Clique-Width: Harnessing the Power of Atoms[⋆,⋆⋆]

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Abstract. Many NP-complete graph problems are polynomial-time solvable on graph classes of bounded clique-width. Several of these problems are polynomial-time solvable on a hereditary graph class $\mathcal{G}$ if they are so on the atoms (graphs with no clique cut-set) of $\mathcal{G}$. Hence, we initiate a systematic study into boundedness of clique-width of atoms of hereditary graph classes. A graph G is *H-free* if H is not an induced subgraph of G, and it is *(H_1, H_2)-free* if it is both H_1-free and H_2-free. A class of H-free graphs has bounded clique-width if and only if its atoms have this property. This is no longer true for (H_1, H_2)-free graphs, as evidenced by one known example. We prove the existence of another such pair (H_1, H_2) and classify the boundedness of clique-width on (H_1, H_2)-free atoms for all but 18 cases.

1 Introduction

Many hard graph problems become tractable when restricting the input to some graph class. The two central questions are "for which graph classes does a graph problem become tractable" and "for which graph classes does it stay computationally hard?" Ideally, we wish to answer these questions for a large set of problems simultaneously instead of considering individual problems one by one.

Graph width parameters [26,39,41,45,54] make such results possible. A graph class has *bounded* width if there is a constant c such that the width of all its members is at most c. There are several meta-theorems that provide sufficient conditions for a problem to be tractable on a graph class of bounded width.

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Two popular width parameters are treewidth (tw) and clique-width (cw). For every graph G the inequality $\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)-1}$ holds [19]. Hence, every problem that is polynomial-time solvable on graphs of bounded clique-width is also polynomial-time solvable on graphs of bounded treewidth. However, the converse statement does not hold: there exist graph problems, such as List Colouring, which are polynomial-time solvable on graphs of bounded treewidth [44], but NP-complete on graphs of bounded clique-width [23]. Thus, the trade-off between treewidth and clique-width is that the former can be used to solve more problems, but the latter is *more powerful* in the sense that it can be used to solve problems for larger graph classes.

Courcelle [20] proved that every graph problem definable in MSO_2 is linear-time solvable on graphs of bounded treewidth. Courcelle, Makowsky and Rotics [22] showed that every graph problem definable in the more restricted logic MSO_1 is polynomial-time solvable even for graphs of bounded clique-width (see [21] for details on MSO_1 and MSO_2). Since then, several clique-width meta-theorems for graph problems not definable in MSO_1 have been developed [32,36,46,51].

All of the above meta-theorems require a constant-width decomposition of the graph. We can compute such a decomposition in polynomial time for treewidth [4] and clique-width [50], but not for all parameters. For instance, unless $\mathsf{NP} = \mathsf{ZPP}$, this is not possible for mim-width [52], another well-known graph parameter, which is even more powerful than clique-width [54]. Hence, meta-theorems for mim-width [2,16] require an appropriate constant-width decomposition as part of the input (which may still be found in polynomial time for some graph classes).

Our Focus. In our paper we concentrate on *clique-width*[6] in an attempt to find *larger* graph classes for which certain NP-complete graph problems become tractable without the requirement of an appropriate decomposition as part of the input. The type of graph classes we consider all have the natural property that they are closed under vertex deletion. Such graph classes are said to be *hereditary* and there is a long-standing study on boundedness of clique-width for hereditary graph classes (see, for example, [3,6,7,8,10,11,12,13,24,25,27,28,30,31,39,45,48]).

Besides capturing many well-known classes, the framework of hereditary graph classes also enables us to perform a *systematic* study of a width parameter or graph problem. This is because every hereditary graph class $\mathcal{G}$ is readily seen to be uniquely characterized by a minimal (but not necessarily finite) set $\mathcal{F}_\mathcal{G}$ of forbidden induced subgraphs. If $|\mathcal{F}_\mathcal{G}| = 1$ or $|\mathcal{F}_\mathcal{G}| = 2$, then $\mathcal{G}$ is said to be *monogenic* or *bigenic*, respectively. Monogenic and bigenic graph classes already have a rich structure, and studying their properties has led to deep insights into the complexity of bounding graph parameters and solving graph problems; see e.g. [18,26,37,40] for extensive algorithmic and structural studies and surveys.

