change of the parameters, as the re-weighting in Lemma 4.1 is smooth.

PROOF OF THEOREM 3.1. It suffices to show that almost all choices of decision points d_i lie outside the A^t they generate after some finite t . It follows from Lemmas 4.4 and 4.2 that $\mu(A^t) \rightarrow 0$ as $t \rightarrow \infty$ for any decision points d_i , but this is not sufficient since $\liminf_t A^t$ is a function of the d_i . What we show is that, for any given parameters ρ_i , there is a measure zero set of locations for the decision points where they could have infinitely many pre-images.

We start with the case $e_i \notin {}^V A$ or, equivalently, $J_i \neq \emptyset$, $i = 1, 2, 3$. Let $J_i^1 = f_i(J_i)$ and $J_i^{t+1} = f_i(J_j^t \cup J_k^t)$ for $i = 1, 2, 3$ and $t = 1, 2, \dots$ (here $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$). These sets are well defined as $J_i^1 \subset {}^V A_i$ and, hence, $J_i^t \subset {}^V A_i$ for all t . Further, $J_i \cap J_i^1 = \emptyset$ and, hence, $J_i^{t+1} \cap \bigcup_{n=1}^t J_i^n = \emptyset$ for all t . Let $I_i^t = {}^V A_i \setminus \bigcup_{n=1}^t J_i^n$ and note that $I_i^{t+1} = f_i(I_j^t \cup I_k^t)$.

We initially consider only ${}^V A_i$. At stage t' , the set $\bigcup_1^{t'} J_i^n$ consists of $2^{t'+1} - 1$ disjoint intervals interleaved with the $2^{t'+1}$ intervals forming $I_i^{t'}$. The set $J_i^{t'+1}$ contains $2^{t'+1}$ intervals, each interior to one of those of $I_i^{t'}$. Recall that μ is the measure induced by the distance $|\cdot|_1$. Let $K > 0$ be such that $\mu(I_j^{t'}) < K\mu(J_j^{t'+1})$, $j = 1, 2, 3$. As $\sum_{n=1}^{t'} \mu(J_i^n) \leq \mu({}^V A_i) \leq 1$, it follows that $\mu(J_i^{t'}) \rightarrow 0$ as $t \rightarrow \infty$.

From Lemma 4.3(i), each subinterval of $I_j^{t'}$, $j = 1, 2, 3$, has length bounded by $\gamma^{t'}$ for some $\gamma < 1$. Now from Lemma 4.3(ii), it follows, by summing over all the subintervals in $I_i^{t'}$ and mapping sequences, that

$$\mu(I_i^{t'+t}) < e^{\kappa\gamma^{t'}} K \mu(J_i^{t'+t+1}), \quad t = 2, 3, \dots$$

and, hence, $\mu(I_i^t) \rightarrow 0$ as $t \rightarrow \infty$. The set I_i^t contains all points in A_i^0 that could have t or more pre-images, given appropriately chosen decision points. Hence, $I_i^\infty = \bigcap_1^\infty I_i^t$ contains those points that could have infinitely many pre-images and $\mu(I_i^\infty) = 0$.

This construction is independent of the choice of the decision points and works on all three sides of ${}^V A$. With $I^\infty = \bigcup_1^3 I_i^\infty$, we have shown $\mu(I^\infty) = 0$, that is, μ -almost all choices of the decision points have only finitely many pre-images under the mapping φ that they define. The theorem now follows in this case from Lemma 4.7.

It remains to modify this argument in the case where one or more of the $e_i \in {}^V A$. For any such corner, $J_i = \emptyset$, so let $J_i^0(\varepsilon) = \{|z - e_i|_1 < \varepsilon\}$ when $e_i \in {}^V A$, $J_i^0 = J_i$ otherwise, where $\varepsilon > 0$ is small. The previous double mapping process will produce overlapping sets, so we modify it by setting $J_i^1 = f_i(J_i^0) \setminus (J_j^0 \cup J_k^0)$ and then $J_i^{t+1} = f_i(J_j^t \cup J_k^t) \setminus (J_j^0 \cup J_k^0)$ for $i = 1, 2, 3$ and $t = 1, 2, \dots$. This change ensures the previous disjointness properties and the construction goes through without further change, except that now some of the sub-intervals of I_i^t may be empty. As before, we conclude that $\mu(I^\infty) = 0$, though here I^∞ does depend upon ε .

To relate this construction to the existence of pre-images, let $\mathcal{D}_n = \{(d_1, d_2, d_3) : \text{all } z \in J_i^0(1/n), i = 1, 2, 3, \text{ have no legitimate pre-images under } \varphi\}$. Then $\bigcup_n \mathcal{D}_n = A_1^0 \times A_2^0 \times A_3^0$. For any fixed n , $\mathcal{D}_n^\infty = \mathcal{D}_n \cap I^\infty$ has $\mu(\mathcal{D}_n^\infty) = 0$ and hence $\mu(\bigcup_1^\infty \mathcal{D}_n^\infty) \leq \sum_1^\infty \mu(\mathcal{D}_n^\infty) = 0$. Now the result follows as before. \square

PROOF OF THEOREM 3.2. We consider first the case where $\mathcal{P} = \{\varphi^{(-t)}(d_i) : i = 1, 2, 3; t = 0, 1, 2, \dots\}$, the set of pre-images of the decision points, is finite. A point $z \in \mathcal{P}$ is of type i when it is a pre-image of d_i . \mathcal{P} splits A^0 into closed intervals $M_n, n = 1, 2, \dots, |\mathcal{P}|$ which are never split by iterated mappings with φ . Hence, each M_n contains points from at most one periodic orbit and as φ is contracting on ${}^V A$, each M_n can contain only a single point from an orbit—in cases where the shared endpoint of M_n and M_{n+1} lies on an orbit, that is, this orbit contains one or more decision points, there can be at most one other orbit with a point in $M_n \cup M_{n+1}$.

The number of orbits is limited by each d_i being at the boundary of just two intervals. Consider an orbit with a point $u \in M_t$ which has an endpoint $\varphi^{(-r)}(d_i)$. The point $\varphi^{(r)}(u)$ lies on the same orbit but occupies one of the M_n neighboring d_i , or equals d_i . By the preceding paragraph, there are at most six orbits.

We can reduce the bound as there are at least two intervals M_n with endpoints of different type, ij and ik , say. Suppose the first is M_i with endpoints $\varphi^{(-r)}(d_i)$ and $\varphi^{(-s)}(d_j)$ with $r < s$. The interval $\varphi^{(r)}(M_i)$ has endpoints d_i and $\varphi^{(r-s)}(d_j)$ so it neighbors d_i . $\varphi^{(s)}(M_i)$ sits inside an interval neighboring d_j . If M_i contains a point from an orbit, then this orbit occupies two of the six intervals neighboring the decision points. If M_i contains no point from any orbit, then neither can one of the intervals neighboring d_i . Either way, the maximum number of orbits is now only five. Repeat the argument with the interval with endpoints ik and there are at most four orbits in the stable case, that is, where \mathcal{P} is finite.

We now extend this argument to the case where \mathcal{P} is infinite—let $\mathcal{P}' = \{z \in \mathcal{P} : z \text{ has infinitely many pre-images}\}$. Suppose there are some finite orbits and consider any of them. It cannot include any points in \mathcal{P}' , so choose $\varepsilon > 0$ small enough that $|u_n - d_i|_1 > \varepsilon$ for every u_n on the orbit and each $d_i \in \mathcal{P}'$. It follows that $|u_n - \varphi^{(-r)}(d_i)|_1 > \varepsilon$ for every u_n on the orbit and every point in \mathcal{P}' for suppose this was not true. Select the $\varphi^{(-r)}(d_i)$ with minimal r within ε of u_n . By Lemma 4.3(i), it follows that

$$|\varphi^{(r)}(u_n) - d_i|_1 \leq \gamma^r |u_n - \varphi^{(-r)}(d_i)|_1 < \varepsilon,$$

in contradiction to our choice of ε . Now our argument for the stable case applies with the added possibility that one or more d_i may be limit points (from one or both sides) of \mathcal{P} further restricting the opportunities for orbits. \square

We show later that it is possible for finite orbits to exist simultaneously with nonperiodic trajectories in this nonstable case. In our study of this model we simulated the triangle process over the whole range of its parameters ρ_i and d_i . To avoid problems with rounding, we used an algebraic representation of the process which is described below in Section 5. The representation there of the decision points has to be finite in practice, so the results of the computer analysis were sometimes inconclusive, for example, the implemented algorithm sometimes produced trajectories that were not trapped by an orbit in the number of steps we could accurately calculate. Otherwise, the algorithm identified either 1, 2 or 3 orbits and never more. Therefore, we *conjecture* that Theorem 3.1 can be strengthened, by replacing *at most four orbits* by *three orbits*. We did not manage to prove this analytically, though. That three orbits can exist is easily seen after a little numerical work with Figure 5 as a guide.

