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QCSP on Reflexive Tournaments[*]

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We give a complexity dichotomy for the Quantified Constraint Satisfaction Problem QCSP(H) when H is a reflexive tournament. It is well-known that reflexive tournaments can be split into a sequence of strongly connected components $H_1, \ldots, H_n$ so that there exists an edge from every vertex of H_i to every vertex of H_j if and only if $i < j$. We prove that if H has both its initial and final strongly connected component (possibly equal) of size 1, then QCSP(H) is in NL and otherwise QCSP(H) is NP-hard.

CCS Concepts: • **Theory of computation** → *Design and analysis of algorithms*; *Logic*; *Computational complexity and cryptography*.

Additional Key Words and Phrases: quantified constraints, constraint satisfaction, graph theorem, logic, computational complexity

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1 INTRODUCTION

The *Quantified Constraint Satisfaction Problem* QCSP(B), for a fixed *template* (structure) B, is a popular generalisation of the *Constraint Satisfaction Problem* CSP(B). In the latter, one asks if a primitive positive sentence (the existential quantification of a conjunction of atoms) φ is true

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on B, while in the former this sentence may also have universal quantification[1]. Much of the theoretical research into (finite-domain[2]) CSPs has been in respect of a complexity classification project [5, 11], recently completed by [4, 22, 24], in which it is shown that all such problems are either in P or NP-complete. Various methods, including combinatorial (graph-theoretic), logical and universal-algebraic were brought to bear on this classification project, with many remarkable consequences.

Complexity classifications for QCSPs appear to be harder than for CSPs. Indeed, a classification for QCSPs will give a fortiori a classification for CSPs (if $\mathrm{B} \uplus \mathrm{K}_1$ is the disjoint union of B with an isolated element, then $\mathrm{QCSP}(\mathrm{B} \uplus \mathrm{K}_1)$ and CSP(B) are polynomial-time many-one equivalent). Just as CSP(B) is always in NP, so QCSP(B) is always in Pspace. However, no polychotomy has been conjectured for the complexities of QCSP(B), though, until recently, only the complexities P, NP-complete and Pspace-complete were known. Recent work [25] has shown that this complexity landscape is considerably richer, and that dichotomies of the form P versus NP-hard (using Turing reductions) might be the sensible place to be looking for classifications.

CSP(B) may equivalently be seen as the *homomorphism* problem which takes as input a structure A and asks if there is a homomorphism from A to B. The *surjective CSP*, SCSP(B), is a cousin of CSP(B) in which one requires that this homomorphism from A to B be surjective. From the logical perspective this translates to the stipulation that all elements of B be used as witnesses to the (existential) variables of the primitive positive input φ. The surjective CSP appears in the literature under a variety of names, including *surjective homomorphism* [2], *surjective colouring* [12, 15] and *vertex compaction* [19, 20]. CSP(B) and SCSP(B) have various other cousins: see the survey [2] or, in the specific context of reflexive tournaments, [15]. The only one we will dwell on here is the *retraction* problem $\mathrm{CSP}^c(\mathrm{B})$ which can be defined in various ways but, in keeping with the present narrative, we could define logically as allowing atoms of the form $v = b$ in the input sentence φ where b is some element of B (the superscript c indicates that constants are allowed). It has only recently been shown that there exists a B so that SCSP(B) is in P while $\mathrm{CSP}^c(\mathrm{B})$ is NP-complete [23]. It is still not known whether such an example exists among the (partially reflexive) graphs.

It is well-known that the binary *cousin* relation is not transitive, so let us ask the question as to whether the surjective CSP and QCSP are themselves cousins? The algebraic operations pertaining to the CSP are *polymorphisms* and for QCSP these become *surjective* polymorphisms. On the other hand, a natural use of universal quantification in the QCSP might be to ensure some kind of surjective map (at least under some evaluation of many universally quantified variables). So it is that there may appear to be some relationship between the problems. Yet, there are known irreflexive graphs H for which QCSP(H) is in NL, while SCSP(H) is NP-complete (take the 6-cycle [18, 20]). On the other hand, one can find a 3-element B whose relations are preserved by a *semilattice-without-unit* operation such that both $\mathrm{CSP}^c(\mathrm{B})$ and SCSP(B) are in P but QCSP(B) is Pspace-complete. We are not aware of examples like this among graphs and it is perfectly possible that for (partially reflexive) graphs H, SCSP(H) being in P implies that QCSP(H) is in P.

Tournaments, both irreflexive and reflexive (and sometimes in between), have played a strong role as a testbed for conjectures and a habitat for classifications, for relatives of the CSP both complexity-theoretic [1, 10, 15] and algebraic [14, 21]. Looking at Table 1 one can see the last unresolved case is precisely QCSP on reflexive tournaments. This is the case we address in this paper. For irreflexive tournaments H, QCSP(H) is in P if and only if SCSP(H) in P, but for reflexive

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[1] Typically, primitive positive logic also possesses equality, but these can be propagated out by substitution, or removed in the case $x = x$. In the presence of universal quantification, any atom $x = y$ whose innermost variable is universal is false (unless x and y are the same variable). Other instances of equality may be propagated out as before. It follows that the complexity of QCSP(B) is not affected by the presence or absence of equality, up to logarithmic space reducability.