It is well known (see e.g. [31]) that a monogenic class of graphs has bounded clique-width if and only if it is a subclass of the class $\mathcal{G}$ with $\mathcal{F}_\mathcal{G} = \{P_4\}$. The survey [26] gives a state-of-the-art theorem on the boundedness and unboundedness

[6] See Section 2 for a definition of clique-width and other terminology used in Section 1.

of clique-width of bigenic graph classes. Unlike treewidth, for which a complete dichotomy is known [5], and mim-width, for which there is an infinite number of open cases [15], this state-of-the-art theorem shows that there are still five open cases (up to an equivalence relation). From the same theorem we observe that many graph classes are of unbounded clique-width. However, if a graph class has unbounded clique-width, then this does not mean that a graph problem must be NP-hard on this class. For example, Colouring is polynomial-time solvable on the (bigenic) class of (C_4, P_6)-free graphs [35], which contains the class of split graphs and thus has unbounded clique-width [48]. In this case it turns out that the *atoms* (graphs with no clique cut-set) in the class of (C_4, P_6)-free graphs *do* have bounded clique-width. This immediately gives us an algorithm for the whole class of (C_4, P_6)-free graphs due to Tarjan's decomposition theorem [53].

In fact, Tarjan's result holds not only for Colouring, but also for many other graph problems. For instance, several other classical graph problems, such as Minimum Fill-In, Maximum Clique, Maximum Weighted Independent Set [53] (see [1] for the unweighted variant) and Maximum Induced Matching [14] are polynomial-time solvable on a hereditary graph class $\mathcal{G}$ if and only if this is the case on the atoms of $\mathcal{G}$. Hence, we aim to investigate, in a systematic way, the following natural research question:

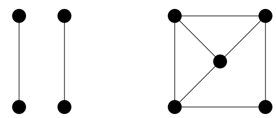
Which hereditary graph classes of unbounded *clique-width have the property that their atoms have* bounded *clique-width?*

Known Results. For monogenic graph classes, the restriction to atoms does not yield any algorithmic advantages, as shown by Gaspers et al. [35].

Theorem 1 ([35]). *Let H be a graph. The class of H-free atoms has bounded clique-width if and only if the class of H-free graphs has bounded clique-width (so, if and only if H is an induced subgraph of P_4).*

The result for (C_4, P_6)-free graphs [35] shows that the situation is different for bigenic classes. We are aware of two more hereditary graph classes $\mathcal{G}$ with this property, but in both cases $|\mathcal{F}_\mathcal{G}| > 2$. Split graphs, or equivalently, $(C_4, C_5, 2P_2)$-free graphs have unbounded clique-width [48], but split atoms are complete graphs and have clique-width at most 2. Cameron et al. [17] proved that (cap, C_4)-free odd-signable atoms have clique-width at most 48, whereas the class of all (cap, C_4)-free odd-signable graphs contains the class of split graphs and thus has unbounded clique-width. See [33,34] for algorithms for Colouring on hereditary graph classes that rely on boundedness of clique-width of atoms of subclasses.

Our Results. Due to Theorem 1, and motivated by algorithmic applications, we focus on the atoms of bigenic graph classes. Recall that the class of (C_4, P_6)-free graphs has unbounded clique-width but its atoms have bounded clique-width [35]. This also holds, for instance, for its subclass of $(C_4, 2P_2)$-free graphs and thus for (C_4, P_5)-free graphs and $(C_4, P_2 + P_3)$-free graphs. We determine a new, incomparable case where we forbid $2P_2$ and $\overline{P_2 + P_3}$ (also known as the *paraglider* [43]); see Fig. 1 for illustrations of these forbidden induced subgraphs.



$2P_2$ $\overline{P_2 + P_3}$

Fig. 1. The two forbidden induced subgraphs from Theorem 2.

Theorem 2. *The class of* $(2P_2, \overline{P_2 + P_3})$*-free atoms has bounded clique-width (whereas the class of* $(2P_2, \overline{P_2 + P_3})$*-free graphs has unbounded clique-width).*

We sketch the proof of Theorem 2 in Section 3 after first giving an outline. Our approach shares some similarities with the approach Malyshev and Lobanova [49] used to show that (Weighted) Colouring is polynomial-time solvable on $(P_5, \overline{P_2 + P_3})$-free graphs. We explain the differences between both approaches and the new ingredients of our proof in detail in Section 3. Here, we only discuss a complication that makes proving boundedness of clique-width of atoms more difficult in general. Namely, when working with atoms, we need to be careful with performing complementation operations. In particular, a class of (H_1, H_2)-free graphs has bounded clique-width if only if the class of $(\overline{H_1}, \overline{H_2})$-free graphs has bounded clique-width. However, this equivalence relation no longer holds for classes of (H_1, H_2)-free atoms. For example, (C_4, P_5)-free (and even (C_4, P_6)-free) atoms have bounded clique-width [35], but we prove that $(\overline{C_4}, \overline{P_5})$-free atoms have unbounded clique-width.