5. The triangle process in the nonstable case. We now show that there are locations for the decision points where at least one of them has infinitely many pre-images and nonperiodic trajectories exist. This argument necessarily uses yet another way of describing the triangle process. We will consider only the case where each $J_i = \emptyset$, but our construction can be carried out in other cases too, with some limitations.

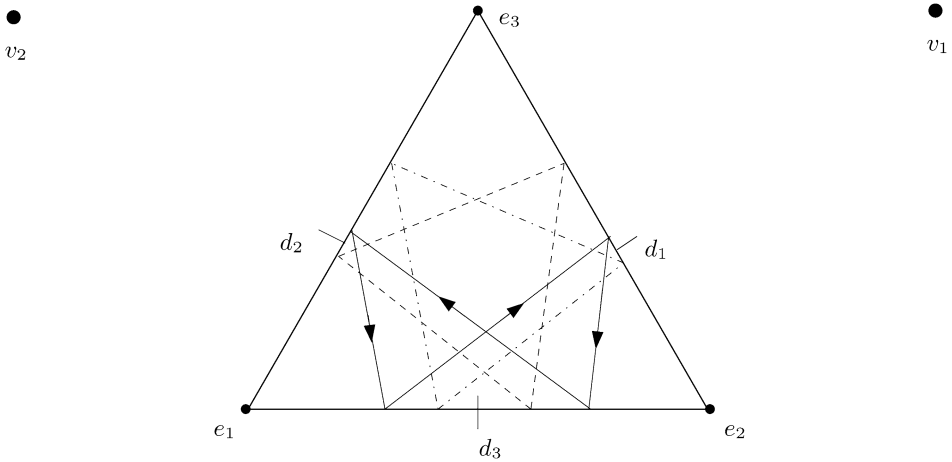


FIG. 5. An example with three orbits.

We first construct a binary representation $z = \hat{i} : x_1 x_2 x_3 \dots$ with each $x_t \in \{0, 1\}$ for all points of A^0 . For each $z \in A^0_{\hat{i}}$, $\hat{i} = 1, 2, 3$, set [recall the notation $\pi(\cdot)$ from Section 3]

$$x_1 = \begin{cases} 0, & \text{if } \pi(z) < \pi(f_{\hat{i}}(e_{\hat{i}})), \\ 1, & \text{if } \pi(z) \geq \pi(f_{\hat{i}}(e_{\hat{i}})). \end{cases}$$

To continue this construction, set $B^1_{\hat{i}} = \{f_{\hat{i}}(e_{\hat{i}})\}$ and $B^{t+1}_{\hat{i}} = f_{\hat{i}}(B^t_j \cup B^t_k)$ for $\hat{i} = 1, 2, 3$ and $t = 1, 2, \dots$

Each set $B^t_{\hat{i}}$ contains 2^{t-1} points which, by monotonicity of the mappings $f_{\hat{i}}$, interleave the $1 + 2^{t-1}$ points of $\bigcup_{r=1}^{t-1} B^r_{\hat{i}}$ and, hence, split in two each of the 2^{t-1} intervals created up to stage $t - 1$. Label the $b_r \in B^t_j \cup B^t_k$ clockwise and the $a_r \in \bigcup_{r=1}^{t-1} B^r_{\hat{i}}$ anticlockwise so that $a_0 = e_j$ and

$$\pi(a_{r-1}) < \pi(f_{\hat{i}}(b_r)) < \pi(a_r), \quad r = 1, 2, \dots, 2^{t-1},$$

and for $z \in [a_{r-1}, a_r)$, set

$$x_t = \begin{cases} 0, & \text{if } \pi(z) < \pi(f_{\hat{i}}(b_r)), \\ 1, & \text{if } \pi(z) \geq \pi(f_{\hat{i}}(b_r)). \end{cases}$$

This encodes every point of A^0 with a unique binary code up to the usual indeterminacy for points terminating with infinite strings of zeros or ones, for example, $e_1 = 2 : 111 \dots = 3 : 000 \dots$.

This binary encoding can be used to define a distance between points $z = i : x_1 x_2 \dots, z' = i' : x'_1 x'_2 \dots$ by

$$(11) \quad |z - z'|_b \equiv \sum_{t=1}^{\infty} |x_t - x'_t| \cdot 2^{-t}$$

and we note the following: (i) ordering of points on A_i^0 by $\pi(z)$ corresponds to lexicographic ordering on the binary sequence x ; (ii) any sequence of points which is convergent under $|\cdot|_1$ remains convergent under $|\cdot|_b$ and vice versa; (iii) this metric induces a measure μ_b on A_i^0 .

In defining the required decision points, we will make much use of the inverse mapping ψ , where

$$\psi(z) = f_i^{(-1)}(z), \quad z \in A_i^0$$

for $i = 1, 2, 3$. When $\varphi^{(-1)}(z)$ exists, then we will say ψ is *legitimate* and we have $\psi(z) = \varphi^{(-1)}(z)$. The mapping ψ can be described quite simply using the binary encoding. For $z = \hat{i} : x_1 x_2 \dots$ with $x_1 = 0$, then $\psi(z) = \hat{k} : y_1 y_2 \dots$, where $y_1 = 1 - x_2$, $y_2 = 1 - x_3$ and so on. In general, let $\bar{w} = 1 - w$ for $w \in \{0, 1\}$, $\bar{x} = \bar{x}_1 \bar{x}_2 \dots$ and with this notation ψ maps

$$(12) \quad \hat{i} : 0x \longrightarrow \hat{k} : \bar{x} \quad \text{and} \quad \hat{i} : 1x \longrightarrow \hat{j} : \bar{x}$$

for $\hat{i} = 1, 2, 3$.

The forward mapping φ thus can be represented as

$$\hat{i} : x \longrightarrow \hat{j} : 0\bar{x} \quad \text{if } x \leq d_i \quad \text{and} \quad \hat{i} : x \longrightarrow \hat{k} : 1\bar{x} \quad \text{if } x \geq d_i.$$

Now represent each point of A^0 by a point in the interval $[0, 1]$ using the mapping

$$i : x_1 x_2 \dots \longrightarrow \frac{i - 1}{3} + \frac{1}{3} \sum_{j=1}^{\infty} 2^{-j} x_j$$

and ignore, for the moment, the branching at the decision points. The forward mapping φ becomes

$$(13) \quad \varphi(x) = \begin{cases} -\frac{1}{2}x + 1, & \text{for } x \in [\tilde{d}_1, \tilde{d}_2) \cup [\tilde{d}_3, 1), \\ -\frac{1}{2}x + \frac{1}{2}, & \text{for } x \in [0, \tilde{d}_1) \cup [\tilde{d}_2, \tilde{d}_3), \end{cases}$$

for $x \in [0, 1)$, where $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3 \in [0, 1)$ are the points corresponding to the decision points. The mapping above can be regarded as a particular case of the affine interval exchange transformation model (see, e.g., [2, 7]). The mapping defined by (13) is contracting and is not order-preserving, which makes it different from the usual and well-studied case of interval exchange transformations (see, e.g., [3, 11] and references therein). Note also that, for interval exchange transformations with contracting and/or flips, it is natural that cycles do exist (see [3]), as happens in our model.

5.1. *A decision point with infinitely many pre-images.* We are now ready to construct decision points providing a nonstable case. We will use $q = 1001$ and $r = 0110$, two four digit sequences such that $\bar{q} = r$ and such that ψ successively maps $z = \hat{i} : qx$ and $z = \hat{i} : rx$ as follows:

$$z = \hat{i} : 1001x \longrightarrow \hat{j} : 110\bar{x} \longrightarrow \hat{k} : 01x \longrightarrow \hat{j} : 0\bar{x} \longrightarrow \hat{i} : x = \psi^{(4)}(z)$$

and, similarly,

$$z = \hat{t} : 0110x \longrightarrow \hat{k} : 001\bar{x} \longrightarrow \hat{j} : 10x \longrightarrow \hat{k} : 1\bar{x} \longrightarrow \hat{t} : x = \psi^{(4)}(z).$$

The decision points have the form

$$(14) \quad \begin{aligned} d_1 &= 1 : qrx \dots, \\ d_2 &= 2 : 1010100000 \dots, \\ d_3 &= 3 : 0100000 \dots, \end{aligned}$$

where x , the infinite tail of the sequence for d_1 , will be constructed recursively at a later point to ensure that d_1 has infinitely many legitimate ψ -images. As the notation d_i identifies the triangle side, we will sometimes use it to denote just the binary sequence. Note that, under lexicographic ordering, $r = 0110 < 1001 = q$.

For d_1 to have infinitely many legitimate ψ -images, it must certainly have one and the intervals with at least one legitimate ψ -image are

$$\begin{aligned} \text{on } A_1^0 \quad & [0\bar{d}_3, 1\bar{d}_2] = [0101111 \dots, 1010101111 \dots], \\ \text{on } A_2^0 \quad & [0\bar{d}_1, 1\bar{d}_3] = [0rq\bar{x} \dots, 110111111 \dots], \\ \text{on } A_3^0 \quad & [0\bar{d}_2, 1\bar{d}_1] = [00101011111 \dots, 1rq\bar{x} \dots], \end{aligned}$$

which the reader can readily check certainly contain d_1 and the $\psi^{(t)}(d_1)$ for $t = 1, 2, \dots, 8$. We will construct the rest of the binary sequence for d_1 so that its ψ -images always fall in these three intervals by finding a sequence of quadruples q and r that guarantees legitimacy.

Extended legitimacy property. The sequence $d_1 = y_1y_2y_3 \dots$ where $y_1 = q$, $y_2 = r$ and $y_t \in \{q, r\}$ for $t = 3, 4, \dots$ has *extended legitimacy* if, under lexicographical ordering,

- (a) when $y_t = r$, then $y_{t+1}y_{t+2} \dots > d_1$,
- (b) when $y_t = q$, then $y_{t+1}y_{t+2} \dots < d_1$,

where $q > r$ as remarked above.

At each $t \geq 3$ this property ensures the legitimacy of $\psi^{(4t-r)}(d_1)$ for $r = 3, 2, 1, 0$. We will now construct an aperiodic sequence with extended legitimacy and show that this means there are decision points leading to the existence of aperiodic orbits which is crucial for establishing Theorem 3.3.

Fix an *irrational* $\alpha \in (1, 2)$ and some $\beta \in [-1, 1]$. Let $y = y(\alpha, \beta) = y_1y_2y_3 \dots$ be a sequence of quadruples q and r with $y_1 = q$, $y_2 = r$ and for $t = 3, 4, \dots$,

$$y_{t+1} = \begin{cases} q, & \text{if } 1 + Q_t < \alpha(1 + R_t) + \beta, \\ r, & \text{otherwise,} \end{cases}$$

where $Q_t = Q_t(y) = \sum_{r=1}^t \mathbf{1}_{\{y_r=q\}}$ and $R_t = t - Q_t$, that is, Q_t is the number of times q appears in the sequence $y_1y_2 \dots y_t$. This sequence is defined using successive rational approximations to the irrational number α as shown in Figure 6, which shows a *staircase* approximating the line $y = \alpha x + \beta$.

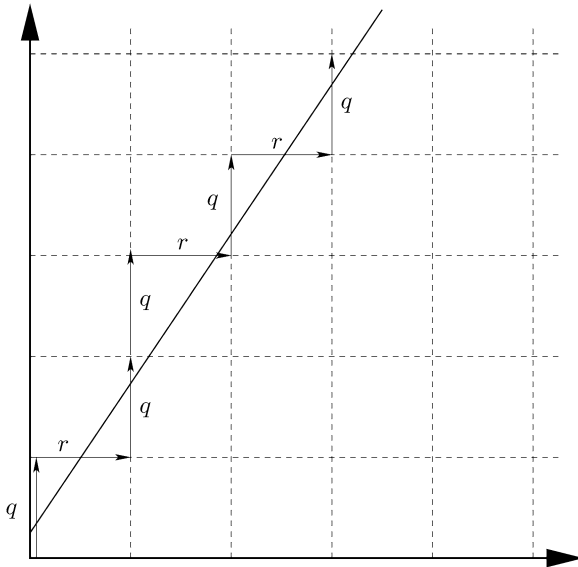


FIG. 6. Staircase diagram for sequence $qrqqrqqrq \dots$

LEMMA 5.1. *There are uncountably many sets of decision points where at least one of them has infinitely many pre-images.*

PROOF. We first show that, for any irrational $\alpha > 0$, the sequence $y' = y(\alpha, \beta')$ is lexicographically less than $y'' = y(\alpha, \beta'')$ whenever $-1 \leq \beta' < \beta'' \leq 1$.

Suppose y' and y'' are not equal. Then there is $t \geq 1$ such that $y'_n = y''_n$ for $n \leq t$, but $y'_{t+1} \neq y''_{t+1}$. Evidently, $Q_t(y') = Q_t(y'')$ and $R_t(y') = R_t(y'')$, so the only way for $y'_{t+1} \neq y''_{t+1}$ is

$$\begin{aligned} \alpha(1 + R_t(y')) + \beta' &< 1 + Q_t(y') \\ &= 1 + Q_t(y'') \\ &< \alpha(1 + R_t(y'')) + \beta'', \end{aligned}$$

in which case $y'_{t+1} = r < q = y''_{t+1}$. As the fractional parts of $m\alpha$, $m = 1, 2, \dots$, are dense in $(0, 1)$, the inequalities $\beta' < n - m\alpha < \beta''$ are satisfied for infinitely many pairs of integers m, n and so the above inequalities also show that $y' = y''$ is not possible.

Now consider the sequence $y = y(\alpha, 0)$ which starts $qr \dots$. Each r is followed by a q and there cannot be more than two qs in succession as $\alpha < 2$. Suppose $y_t = r$. Then the sequence $y_{t+1}y_{t+2}y_{t+3} \dots$ coincides with $y(\alpha, \beta_t)$, where

$$\beta_t = \alpha R_t - Q_t > 0$$

and so $y_{t+1}y_{t+2}y_{t+3}\dots > y$. If instead $y_t = q$, then either $y_{t+1} = r$ so that immediately we have $y_{t+1}y_{t+2}y_{t+3}\dots < y$ or $y_{t+1} = q$, in which case $y_{t+1}y_{t+2}y_{t+3}\dots$ coincides with $y(\alpha, \beta_t)$, where

$$\beta_t = \alpha R_t - Q_t < 0$$

and so $y_{t+1}y_{t+2}y_{t+3}\dots < y$. In all cases we see that the sequence y has extended legitimacy at t for all $t \geq 3$. As α is irrational, $\alpha R_t - Q_t = 0$ is not possible and, hence, there is no t such that $y_{t+1}y_{t+2}y_{t+3}\dots = y$ from which it follows immediately that y is aperiodic.