We also identify a number of new bigenic graph classes whose atoms already have unbounded clique-width. We prove this by modifying existing graph constructions for proving unbounded clique-width of the whole class (proofs omitted due to space restrictions). Combining these constructions with Theorem 2 and the state-of-art theorem on clique-width from [26] yields the following summary.

Theorem 3. *For graphs* H_1 *and* H_2*, let* $\mathcal{G}$ *be the class of* (H_1, H_2)*-free graphs.*

1. *The class of atoms in* $\mathcal{G}$ *has bounded clique-width if*
   (i) H_1 *or* $H_2 \subseteq_i P_4$
   (ii) $H_1 = \text{paw}$ *or* K_s *and* $H_2 = P_1 + P_3$ *or* tP_1 *for some* $s, t \geq 1$
   (iii) $H_1 \subseteq_i \text{paw}$ *and* $H_2 \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + P_2 + P_3,\ P_1 + P_5,\ P_1 + S_{1,1,2},\ P_2 + P_4,\ P_6,\ S_{1,1,3}$ *or* $S_{1,2,2}$
   (iv) $H_1 \subseteq_i P_1 + P_3$ *and* $H_2 \subseteq_i \overline{K_{1,3} + 3P_1},\ \overline{K_{1,3} + P_2},\ \overline{P_1 + P_2 + P_3},\ \overline{P_1 + P_5},\ \overline{P_1 + S_{1,1,2}},\ \overline{P_2 + P_4},\ \overline{P_6},\ \overline{S_{1,1,3}}$ *or* $\overline{S_{1,2,2}}$
   (v) $H_1 \subseteq_i \text{diamond}$ *and* $H_2 \subseteq_i P_1 + 2P_2,\ 3P_1 + P_2$ *or* $P_2 + P_3$
   (vi) $H_1 \subseteq_i 2P_1 + P_2$ *and* $H_2 \subseteq_i \overline{P_1 + 2P_2},\ \overline{3P_1 + P_2}$ *or* $\overline{P_2 + P_3}$
   (vii) $H_1 \subseteq_i \text{gem}$ *and* $H_2 \subseteq_i P_1 + P_4$ *or* P_5
   (viii) $H_1 \subseteq_i P_1 + P_4$ *and* $H_2 \subseteq_i \overline{P_5}$
   (ix) $H_1 \subseteq_i K_3 + P_1$ *and* $H_2 \subseteq_i K_{1,3}$*,*
   (x) $H_1 \subseteq_i \overline{2P_1 + P_3}$ *and* $H_2 \subseteq_i 2P_1 + P_3$

(xi) $H_1 \subseteq_i P_6$ *and* $H_2 \subseteq_i C_4$, *or*
(xii) $H_1 \subseteq_i 2P_2$ *and* $H_2 \subseteq_i \overline{P_2 + P_3}$.

2. *The class of atoms in* $\mathcal{G}$ *has unbounded clique-width if*
   (i) $H_1 \notin \mathcal{S}$ *and* $H_2 \notin \mathcal{S}$
   (ii) $H_1 \notin \overline{\mathcal{S}}$ *and* $H_2 \notin \overline{\mathcal{S}}$
   (iii) $H_1 \supseteq_i K_3 + P_1$ *and* $H_2 \supseteq_i 4P_1$ *or* $2P_2$
   (iv) $H_1 \supseteq_i K_{1,3}$ *and* $H_2 \supseteq_i K_4$ *or* C_4
   (v) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ *or* $P_1 + P_6$
   (vi) $H_1 \supseteq_i 2P_1 + P_2$ *and* $H_2 \supseteq_i K_3 + P_1,\ K_5,\ \overline{P_2 + P_4}$ *or* $\overline{P_6}$
   (vii) $H_1 \supseteq_i K_3$ *and* $H_2 \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ *or* $2P_3$
   (viii) $H_1 \supseteq_i 3P_1$ *and* $H_2 \supseteq_i \overline{2P_1 + 2P_2},\ \overline{2P_1 + P_4},\ \overline{4P_1 + P_2},\ \overline{3P_2}$ *or* $\overline{2P_3}$
   (ix) $H_1 \supseteq_i K_4$ *and* $H_2 \supseteq_i P_1 + P_4,\ 3P_1 + P_2$ *or* $2P_2$
   (x) $H_1 \supseteq_i 4P_1$ *and* $H_2 \supseteq_i$ gem, $\overline{3P_1 + P_2}$ *or* C_4
   (xi) $H_1 \supseteq_i$ gem, $\overline{P_1 + 2P_2}$ *or* $\overline{P_2 + P_3}$ *and* $H_2 \supseteq_i P_1 + 2P_2$ *or* P_6
   (xii) $H_1 \supseteq_i P_1 + P_4$ *and* $H_2 \supseteq_i \overline{P_1 + 2P_2}$, *or*
   (xiii) $H_1 \supseteq_i 2P_2$ *and* $H_2 \supseteq_i \overline{P_2 + P_4},\ \overline{3P_2}$ *or* $\overline{P_5}$.