To establish this lemma, let $d_1 = 1 : y(\alpha, 0)$ and d_2, d_3 be as defined in (14) for some irrational $\alpha \in (1, 2)$. By construction, d_1 has an infinite sequence of legitimate ψ -images or, equivalently, an infinite sequence of pre-images under φ . □

REMARKS. For the above set of the decision points, there is also a period 3 orbit: (a, c, b) , with the points of the orbit given by

$$(15) \quad \begin{aligned} 1 &: 101010\dots 10\dots, \\ 3 &: 101010\dots 10\dots, \\ 2 &: 101010\dots 10\dots. \end{aligned}$$

We mention here that originally we were not sure that the nonstable case was possible and unsuccessfully tried to prove this using the triangle process description of this section. The specific combination of decision points producing the nonperiodic orbits described above was discovered by studying computer output (of a programme written to generate sample trajectories) to assess various lines of investigation.

5.2. *The nonstable case has μ_b -measure 0.* The measure μ_b on A^0 induced by the metric $|\cdot|_b$ is generally singular with respect to Lebesgue measure, but the set of decision points with infinitely many φ pre-images has μ_b -measure 0.

Suppose we select the decision points uniformly, that is, each digit in the binary sequence is 0 or 1 with chance $1/2$ independently of the other digits. Then μ_b a.s. we can write, for some $k \geq 0$,

$$d_2 = 2 : \underbrace{11\dots 11}_k 0\dots$$

This means the interval in A_1^0 with legitimate ψ -images has left endpoint

$$\underbrace{00\dots 00}_{k+1} 1\dots$$

The Borel–Cantelli lemma tells us that μ_b a.s. there is a subsequence in the binary sequence for d_1 of the form

$$y = \underbrace{00 \cdots 00}_m \underbrace{10 \ 00 \cdots 00}_m \underbrace{10 \ 00 \cdots 00}_m,$$

where $m > k + 3$ is even. Now consider the sequence $\psi^{(t)}(d_1), t \geq 1$. For some r , $\psi^{(r)}(d_1)$ starts with either the above sequence y or with \bar{y} and for some $s \leq 2m + 6$, $\psi^{(t+s)}(d_1)$ is a point on A_1^0 with binary sequence starting with at least $k + 2$ zeros and so its next ψ -image is not legitimate. It follows immediately that the set of decision points d_i with infinitely many pre-images has μ_b -measure 0.

5.3. *Properties of the deterministic system when \mathcal{P} is infinite.* For the triple of the decision points constructed in (14), d_1 has infinitely many pre-images and since $\psi^{-5}(d_2) = 1:000 \cdots = e_2$ and $\psi^{-2}(d_3) = 1:111 \cdots = e_3$, both d_2 and d_3 have finitely many pre-images under φ , unlike d_1 . Throughout this section, the triple of the decision points is assumed to be such that d_1 has infinitely many pre-images, while d_2 and d_3 have only finitely many. We also assume in this subsection that ${}^VA = A^0$ so that φ is contracting everywhere.

Let $\mathcal{P}_1 = \{d_1, \psi(d_1), \psi^{(2)}(d_1), \dots\}$ be the infinite set of pre-images of d_1 under φ which, by assumption, are all legitimate. We will study the properties of this set. Let $\overline{\mathcal{P}}_1$ denote the closure of the set \mathcal{P}_1 .

LEMMA 5.2. *Each $u \in \mathcal{P}_1$ is a limiting (accumulation) point of \mathcal{P}_1 .*

PROOF. It is sufficient to show that d_1 is a limiting point for \mathcal{P}_1 . Indeed, since the mapping ψ doubles the distance $|\cdot|_b$ between points, if d_1 is limiting for a sequence $u_1, u_2, \dots \subset \mathcal{P}_1$, then $\psi^{(i)}(d_1)$ is a limiting point for $\psi^{(i)}(u_1), \psi^{(i)}(u_2), \dots \subset \mathcal{P}_1$ [as mentioned just after (11), the distances $|\cdot|_1$ and $|\cdot|_b$ are topologically equivalent so sets of points generate the same limit points under both].

Consider any point $u \in A^0$ which is a limiting point for \mathcal{P}_1 , that is, there is a sequence of indices $t_1 < t_2 < t_3 < \dots$ such that $\lim_n \psi^{(t_n)}(d_1) = u$. No such u can be a pre-image of d_2 or d_3 , as for each t , $A^0 \setminus A^t$, the set of points with fewer than t legitimate pre-images, is open. Hence, around each pre-image of d_2 or d_3 there is an open interval of points which have only finitely many pre-images and so cannot contain any $\psi^{(t)}(d_1)$.

If $u \in \mathcal{P}_1$, then $u = \psi^{(m)}(d_1)$ for some m which is unique (otherwise u is in a finite orbit and \mathcal{P}_1 cannot be infinite). Without loss of generality, we can assume $m < t_1$ and then, since the mapping $\varphi^{(m)}$ is continuous except at $\psi^{(n)}(d_1)$ for $n < m$, $d_1 = \varphi^{(m)}(u) = \lim_n \psi^{(t_n - m)}(d_1)$.

If $u \notin \mathcal{P}_1$, fix a small $\varepsilon > 0$ and consider the ε -neighborhood $\mathcal{N}_\varepsilon(u)$ of u as defined in (8). Assume that ε is so small that there are no pre-images of d_2 or d_3 in $\mathcal{N}_\varepsilon(u)$. Let m be the smallest index such that $\psi^{(m)}(d_1) \in \mathcal{N}_\varepsilon(u)$. Then $\mathcal{N}_\varepsilon(u)$

contains infinitely many points of \mathcal{P}_1 and each is a pre-image of d_1 with the index of at least m . Now map $\mathcal{N}_\varepsilon(u)$ with $\varphi^{(m)}(\cdot)$ which sends $\psi^{(m)}(d_1)$ to d_1 and all the points of the neighborhood $\mathcal{N}_\varepsilon(u)$ follow the same trajectory. As $\varphi^{(m)}(\cdot)$ is contracting here, $\varphi^{(m)}(\mathcal{N}_\varepsilon(u))$ is an interval within $\mathcal{N}_\varepsilon(d_1)$. This implies that there are infinitely many points of \mathcal{P}_1 in $\mathcal{N}_\varepsilon(d_1)$. Letting $\varepsilon \downarrow 0$, we obtain that d_1 is a limiting point for \mathcal{P}_1 . \square

The following statement immediately follows from Lemma 5.2, properties of perfect sets and the fact that \mathcal{P}_1 is infinite and countable.

COROLLARY 5.1. *The set $\overline{\mathcal{P}_1}$ is a perfect set (\equiv it is closed and every point of it is an accumulation point). Therefore, $\overline{\mathcal{P}_1}$ is uncountable, and hence, $\overline{\mathcal{P}_1} \setminus \mathcal{P}_1$ is also uncountable.*

Recall that \mathcal{P} is the union of \mathcal{P}_1 and the set of d_2 and d_3 and their finitely many pre-images. Then $\overline{\mathcal{P}}$ is the union of $\overline{\mathcal{P}_1}$ and the set of d_2 and d_3 and their pre-images, and is also closed.

LEMMA 5.3. *If $u \in \overline{\mathcal{P}_1}$, then u does not belong to any finite orbit and no trajectory $z(t)$ with $z(0) = u$ can be periodic.*

PROOF. If $u \in \mathcal{P}_1$, then $\varphi^{(n)}(u) = d_1$ for some n . However, if u belonged to a finite orbit, then $\varphi^{(m)}(u) = u$ for some m and, hence, $\varphi^{(m)}(d_1) = d_1$, contradicting the fact that d_1 has more than m distinct pre-images under $\psi_r = \varphi^{(-1)}$.

Now suppose $u \in \overline{\mathcal{P}_1} \setminus \mathcal{P}_1$ and u belongs to a finite orbit. As $d_2, d_3 \notin A^n$ for some finite n and A^n is closed, we can choose $\varepsilon > 0$ so that $|d_i - A^n|_1 > \varepsilon$ for $i = 2, 3$. Since $\overline{\mathcal{P}_1} \subset A^n$, it follows that the neighborhood $\mathcal{N}_\varepsilon(u)$ contains no pre-images of d_2 or d_3 . Let $n = n(\varepsilon)$ be the smallest index such that $\psi^{(n)}(d_1) \in \mathcal{N}_\varepsilon(u)$. As $\varphi^{(n)}$ does not split this neighborhood, $\varphi^{(n)}(\mathcal{N}_\varepsilon(u)) \subset \mathcal{N}_{2\varepsilon}(d_1)$ and so there is a point of the orbit, u' , say, with $|u' - d_1|_1 < 2\varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain a contradiction, since this orbit contains only m distinct points and does not contain d_1 .

Finally, consider any trajectory $z(t)$ with $z(0) = u$ and suppose it is periodic. By Lemma 4.6(i), $z(t)$ converges onto an orbit w_1, \dots, w_m and we have just shown there exists $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(w_1) \cap \overline{\mathcal{P}_1} = \emptyset$. However, $z(t) = \varphi^{(t)}(u) \in \mathcal{N}_\varepsilon(w_1)$ for some large t , while simultaneously, by Lemma 5.2, $\varphi^{(t)}(u)$ is a limiting point and, hence, an element of $\overline{\mathcal{P}_1}$ which is impossible. \square

PROOF OF THEOREM 3.3. Lemma 5.1 explains how to construct d_1 with an infinite sequence of pre-images under φ . Also, by Lemma 5.1, the binary sequence for d_1 is not periodic and, hence, by Lemma 5.3, there are nonperiodic trajectories $\{z(t) : t \geq 0\}$ for the triangle process. \square

This completes the main task of this section, but it is possible to say more about the mixture of periodic and nonperiodic trajectories. We know from Lemma 4.6(ii)

that all periodic trajectories avoiding decision points are stable. It is easy to see that *there can be* stable periodic trajectories even if \mathcal{P} is infinite. For example, for the decision points given by (14), the sequence in (15) is an orbit with period 3.

We say that a point $u \in A_i^0$ is an R -limit (L -limit resp.) for \mathcal{P}_1 if there is a sequence of points u_1, u_2, \dots in $A_i^0 \cap \mathcal{P}_1$ such that $\pi(u_n) \downarrow u$ [$\pi(u_n) \uparrow u$ resp.], that is, for an R -limit, u_{n+1} is clockwise of u_n for each n . The next result strengthens Lemma 5.2.

LEMMA 5.4. *Each point $u \in \mathcal{P}_1$ is limiting for \mathcal{P}_1 on both sides, that is, it is both an R -limit and an L -limit for \mathcal{P}_1 .*

PROOF. It suffices to show that d_1 is both an R -limit and an L -limit, since all other points of \mathcal{P}_1 are ψ -images of d_1 .

By Lemma 5.2, d_1 is a limiting point for \mathcal{P}_1 , but suppose that it is just an R -limit and not an L -limit. Then there is $\delta > 0$ such that:

- (i) there are no \mathcal{P}_1 points on the interval $(d_1 - \delta, d_1)$, and
- (ii) there are no pre-images of d_2 or d_3 in the segment $[d_1, d_1 + \delta]$.

Since mapping ψ reverses the orientation of intervals, if $u = \psi^{(n)}(d_1) \in A_1^0$ and n is even, then u is an R -limit and not an L -limit, while if n is odd, then u is an L -limit and not an R -limit.

Fix $0 < \delta_1 < \delta$. Denote $d_1^n := \psi^{(n)}(d_1)$, $n = 1, 2, \dots$, and define

$$(16) \quad r := \min\{n : \psi^{(n)}(d_1) \in (d_1, d_1 + \delta_1]\},$$

so d_1^r is the point of $\mathcal{P}_1 \cap [d_1, d_1 + \delta_1]$ with the smallest index. If r is even, then we obtain a contradiction since $\varphi^{(r)}((d_1, d_1^r])$ is not split during these mappings and $\varphi^{(r)}(d_1^r) = d_1$. Since r is even, $\varphi^{(r)}$ preserves orientation and is contracting so that $\varphi^{(r)}((d_1, d_1^r]) \subset (d_1 - \delta, d_1]$, which contradicts the assumption that $(d_1 - \delta, d_1) \cap \mathcal{P}_1 = \emptyset$.

Therefore, in any small interval $(d_1, d_1 + \delta_1)$, d_1^r has r odd and there will be infinitely many d_1^n with odd n within $(d_1, d_1 + \delta_1)$. This implies there will also be d_1^m with even m in $(d_1, d_1 + \delta_1)$ since $\varphi^{(r)}$ reverses orientation and is contracting, so $\varphi^{(r)}((d_1, d_1^r]) \subset [d_1, d_1^r)$ and all the $d_1^n \in (d_1, d_1^r)$ with odd $n > r$ have been mapped to d_1^{n-r} with $n - r$ even.

Now choose d_1^m in $(d_1, d_1 + \delta)$ with the smallest possible even index $m \geq 2$ and choose odd index $t < m$ such that $d_1^t \in (d_1, d_1^m)$ with d_1^t the closest of such points to d_1^m . Such a $t \geq r$ exists by the arguments following (16) above. The interval $(d_1^t, d_1^m]$ contains no d_1^n with index less than m so, as $\varphi^{(t)}$ reverses orientation and is contracting, $\varphi^{(t)}(d_1^m) = d_1^{m-t} \in (d_1 - \delta, d_1)$, contradicting the assumption that $(d_1 - \delta, d_1) \cap \mathcal{P}_1 = \emptyset$. \square

LEMMA 5.5. Fix a point u and suppose $u \notin \overline{\mathcal{P}}$. Then u belongs to the segment $[\tilde{l}(u), \tilde{r}(u)] = [\tilde{l}, \tilde{r}]$ such that each of \tilde{l} and \tilde{r} is either a pre-image of d_2 or d_3 , or belongs to $\overline{\mathcal{P}}_1$ and $\tilde{l}(u)$ is the largest of such points and $\tilde{r}(u)$ is the smallest of such points:

$$\begin{aligned} \tilde{l}(u) &= \max\{\pi(s) : \text{Side}(s) = \text{Side}(u), \pi(s) < \pi(u), s \in \overline{\mathcal{P}}\}, \\ \tilde{r}(u) &= \min\{\pi(s) : \text{Side}(s) = \text{Side}(u), \pi(s) > \pi(u), s \in \overline{\mathcal{P}}\}. \end{aligned}$$

Moreover, the trajectory started from u can be periodic only if both $\tilde{l}(u)$ and $\tilde{r}(u)$ are pre-images of d_2 or d_3 .

PROOF. The existence of $\tilde{l}(u)$ and $\tilde{r}(u)$ follows from the fact that $\overline{\mathcal{P}}$ is closed. Also, $\pi(\tilde{l}(u)) < \pi(\tilde{r}(u))$ since the complement of the set $\overline{\mathcal{P}}$ is open.

Suppose, for example, that $\tilde{l}(u)$ is not a pre-image of d_2 or d_3 , that is, $\tilde{l}(u) \in \overline{\mathcal{P}}_1$. Since there are no pre-images of the decision points on $[\tilde{l}(u), u]$, u and $\tilde{l}(u)$ can follow the same trajectory under mapping φ . On the other hand, by Lemma 5.3, no trajectory started from $\tilde{l}(u)$ is periodic, hence, the trajectory starting from u is not periodic. \square

Lemma 5.5 implies that the set of points from which periodic trajectories can start—call this the *periodic set*—consists of finitely many intervals, since d_2 and d_3 have only finitely many pre-images. Also, the periodic set has measure smaller than the total length of the sides of the triangle, since \mathcal{P}_1 is infinite. At the same time, Lemma 5.4 yields that, for any $u \notin \overline{\mathcal{P}}$, if $\tilde{l}(u) \in \overline{\mathcal{P}}_1$, then $\tilde{l}(u) \notin \mathcal{P}_1$, and the same is true for $\tilde{r}(u)$.