Due to Theorem 3, we are left with 18 open cases, listed in Section 4, where we discuss directions for future work.

2 Preliminaries

Let G be a graph. For a subset $S \subseteq V(G)$, the subgraph of G *induced by* S is the graph $G[S]$, which has vertex set S and edge set $\{uv \mid uv \in E(G), u, v \in S\}$. If $S = \{s_1, \ldots, s_r\}$, we may write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. We write $F \subseteq_i G$ to denote that F is an induced subgraph of G. We say that G is H*-free* if G does not contain H as an induced subgraph, and that G is $(H_1, \ldots, H_p)$*-free* if it is H_i-free for all $i \in \{1, \ldots, p\}$. A *(connected) component* of G is a maximal connected subgraph of G. A clique $K \subseteq V(G)$ is a *clique cut-set* of G if $G \setminus K = G[V(G) \setminus K]$ is disconnected. A graph with no clique cut-sets is an *atom*; note that such graphs are connected. The *complement* $\overline{G}$ of G has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{uv \mid u, v \in V(G), u \neq v, uv \notin E(G)\}$. The *neighbourhood* of a vertex $u \in V(G)$ is the set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. Let X and Y be two disjoint vertex subsets of G. A vertex $x \in V(G) \setminus Y$ is *(anti-)complete* to Y if it is (non-)adjacent to every vertex in Y. Similarly, X is *complete* to Y if every vertex of X is complete to Y and *anti-complete* to Y if every vertex of X is anti-complete to Y.

The graph $G_1 + G_2$ is the *disjoint union* of two vertex-disjoint graphs G_1 and G_2 and has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graph rG is the disjoint union of r copies of a graph G. The graphs C_t, K_t, and P_t denote the cycle, complete graph, and path on t vertices, respectively. The paw is the graph $\overline{P_1 + P_3}$, the diamond is the graph $\overline{2P_1 + P_2}$, and the gem is the graph $\overline{P_1 + P_4}$. The *subdivided claw* $S_{h,i,j}$, for $1 \leq h \leq i \leq j$ is the tree with one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. We let $\mathcal{S}$ denote the class of graphs every connected component of which is either a subdivided claw or a path on at least one vertex. Note that $S_{1,1,1} = K_{1,3}$.

The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G using the following four operations:

1. create a new graph consisting of a single vertex v with label i;
2. take the disjoint union of two labelled graphs G_1 and G_2;
3. add an edge between every vertex with label i and every vertex with label j ($i \neq j$);
4. relabel every vertex with label i to have label j.

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that $\mathrm{cw}(G) \leq c$ for every $G \in \mathcal{G}$; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

For an induced subgraph G' of a graph G, the *subgraph complementation* acting on G with respect to G' replaces every edge of G' by a non-edge, and vice versa. Hence, the resulting graph has vertex set $V(G)$ and edge set $(E(G) \setminus E(G')) \cup E(\overline{G'})$. For two disjoint vertex subsets S and T in G, the *bipartite complementation* acting on G with respect to S and T replaces every edge with one end-vertex in S and the other in T by a non-edge and vice versa.

For a constant $k \geq 0$ and a graph operation γ, a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if (i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and (ii) for every $G \in \mathcal{G}$, there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times. Then γ *preserves* boundedness of clique-width if for every constant k and every graph class $\mathcal{G}$, every graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [47].
Fact 2. Subgraph complementation preserves boundedness of clique-width [45].
Fact 3. Bipartite complementation preserves boundedness of clique-width [45].

A graph is *split* if its vertex set can be partitioned into a clique K and an independent set I. Note that if there is a vertex $v \in I$ with $N(v) \subsetneq K$, then $N(v)$ is a clique cut-set. Furthermore, if $|I| > 1$ then K is a clique cut-set. It follows that split atoms are complete graphs. Since complete graphs have clique-width at most 2, this means that split atoms have bounded clique-width.

3 The Proof of Theorem 2

Here, we prove Theorem 2, namely that the class of $(2P_2, \overline{P_2 + P_3})$-free atoms has bounded clique-width. Our approach is based on the following three claims:

(i) $(2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_5 have bounded clique-width.
(ii) $(2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_4 have bounded clique-width.
(iii) $(C_4, C_5, 2P_2, \overline{P_2 + P_3})$-free atoms have bounded clique-width.