Call the set of points u for which one of $\tilde{l}(u)$ or $\tilde{r}(u)$ is in $\overline{\mathcal{P}}_1$, while the other is a pre-image of d_2 or d_3 , *semi-periodic*. The endpoint which is in $\overline{\mathcal{P}}_1$ will be referred to as an *aperiodic endpoint*.

Since there are finitely many pre-images of d_2 and d_3 , the length $\Delta > 0$ of the shortest interval in either in the periodic set or the semi-periodic set is properly defined. We say that a point belongs to the *aperiodic set* if it does belong either to the periodic or the semi-periodic set.

6. Behavior of the stochastic process when \mathcal{P} is finite. We can now show how the behavior of the stochastic process Ξ introduced in Section 2 is influenced by the behavior of the deterministic triangle process. Recalling (9), we consider

$$\zeta(n) = \Lambda(\xi(\tau_n)),$$

the projection onto \mathcal{S}_1^+ of Ξ at the random times $\tau_n, n = 0, 1, 2, \dots$, when the server switches from one queue to another (where $\tau_0 = t_0$, a constant to be determined), under the transience conditions in (5), that is, $\lambda_i/\mu_i = \rho_i < 1$ for each i and $\rho = \sum \rho_i > 1$.

We wish to prove Theorem 3.4, namely, that the trajectories of $\zeta(n)$ a.s. converge onto one of the stable orbits of the triangle process, under the assumption that \mathcal{P} , the set of pre-images of the decision points under φ , is finite and $\mathcal{P} \cap A^t = \emptyset$ for some finite t .

Recall the assumption that the service time distribution with the server at queue i has finite variance σ_i^2 . It is convenient, as with the triangle process, to use the transformation in Lemma 4.1, so we can assume that each $\mu_i = 1$. We now spell out some elementary properties of random walks and Poisson processes.

Preliminaries. Consider a random walk $S_n = \sum_{t=1}^n X_t$ on \mathbb{Z} , where $S_0 = 0$, $\mathbf{E}(X_t) = \rho_j - 1 < 0$ and $\text{Var}(X_t) = \sigma^2$ and $X_t \geq -1$. Define the stopping times $T_c = \min\{n : S_n = -c\}$ for $c = 1, 2, \dots$ and let $T_{1,n}$ be independent copies of T_1 so that T_c is equal in distribution to $\sum_{t=1}^c T_{1,t}$. Using the standard martingales $S_n + n(\rho_j - 1)$ and $(S_n + n(\rho_j - 1))^2 - n\sigma^2$, we find that $\mathbf{E}(T_1) = 1/(1 - \rho_j)$ and $\text{Var}(T_1) = \sigma^2/(1 - \rho_j)^3$ and, hence, that

$$\mathbf{E}(T_c) = c/(1 - \rho_j)$$

and

$$\text{Var}(T_c) = c\sigma^2/(1 - \rho_j)^3.$$

Chebyshev’s inequality provides the bound $\mathbf{P}[|T_c - c/(1 - \rho_j)| > c^{2/3}] \leq c\sigma^2/c^{4/3}(1 - \rho_j)^3 = c^{-1/3}\sigma^2/(1 - \rho_j)^3$ on the likely size of deviations from the mean.

Next consider a homogeneous Poisson process with arrivals at rate ρ and let $L = L_0 + N(T_c)$, where L_0 is constant and $N(T_c)$ the number of arrivals by time T_c . Standard calculations give $\mathbf{E}(L) = L_0 + \rho\mathbf{E}(T_c)$ and $\text{Var}(L) = \rho^2 \text{Var}(T_c) + \rho\mathbf{E}(T_c)$. Applying these to the process Ξ over the interval $\tau_{n+1} - \tau_n$ with the server at node j , we have, for $i \neq j$,

$$(17) \quad \mathbf{E}(\xi_i(\tau_{n+1})|\xi(\tau_n)) = \xi_i(\tau_n) + \frac{\rho_i \xi_j(\tau_n)}{(1 - \rho_j)}$$

and

$$(18) \quad \text{Var}(\xi_i(\tau_{n+1})|\xi(\tau_n)) = \xi_j(\tau_n) \frac{\rho_i^2 \sigma_j^2 + \rho_i(1 - \rho_j)^2}{(1 - \rho_j)^3}.$$

Using Chebyshev’s inequality, we have

$$(19) \quad \mathbf{P} \left[\left| \xi_i(\tau_{n+1}) - \xi_i(\tau_n) - \frac{\rho_i \xi_j(\tau_n)}{(1 - \rho_j)} \right| > \xi_j(\tau_n)^{2/3} \middle| \xi(\tau_n) \right] \\ \leq \xi_j(\tau_n)^{-1/3} \frac{\rho_i^2 \sigma_j^2 + \rho_i(1 - \rho_j)^2}{(1 - \rho_j)^3}.$$

While typical deviations on the number of arrivals are of order $\sqrt{\xi_j(\tau_n)}$, this simple bound for larger deviations will be enough for our purposes.

To estimate the effect of the projection, we apply Taylor’s theorem to the function

$$g(u_1, u_2; a, b) = \frac{a + u_1}{b + u_1 + u_2},$$

where $0 \leq a \leq b$ are constants [compare with equation (10)], which provides the formula

$$g(u_1 + h_1, u_2 + h_2; a, b) - g(u_1, u_2; a, b) = \frac{h_1 u_2 - h_2(a + u_1)}{(b + u_1 + u_2 + \alpha(h_1 + h_2))^2},$$

where $\alpha \in (0, 1)$. Applying this at $u_1 + u_2 = u$ with $|h_i| \leq u^{2/3}$ for $i = 1, 2$ —the deviation considered in (19)—we obtain the bound

$$\begin{aligned} & |g(u_1 + h_1, u_2 + h_2; a, b) - g(u_1, u_2; a, b)| \\ & \leq \frac{(a + u_1 + u_2)(|h_1| + |h_2|)}{(b + u_1 + u_2 + \alpha(h_1 + h_2))^2} \\ (20) \quad & \leq \frac{|h_1| + |h_2|}{b + u_1 + u_2} \left(1 + 2 \frac{|h_1| + |h_2|}{b + u_1 + u_2} + O(u^{-2/3}) \right) \\ & \leq 2u^{-1/3} \quad \text{as } u \rightarrow \infty. \end{aligned}$$

Deviations $h_i = O(\sqrt{u})$ give a smaller bound, but what we have will suffice.

PROOF OF THEOREM 3.4. We start with the stochastic process $W_n = \sum_{i=1}^3 \xi_i(\tau_n)$, $n = 0, 1, 2, \dots$, the total number of jobs at the queues at switching epochs subsequent to $\tau_0 = t_0$, some initial time to be chosen below. From (17) and (18), we have

$$\begin{aligned} \mathbf{E}(W_{n+1} | \xi(\tau_n), \mathfrak{R}(\xi(\tau_n)) = j) &= \frac{\rho_i + \rho_k}{1 - \rho_j} \xi_j(\tau_n) + \xi_i(\tau_n) + \xi_k(\tau_n) \\ &= W_n + \frac{\theta}{1 - \rho_j} \xi_j(\tau_n), \end{aligned}$$

where $\theta = \sum_1^3 \rho_k - 1 > 0$. As we are only considering threshold switching rules, there is some value $D > 0$ (dependent on the decision points) such that if the server switches to queue j at time τ_n , then $\xi_j(\tau_n) > DW_n$ for each j and every n . Hence, with $v = \frac{1}{2} \min_j D\theta / (1 - \rho_j) > 0$, this implies that $\{(1 + 2v)^{-n} W_n\}$ is a nonnegative submartingale and so W_n grows exponentially in mean. At this point we also introduce $v' > 0$ with $(1 + v')^3 = 1 + v$.

We are assuming that $\mathcal{P} \cap A^t = \emptyset$ eventually so, for any sufficiently small $\varepsilon_0 > 0$, there exists $n_0(\varepsilon_0)$ such that $|z - z'|_1 > 2\varepsilon_0$ whenever $z \in \mathcal{P}$, $z' \in A^{n_0}$,

that is, the set A^{n_0} is at least $2\varepsilon_0$ away from the decision points and their pre-images. Also, by Lemma 4.2, we can choose n_0 large enough that $A^{n_0} \subset C(\gamma)$, the set where φ is contracting. We want to show that any stochastic trajectory (in projection) enters A^{n_0} and then converges onto a trajectory of the triangle process.