We prove Claims (i) and (ii) in Lemmas 4 and 5, respectively, whereas Claim (iii) follows from the fact that $(C_4, C_5, 2P_2)$-free graphs are split graphs and so the atoms in this class are complete graphs, which therefore have clique-width at most 2. We partition the vertex set of an arbitrary $(2P_2, \overline{P_2 + P_3})$-free atom G into a number of different subsets with according to their neighbourhoods in an induced C_5 in Lemma 4 or an induced C_4 in Lemma 5. We then analyse the properties of these different subsets of $V(G)$ and how they are connected to each other, and use this knowledge to apply a number of appropriate vertex deletions, subgraph complementations and bipartite complementations. These operations will modify G into a graph G' that is a disjoint union of a number of smaller "easy" graphs known to have "small" clique-width. We then use Facts 1–3 to conclude that G also has small clique-width.

This approach works, as we will:

– apply the vertex deletions, subgraph complementations, and bipartite complementations only a constant number of times;
– not use the properties of being an atom or being $(2P_2, \overline{P_2 + P_3})$-free once we "leave the graph class" due to applying the above graph operations.

Our approach is similar to the approach used by Malyshev and Lobanova [49] for showing that Colouring is polynomial-time solvable on the superclass of $(P_5, \overline{P_2 + P_3})$-free graphs. However, we note the following two differences:

1. Prime atoms restriction: OK for Colouring, but not for clique-width. A set $X \subseteq V(G)$ is said to be a *module* if all vertices in X have the same set of neighbours in $V(G) \setminus X$. A module X in a graph G is *trivial* if it contains either all or at most one vertex of G. A graph G is *prime* if it has no non-trivial modules. To solve Colouring in polynomial time on some hereditary graph class $\mathcal{G}$, one may restrict to prime atoms from $\mathcal{G}$ [42]. Malyshev and Lobanova proved that $(P_5, \overline{P_2 + P_3})$-free prime atoms with an induced C_5 are $3P_1$-free or have a bounded number of vertices. In both cases, Colouring can be solved in polynomial time. We cannot make the pre-assumption that our atoms are prime. To see this, let G be a split graph. Add two new non-adjacent vertices to G and make them complete to the rest of $V(G)$. Let $\mathcal{G}$ be the (hereditary) graph class that consists of all these "enhanced" split graphs and their induced subgraphs. These enhanced split graphs are atoms, which have unbounded clique-width due to Fact 1 and the fact that split graphs have unbounded clique-width [48]. However, the prime atoms of $\mathcal{G}$ are the complete graphs, which have clique-width at most 2.

2. Perfect graphs restriction: OK for Colouring, but not for clique-width. Malyshev and Lobanova observed that $(P_5, \overline{P_2 + P_3}, C_5)$-free graphs are perfect. Hence, Colouring can be solved in polynomial time on such graphs [38]. However, being perfect does not imply boundedness of clique-width (for instance, split graphs are perfect graphs with unbounded clique-width).

We omit the proof of the next lemma.

Lemma 4. *The class of* $(2P_2, \overline{P_2+P_3})$*-free atoms that contain an induced* C_5 *has bounded clique-width.*

Lemma 5. *The class of* $(2P_2, \overline{P_2+P_3})$*-free atoms that contain an induced* C_4 *has bounded clique-width.*

Proof. Suppose G is a $(2P_2, \overline{P_2+P_3})$-free atom containing an induced cycle C on four vertices, say $v_1, \ldots, v_4$ in that order. By Lemma 4, we may assume that G is C_5-free. For $S \subseteq \{1, \ldots, 4\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i \mid i \in S\}$.

To simplify notation, in the following claims, subscripts on vertices and vertex sets should be interpreted modulo 4 and whenever possible we will write V_i instead of $V_{\{i\}}$, write $V_{i,j}$ instead of $V_{\{i,j\}}$, and so on.

Claim 1. *For* $i \in \{1, \ldots, 4\}$*,* $V_{i,i+1,i+2}$ *is empty.*

Proof of Claim. Suppose, for contradiction, that $x \in V_{1,2,3}$. Then $G[v_1, v_3, v_2, v_4, x]$ is a $\overline{P_2+P_3}$, a contradiction. The claim follows by symmetry. ⋄

Claim 2. *For* $i \in \{1, \ldots, 4\}$*,* $V_\emptyset \cup V_i \cup V_{i+1} \cup V_{i,i+1}$ *is an independent set.*

Proof of Claim. Suppose, for contradiction, that $x, y \in V_\emptyset \cup V_1 \cup V_2 \cup V_{1,2}$ are adjacent. Then $G[x, y, v_3, v_4]$ is a $2P_2$, a contradiction. The claim follows by symmetry. ⋄

Claim 3. *For* $i \in \{1, \ldots, 4\}$*,* $V_{i,i+1} \cup V_{i,i+2}$ *and* $V_{i,i+1} \cup V_{i+1,i+3}$ *are independent sets.*

Proof of Claim. Suppose, for contradiction, that $x, y \in V_{1,2} \cup V_{1,3}$ are adjacent. By Claim 2, x and y cannot both be in $V_{1,2}$, so assume without loss of generality that $x \in V_{1,3}$. Now $G[x, v_2, v_1, v_3, y]$ or $G[v_1, v_3, x, v_2, y]$ is a $\overline{P_2+P_3}$ if $y \in V_{1,2}$ or $y \in V_{1,3}$, respectively, a contradiction. The claim follows by symmetry. ⋄

Claim 4. $G[V_{1,2,3,4}]$ *is* (P_1+P_2)*-free and so it has bounded clique-width.*

Proof of Claim. Suppose, for contradiction, that $x, y, y' \in V_{1,2,3,4}$ induce a P_1+P_2 in G. Then $G[v_1, v_3, y, x, y']$ is a $\overline{P_2+P_3}$, a contradiction. Therefore $G[V_{1,2,3,4}]$ is (P_1+P_2)-free and so P_4-free, so it has bounded clique-width by Theorem 1. ⋄

Claim 5. *For* $i \in \{1, 2\}$*,* $V_{i,i+2}$ *is complete to* $V_{1,2,3,4}$*.*

Proof of Claim. Suppose, for contradiction, that $x \in V_{1,3}$ is non-adjacent to $y \in V_{1,2,3,4}$. Then $G[v_1, v_3, v_2, x, y]$ is a $\overline{P_2+P_3}$, a contradiction. The claim follows by symmetry. ⋄

Claim 6. *For* $i \in \{1, 2, 3, 4\}$ *either* $V_{i-1} \cup V_{i-1,i}$ *or* $V_{i,i+1} \cup V_{i+1}$ *is empty.*

Proof of Claim. Suppose, for contradiction, that $x \in V_1 \cup V_{1,2}$ and $y \in V_{2,3} \cup V_3$. Then $G[v_1, x, y, v_3, v_4]$ is a C_5 or $G[x, v_1, y, v_3]$ is a $2P_2$ if x is adjacent or non-adjacent to y, respectively, a contradiction. The claim follows by symmetry. ⋄

Claim 7. *If $x \in V_\emptyset$ then x has at least two neighbours in one of $V_{1,3}$ and $V_{2,4}$ and is anti-complete to the other. Furthermore, in this case x is complete to $V_{1,2,3,4}$.*

Proof of Claim. Suppose $x \in V_\emptyset$. Since G is not an atom, $N(x)$ cannot be a clique, and so must contain two non-adjacent vertices y, y'. By Claims 1 and 2, and the definition of $V_\emptyset$, it follows that $y, y' \in V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$. If $y, y' \in V_{1,2,3,4}$, then $G[y, y', v_1, x, v_2]$ is a $\overline{P_2 + P_3}$, a contradiction. By Claim 5, $V_{1,2,3,4}$ is complete to $V_{1,3} \cup V_{2,4}$, so it follows that $y, y' \in V_{1,3} \cup V_{2,4}$. If $y \in V_{1,3}$ and $y' \in V_{2,4}$, then $G[v_1, v_2, y', x, y']$ is a C_5, a contradiction. It follows that $y, y' \in V_{1,3}$ or $y, y' \in V_{2,4}$.

Suppose $y, y' \in V_{1,3}$. If $z \in V_{2,4}$ is a neighbour of x, then z must be adjacent to y and y' (since, as shown above, x cannot have a pair of non-adjacent neighbours one of which is in $V_{1,3}$ and the other of which is in $V_{2,4}$), in which case $G[y, y', x, v_1, z]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore x cannot have a neighbour in $V_{2,4}$. If $z \in V_{1,2,3,4}$ is a non-neighbour of x, then z must be adjacent to y and y' by Claim 5, so $G[y, y', v_1, x, z]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore x is complete to $V_{1,2,3,4}$. The claim follows by symmetry. ⋄