Define a sequence of events $G_n, n = 0, 1, 2, \dots$, depending on parameters w_0, ε_0 by [recall (9)] $G_0(w_0) = \{W_0 \geq w_0\} \cap \{|\zeta(0) - z|_1 < \varepsilon_0 \text{ for some } z \in A^{n_0}\}$ and

$$G_n = \{W_n > w_0(1 + \nu)^n\} \cap \{|\zeta(n) - \varphi(\zeta(n - 1))|_1 \leq \varepsilon_0 \nu'(1 + \nu')^{-n}\}.$$

We first bound $\mathbf{P}[G_n | \bigcap_1^{n-1} G_i]$. Choose w_0 large enough that $w^{1/3} > 1/\nu$ or, equivalently, $w^{2/3} < \nu w$ for all $w \geq w_0$. As $\mathbf{E}(W_n | W_{n-1}) \geq (1 + 2\nu)W_{n-1}$, we have from (19)

$$\begin{aligned} &\mathbf{P}[W_n \leq w_0(1 + \nu)^n | W_{n-1} > w_0(1 + \nu)^{n-1}] \\ &< \mathbf{P}[|W_n - \mathbf{E}(W_n | W_{n-1})| > \nu w_0(1 + \nu)^{n-1} | W_{n-1} > w_0(1 + \nu)^{n-1}] \\ &< \eta \nu^{-1/3} w_0^{-1/3} (1 + \nu')^{-(n-1)}, \end{aligned}$$

where $\eta = \max_{i,j,k} ((\rho_i + \rho_j)^2 \sigma_k^2 + (\rho_i + \rho_j)(1 - \rho_k)^2) / (1 - \rho_k)^3$, a loose but adequate bound which shows that W_n grows geometrically quickly with large probability.

Now we deal with deviation of the stochastic triangle process from the deterministic one. From any $y \in \mathbb{R}_+^3$ with $\Lambda(y) = z \in A^0 \setminus A_j^0$ with the server at node \hat{j} , we have

$$f_j(z) = \sum_{i \neq \hat{j}} \frac{(1 - \rho_j)z_i + \rho_i z_{\hat{j}}}{(1 - \rho_j) + \theta_{\hat{j}} z_{\hat{j}}} e_i \equiv \left(\frac{(1 - \rho_j)z_i + \rho_i z_{\hat{j}}}{(1 - \rho_j)(z_i + z_{\hat{k}}) + (\rho_i + \rho_{\hat{k}})z_{\hat{j}}}, \hat{j} \right)$$

and multiplying the fraction by $\sum y_i / \sum y_i$ to switch to un-normalized values, we have

$$(21) \quad \frac{(1 - \rho_j)z_i + \rho_i z_{\hat{j}}}{(1 - \rho_j)(z_i + z_{\hat{k}}) + (\rho_i + \rho_{\hat{k}})z_{\hat{j}}} = \frac{(1 - \rho_j)y_i + \rho_i y_{\hat{j}}}{(1 - \rho_j)(y_i + y_{\hat{k}}) + (\rho_i + \rho_{\hat{k}})y_{\hat{j}}}.$$

For the stochastic process Ξ starting from $\xi(\tau_{n-1}) = y$ with $\sum_i y_i > w_0(1 + \nu)^{n-1}$ and the server at \hat{j} , the bound (19) implies that, for $i \neq \hat{j}$,

$$\mathbf{P}\left[\left| \xi_i(\tau_n) - \left(y_i + \rho_i \frac{y_{\hat{j}}}{1 - \rho_{\hat{j}}} \right) \right| > y_j^{2/3} \mid \xi(\tau_{n-1}) = y \right] \leq \eta y_j^{-1/3}.$$

Now the identity (21) and the estimate (20), when applied at $\xi(\tau_{n-1}) = y$ with $\sum y_i > w_0(1 + \nu)^{n-1}$ and $\xi_i(\tau_n)$ in place of the $u_i + h_i$, immediately imply

$$\begin{aligned} &\mathbf{P}[|\zeta(n) - \varphi(\zeta(n - 1))|_1 < 2w_0^{-1/3} (1 + \nu')^{-(n-1)} | \xi(\tau_{n-1})] \\ &\geq 1 - 2\eta w_0^{-1/3} (1 + \nu')^{-(n-1)}. \end{aligned}$$

This bound only relies upon $W_{n-1} = \sum_i \xi_i(\tau_{n-1}) > w_0(1 + \nu)^{n-1}$, an event which contains $\bigcap_1^{n-1} G_t$. As the process Ξ is Markov and we can choose w_0 large enough that $2w_0^{-1/3} < \varepsilon_0\nu'/(1 + \nu')$ and, as above, $w_0^{1/3} > 1/\nu$, we have established, for such w_0 , that

$$\mathbf{P} \left[G_n \mid \bigcap_1^{n-1} G_t \right] \geq 1 - \eta\varepsilon_0\nu'(1 + \nu')^{-n}.$$

A very similar argument can be used to show that $\mathbf{P}[G_0] \geq 1 - \eta\varepsilon_0$ when we choose w_0 such that $2n_0w_0^{-1/3} < \varepsilon_0$.

As the stochastic trajectory projects into $C(\gamma)$ (the region of A^0 where φ is contracting), then, as long as the switching decision is the same at $\zeta(t)$ and $\varphi^{(t)}(\zeta(0))$ for every t , we have

$$\begin{aligned} \zeta(n) - \varphi^{(n)}(\zeta(0)) &= \sum_{t=0}^{n-1} \varphi^{(t)}(\zeta(n-t)) - \varphi^{(t+1)}(\zeta(n-t-1)) \\ &\leq \sum_{t=0}^{n-1} \zeta(n-t) - \varphi(\zeta(n-t-1)) \\ &\leq \varepsilon_0\nu' \sum_{t=1}^n (1 + \nu')^{-t} < \varepsilon_0 \quad \text{on } \bigcap_0^n G_t. \end{aligned}$$

This inequality says that the trajectory $\zeta(t)$ is never further than ε_0 from the triangle process trajectory $z(t)$ started at $\zeta(0)$. By choice of G_0 , every point of $z(t)$ is more than ε_0 from any d_i , so the switching decisions for $\zeta(t)$ and $z(t)$ will always be the same. This shows that on $\bigcap_0^\infty G_t$ the projections of the trajectories $\xi(t)$ converge a.s. onto deterministic trajectories of the triangle process which by Theorem 3.1 are periodic.

Finally,

$$\begin{aligned} \mathbf{P} \left[\bigcap_{t=0}^n G_t \right] &= \mathbf{P}[G_0] \prod_{t=1}^n \mathbf{P} \left[G_t \mid \bigcap_0^{t-1} G_s \right] \\ &\geq \prod_{t=0}^{n-1} (1 - \eta\varepsilon_0\nu'(1 + \nu')^{-t}) \\ &\rightarrow p(\varepsilon_0) > 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where $p(\varepsilon_0) \rightarrow 1$ as $\varepsilon_0 \rightarrow 0$. This establishes Theorem 3.4, namely, that the stochastic trajectories $\zeta(t)$ converge onto the stable orbits of z , as we can choose ε_0 as small as we like. \square

REMARK. The argument above is not delicate enough to decide whether trajectories $\zeta(t)$ converge with positive probability onto orbits which are stable on one side, for example, that are of even length and contain a decision point.

7. The stochastic process when \mathcal{P} is infinite. The proofs of the following results employ the results of Section 5.3.