Claim 8. *For $i \in \{1, 2\}$, $|V_{i,i+1} \cup V_{i+2,i+3}| \leq 2$.*

Proof of Claim. Suppose, for contradiction, that $|V_{1,2} \cup V_{3,4}| \geq 3$. First note that if $x \in V_{1,2}$, $y \in V_{3,4}$ are non-adjacent, then $G[v_1, x, v_3, y]$ is a $2P_2$, a contradiction. Therefore $V_{1,2}$ is complete to $V_{3,4}$. By Claim 2, both $V_{1,2}$ and $V_{3,4}$ are independent sets. If $x \in V_{1,2}$ and $y, y' \in V_{3,4}$, then $G[y, y', v_3, x, v_4]$ is a $\overline{P_2 + P_3}$, a contradiction. By symmetry, we conclude that either $V_{1,2}$ or $V_{3,4}$ is empty. Suppose $V_{3,4}$ is empty, so $V_{1,2}$ contains at least three vertices and let $x \in V_{1,2}$ be such a vertex. Since G is an atom, $N(x)$ cannot be a clique, so x must have two neighbours y, y' that are non-adjacent. By Claims 1, 2, 3 and 6, and the definition of $V_{1,2}$, every neighbour of $x \in V_{1,2}$ lies in $\{v_1, v_2\} \cup V_{1,2,3,4}$. Since v_1 is complete to $\{v_2\} \cup V_{1,2,3,4}$ and v_2 is complete to $\{v_1\} \cup V_{1,2,3,4}$, it follows that $y, y' \in V_{1,2,3,4}$. Now $G[y, y', v_1, v_3, x]$ is a $\overline{P_2 + P_3}$, a contradiction. The claim follows by symmetry. ⋄

Claim 9. *For $i \in \{1, 2, 3, 4\}$, V_i is complete to $V_{1,2,3,4}$ and at most one vertex of $V_{i,i+2}$ has neighbours in V_i.*

Proof of Claim. Suppose $x \in V_1$. Since G is an atom, x must have two neighbours y, y' that are non-adjacent. By Claims 1, 2 and 6, and the definition of V_1, every neighbour of x lies in $\{v_1\} \cup V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$. If $y, y' \in V_{1,3} \cup V_{1,2,3,4}$, then $G[y, y', v_1, v_3, x]$ is a $\overline{P_2 + P_3}$, a contradiction. The vertex v_1 is complete to $V_{1,3} \cup V_{1,2,3,4}$. Therefore without loss of generality, we may assume $y \in V_{2,4}$. Furthermore, note that $V_{1,3}$ is an independent set by Claim 3, so x has at most one neighbour in $V_{1,3}$. Since V_1 is an independent set by Claim 2, it follows that $G[V_1 \cup V_{1,3}]$ is a bipartite graph with parts V_1 and $V_{1,3}$. Since G is $2P_2$-free, it follows that no two vertices in V_1 can have different neighbours in $V_{1,3}$. Therefore at most one vertex of $V_{1,3}$ has a neighbour in V_1. Now if $z \in V_{1,2,3,4}$, then z is adjacent to y by Claim 5. If x is non-adjacent to z, then $G[v_1, y, v_2, x, z]$ is a $\overline{P_2 + P_3}$, a contradiction. We conclude that V_1 is complete to $V_{1,2,3,4}$. The claim follows by symmetry. $\diamond$

We now proceed as follows. By Claim 1, the set $V_{1,2,3} \cup V_{2,3,4} \cup V_{1,3,4} \cup V_{1,2,4}$ is empty. By Claims 6 and 8, there are at most two vertices in $V_{1,2} \cup V_{2,3} \cup V_{3,4} \cup V_{1,4}$, so after doing at most two vertex deletions, we may assume these sets are empty (note that the resulting graph may no longer be an atom). Applying four further vertex deletions, we can remove the cycle C from G. By Claim 6, we may assume without loss of generality that V_3 and V_4 are empty. The remaining vertices of G all lie in $V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$ and by Fact 1, it suffices to show that this modified graph has bounded clique-width. By Claims 5, 7 and 9, $V_{1,2,3,4}$ is complete to $V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4}$, and so applying a bipartite complementation between these two sets disconnects $G[V_{1,2,3,4}]$ from the rest of the graph. By Claim 4, $G[V_{1,2,3,4}]$ has bounded clique-width, so by Fact 3, we may assume $V_{1,2,3,4}$ is empty. By Claim 9, at most one vertex of $V_{1,3}$ (resp. $V_{2,4}$) has a neighbour in V_1 (resp. V_2). Applying at most two further vertex deletions, we may assume that $V_{1,3}$ is anti-complete to V_1 and $V_{2,4}$ is anti-complete to V_2. By Claim 7, we can partition $V_\emptyset$ into the set $V_\emptyset^{1,3}$ of vertices that have neighbours in $V_{1,3}$ and the set $V_\emptyset^{2,4}$ of vertices that have neighbours in $V_{2,4}$. Now Claims 2 and 3 imply that $V_\emptyset^{2,4} \cup V_1 \cup V_{1,3}$ and $V_\emptyset^{1,3} \cup V_2 \cup V_{2,4}$ are independent sets, and so $G[V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4}]$ is a $2P_2$-free bipartite graph. Such graphs are also known as bipartite chain graphs and are well known to have bounded clique-width (see e.g. [30, Theorem 2]). By Fact 1, this completes the proof. $\square$