LEMMA 7.1. *Let $\zeta(n)$ denote the state of the stochastic system at time n . Fix $\delta > 0$. Define the event $E_{N,\delta}$ by*

$$(22) \quad E_{N,\delta} = \left\{ \begin{array}{l} |\zeta(N+k) - \varphi^{(k)}(\zeta(N))| < \delta \text{ for all } k \geq 0 \text{ such that} \\ \text{Side}(\zeta(N+i)) = \text{Side}(\varphi^{(i)}(\zeta(N))) \text{ for } i = 0, 1, \dots, k \end{array} \right\}.$$

Then there exists a.s. $N = N(\delta, \omega)$ such that $E_{N,\delta}$ occurs.

PROOF. Analogously to Section 6, one can conclude that there is $0 < \gamma < 1$ such that a.s., for all large n ,

$$(23) \quad \varepsilon_n := |\zeta(n) - \varphi(\zeta(n-1))| < \gamma^n.$$

Let $N_1 = N_1(\omega)$ be the smallest of such n and set $N := \max(N_1, \min\{n : \sum_{i=n}^\infty \gamma^i < \delta\})$. Then as long as $\zeta(N+k)$ and $\varphi^{(k)}(\zeta(N))$ go through the same sequence of sides of the triangle A^0 ,

$$\begin{aligned} & |\zeta(N+k) - \varphi^{(k)}(\zeta(N))| \\ &= |\zeta(N+k) - \varphi(\zeta(N+k-1)) + \varphi(\zeta(N+k-1)) - \varphi(\varphi^{(k-1)}(\zeta(N)))| \\ &\leq \gamma^{n+k} + |\zeta(N+k-1) - \varphi^{(k-1)}(\zeta(N))|, \end{aligned}$$

where we used the fact that φ is contracting. Now the statement of the lemma follows from induction. \square

Recall the definition of the aperiodic set and of Δ at the end of Section 5.3.

LEMMA 7.2. *Define the event E by*

$$E = \{ \text{the stochastic system never comes} \\ \text{closer than } \Delta/2 \text{ to pre-images of } d_2 \text{ or } d_3 \}.$$

Then, given that the stochastic system starts in the aperiodic set, $\mathbf{P}(E) > 0$.

PROOF. First of all, we show that there is a constant $K > 0$ such that if u belongs to the semi-periodic set, then $\varphi^{(k)}(u)$ belongs to the aperiodic set for some $k \leq K$. Indeed, the image of an interval I of the semi-periodic set must be either another semi-periodic interval, or must lie entirely inside an aperiodic interval or

a semi-periodic interval. However, if $\varphi^{(k)}(I)$ is in one of the finitely many semi-periodic intervals for each k , this would imply that the points of I are periodic. This contradicts Lemma 5.5.

Now let $\delta = \Delta/(2K)$ and suppose the system starts in the aperiodic set. By similar arguments to those of Section 6, with positive probability, $\sum_{n=0}^{\infty} |\varepsilon_n| < \delta$, where ε_n is defined in (23). If for some n , $\zeta(n)$ is in the aperiodic set but $\zeta(n + 1)$ is not, then $u := \zeta(n + 1)$ cannot be “far” from the points of \mathcal{P}_1 since $|\varepsilon_n| < \delta$. Because the intervals of the periodic set are separated from the aperiodic set by the intervals of semi-periodic set, u a.s. belongs to a semi-periodic interval, say, $I := [\tilde{l}, \tilde{r}]$, where $\tilde{l} \in \overline{\mathcal{P}}_1$ and \tilde{r} is a pre-image of d_2 or d_3 . We claim that, within the next K applications of φ , u will be “thrown out” of the image of I , or will be in the aperiodic set or some semi-periodic interval not far from its aperiodic endpoint again.

To show this, suppose $\zeta(n + k)$ lies in $\varphi^{(k)}(I)$ for the first K steps. Then, by the choice of K , for some k , it will end up in the aperiodic set, and also it will not approach $\varphi^{(k)}(\tilde{r})$ (which *could* possibly be d_2 or d_3) closer than $K \times \delta \leq \Delta/2$. On the other hand, if $\zeta(n + k - 1) \in \varphi^{(k-1)}(I)$ and yet $\zeta(n + k) \notin \varphi^{(k)}(I)$, it means that $\zeta(n + k)$ is no further from $\varphi^{(k)}(\tilde{r})$ than ε_{n+k} and it lies either in the aperiodic set or in a semi-periodic interval not far from its aperiodic endpoint $\varphi^{(k)}(\tilde{r})$. (Conditioned on $\sum_{n=0}^{\infty} |\varepsilon_n| < \delta$, there is not enough randomness for the stochastic system to exit the interval via the endpoint which is a pre-image of d_2 or d_3 .) \square

LEMMA 7.3. *Conditioned on the event E , periodicity of the stochastic system implies that d_1 is a limiting point for the trajectory of the stochastic system.*

PROOF. Suppose not. Then there is $\delta = \delta(\omega) > 0$ such that the points in the δ -neighborhood of d_1 are never hit by the stochastic system. Without loss of generality, suppose $\delta < \Delta/2$. By Lemma 7.1, there exists an N for this δ such that (22) is fulfilled. Then for any $k \geq 0$, the distance between points $\varphi^{(k)}(\zeta(N))$ and $\zeta(N + k)$ does not exceed δ as long as they follow the same trajectory. On the other hand, they *will* follow the same trajectory, as there will never be the decision point d_1 between them, nor decision points d_2 or d_3 , since we are conditioning on E .

Hence, $\tilde{l}(\varphi^{(k)}(\zeta(N)))$ will be also periodic, following the same path as $\varphi^{(k)}(\zeta(N))$, yielding contradiction with Lemma 5.3. \square

Now we are able to finish the proof of Theorem 3.5.

PROOF OF THEOREM 3.5. First, analogously to the proof of Theorem 3.4, one can easily show that, with positive probability, the stochastic system will converge to one of the finite cycles. So, it remains to show that, with positive probability, the stochastic system is not periodic.

Condition on the event E . Suppose that the stochastic system is periodic with period m , that is, for some N_1 , we have

$$(24) \quad \text{Side}(\zeta(n)) = \text{Side}(\zeta(n+m)) \quad \text{whenever } n \geq N_1.$$

For the moment consider only one image $\varphi(d_1) = f_2(d_1)$ of d_1 . Since $f_2(d_1)$ is not periodic by Lemmas 5.4 and 5.3, there is a positive integer $K_1 = K_1(m) > m$ such that

$$(25) \quad \text{Side}(\varphi^{(K_1)}(d_1)) \neq \text{Side}(\varphi^{(K_1-m)}(d_1)).$$

Similarly, for the other possible image of d_1 , $f_3(d_1)$, there is $K_2 = K_2(m) > m$ such that

$$(26) \quad \text{Side}(\varphi^{(K_2)}(d_1)) \neq \text{Side}(\varphi^{(K_2-m)}(d_1)),$$

under the assumption that $\varphi(d_1) \in A_3^0$.

Choose $\delta > 0$ so small, that each of the one-sided δ -neighborhoods of d_1 which map onto sides A_2^0 and A_3^0 , respectively, does not intersect with d_1 for the first $K = \max\{K_1, K_2\}$ applications of φ , and let $\delta_1 \leq \delta$ be the size of the smaller of these neighborhoods after K mappings by φ .

By Lemma 7.3, the stochastic system will hit the $\delta_1/2$ neighborhood of d_1 at, say, time N . Since, in fact, the stochastic system will hit this neighborhood at arbitrary large times, we can suppose that $N > N_1$ and that $\sum_{n=N}^{\infty} |\varepsilon_n| < \delta_1/2$, where ε_n is defined in (23). Also, for definiteness suppose that $\zeta(n)$ is on the side of d_1 which maps onto A_2^0 . Then the stochastic system will follow the image of d_1 which maps onto A_2^0 for the next K steps, in the sense $\text{Side}(\varphi^{(k)}(\zeta(n))) = \text{Side}(\varphi^{(k)}(d_1))$ since $\delta_1/2 + \delta_1/2 \leq \delta_1$. However, recall that K was chosen in such a way that the sequence $\text{Side}(\varphi^{(k)}(d_1))$, $k = 1, 2, \dots, K$, cannot be m -periodic, creating the contradiction between (24) and (25). \square

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