The class of split graphs is the class of $(C_4, C_5, 2P_2)$-free graphs. Since split graphs therefore form a subclass of the class of $(2P_2, \overline{P_2 + P_3})$-free graphs, and split graphs have unbounded clique-width, it follows that $(2P_2, \overline{P_2 + P_3})$-free graphs also have unbounded clique-width. Recall that split atoms are complete graphs, which therefore have clique-width at most 2. The $(2P_2, \overline{P_2 + P_3})$-free atoms that are not split must therefore contain an induced C_4 or C_5. Applying Lemmas 4 and 5, we obtain Theorem 2, which we restate below.

Theorem 2 (restated). *The class of $(2P_2, \overline{P_2 + P_3})$-free atoms has bounded clique-width (whereas the class of $(2P_2, \overline{P_2 + P_3})$-free graphs has unbounded clique-width).*

4 Conclusions

Motivated by algorithmic applications, we determined a new class of (H_1, H_2)-free graphs of unbounded clique-width whose atoms have *bounded* clique-width, namely when $(H_1, H_2) = (2P_2, \overline{P_2 + P_3})$. We also identified a number of classes of (H_1, H_2)-free graphs of unbounded clique-width whose atoms still have *unbounded* clique-width. The latter results show that boundedness of clique-width of (H_1, H_2)-free atoms does not necessarily imply boundedness of clique-width of $(\overline{H_1}, \overline{H_2})$-free atoms. For example, (C_4, P_5)-free atoms have bounded clique-width [35], but we proved that $(\overline{C_4}, \overline{P_5})$-free atoms have unbounded clique-width (Theorem 3). Note however that while it is not known whether the class of $(K_3, S_{1,2,3})$-free graphs has bounded clique-width, we can show that the class of $(K_3, S_{1,2,3})$-free atoms has bounded clique-width if and only if the class of $(3P_1, \overline{S_{1,2,3}})$-free atoms has bounded clique-width (proof omitted).

We also presented a summary theorem (Theorem 3), from which we can deduce the following list of **18** open cases. The cases marked with a * are those for which even the boundedness of clique-width of the whole class of (H_1, H_2)-free graphs is unknown.

Open Problem 6. *Does the class of* (H_1, H_2)*-free atoms have bounded clique-width if*

(i) $H_1 =$ diamond *and* $H_2 = P_6$
(ii) $H_1 = C_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_4, 3P_2\}$
(iii) $H_1 = \overline{P_1 + 2P_2}$ *and* $H_2 \in \{2P_2, P_2 + P_3, P_5\}$
(iv) $H_1 = \overline{P_2 + P_3}$ *and* $H_2 \in \{P_2 + P_3, P_5\}$
**(v)* $H_1 = K_3$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$
**(vi)* $H_1 = 3P_1$ *and* $H_2 = \overline{P_1 + S_{1,1,3}}$
**(vii)* $H_1 =$ diamond *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$
**(viii)* $H_1 = 2P_1 + P_2$ *and* $H_2 \in \{\overline{P_1 + P_2 + P_3}, \overline{P_1 + P_5}\}$
**(ix)* $H_1 =$ gem *and* $H_2 = P_2 + P_3$, *or*
**(x)* $H_1 = P_1 + P_4$ *and* $H_2 = \overline{P_2 + P_3}$.

In particular, we ask if boundedness of clique-width of $(2P_2, \overline{P_2 + P_3})$-free atoms can be extended to $(P_5, \overline{P_2 + P_3})$-free atoms. Could this explain why Colouring is polynomial-time solvable on $(P_5, \overline{P_2 + P_3})$-free graphs [49]? Is boundedness of clique-width the underlying reason? Brandstädt and Hoàng [9] showed that $(P_5, \overline{P_2 + P_3})$-free atoms with no dominating vertices and no vertex pairs $\{x, y\}$ with $N(x) \subseteq N(y)$ are either isomorphic to some specific graph G^* or all their induced C_5s are dominating. Recently, Huang and Karthick [43] proved a more refined decomposition. However, it is not clear how to use these results to prove boundedness of clique-width of $(P_5, \overline{P_2 + P_3})$-free atoms, and additional insights seem to be needed.

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