[2] All structures considered in this article are finite.

tournaments this is not the case. When H is a reflexive tournament, we prove that QCSP(H) is in NL if H has both initial and final strongly connected components trivial, and is NP-hard otherwise. In contrast to the proof from [10] and like the proof of [15], we will henceforth work largely combinatorially rather than algebraically. Note that we do not investigate beyond NP-hard, so our dichotomy cannot be compared directly to the trichotomy of [10] for irreflexive tournaments which distinguishes between P, NP-complete and Pspace-complete.

	QCSP	CSP	Surjective CSP	Retraction
irreflexive tournaments	trichotomy [10]	dichotomy [1]	dichotomy [1]	dichotomy [1]
reflexive tournaments	**this paper**	all trivial	dichotomy [15]	dichotomy [14]

Table 1. Our result in a wider context. The results for irreflexive tournaments were all proved in the more general setting of irreflexive semicomplete digraphs in the papers cited.

In Section 3 we prove the NP-hard cases of our dichotomy. Our proof method follows that from [15], while adapting the ideas of [8] in order to make what was developed for Surjective CSP applicable to QCSP. The QCSP is not naturally a combinatorial problem but can be seen thusly when viewed in a certain way. We indeed closely mirror [15] with [8] in the strongly connected case. For the not strongly connected case, the adaptation from the strongly connected case was straightforward for the Surjective CSP in [15]. However, the straightforward method does not work for the QCSP. Instead, we seek a direct argument that essentially sees us extending the method from [15].

In Section 4 we prove the NL cases of our dichotomy. Here, we use ideas originally developed in (the conference version of) [8] and first used in the wild in [17]. Thus, we do not introduce new proof techniques as such but rather weave our proof through the reasonably intricate synthesis of different known techniques. In Section 5 we state our dichotomy and give some directions for future work.

2 PRELIMINARIES

For an integer $k \geq 1$, we write $[k] := \{1, \ldots, k\}$. A vertex $u \in V(G)$ in a digraph G is *backwards-adjacent* to another vertex $v \in V$ if $(u, v) \in E$. It is *forwards-adjacent* to another vertex $v \in V$ if $(v, u) \in E$. If a vertex u has a self-loop (u, u), then u is *reflexive*; otherwise u is *irreflexive.* A digraph G is *reflexive* or *irreflexive* if all its vertices are reflexive or irreflexive, respectively.

The *directed path* on k vertices is the digraph with vertices $u_0, \ldots, u_{k-1}$ and edges (u_i, u_{i+1}) for $i = 0, \ldots, k-2$. By adding the edge (u_{k-1}, u_0), we obtain the *directed cycle* on k vertices. A digraph G is *strongly connected* if for all $u, v \in V(\mathrm{G})$ there is a directed path in $E(\mathrm{G})$ from u to v. A *double edge* in a digraph G consists in a pair of distinct vertices $u, v \in V(\mathrm{G})$, so that (u, v) and (v, u) belong to $E(\mathrm{G})$. A digraph G is *semicomplete* if for every two distinct vertices u and v, at least one of (u, v), (v, u) belongs to $E(\mathrm{G})$. A semicomplete digraph G is a *tournament* if for every two distinct vertices u and v, exactly one of (u, v), (v, u) belongs to $E(\mathrm{G})$. A reflexive tournament G is *transitive* if for every three vertices u, v, w with $(u, v), (v, w) \in E(\mathrm{G})$, also (u, w) belongs to $E(\mathrm{G})$. A digraph F is a *subgraph* of a digraph G if $V(\mathrm{F}) \subseteq V(\mathrm{G})$ and $E(\mathrm{F}) \subseteq E(\mathrm{G})$. It is *induced* if $E(\mathrm{F})$ coincides with $E(\mathrm{G})$ restricted to pairs containing only vertices of $V(\mathrm{F})$. A *subtournament* is an induced subgraph of a tournament. It is well known that a reflexive tournament H can be split into a sequence of strongly connected components $\mathrm{H}_1, \ldots, \mathrm{H}_n$ for some integer $n \geq 1$ so that there exists an edge from every vertex of H_i to every vertex of H_j if and only if $i < j$. We will use the notation $\mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ for H and we refer to H_1 and H_n as the *initial* and *final* components of H, respectively.

A *homomorphism* from a digraph G to a digraph H is a function $f : V(\mathrm{G}) \longrightarrow V(\mathrm{H})$ such that for all $u, v \in V(\mathrm{G})$ with $(u, v) \in E(\mathrm{G})$ we have $(f(u), f(v)) \in E(\mathrm{H})$. We say that f is *(vertex)-surjective* if for every vertex $x \in V(\mathrm{H})$ there exists a vertex $u \in V(\mathrm{G})$ with $f(u) = x$. A digraph H' is a *homomorphic image* of a digraph H if there is a surjective homomorphism from H to H' that is also *edge-surjective*, that is, for all $(x', y') \in E(\mathrm{H}')$ there exists an $(x, y) \in E(\mathrm{H})$ with $x' = h(x)$ and $y' = h(y)$.

The problem H-RETRACTION takes as input a graph G of which H is an induced subgraph and asks whether there is a homomorphism from G to H that is the identity on H. This definition is polynomial-time many-one equivalent to the one we suggested in the introduction (see e.g. [2]). The *quantified constraint satisfaction problem* QCSP(H) takes as input a sentence $\varphi := \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \ \Phi(x_1, y_1, \ldots, x_n, y_n)$, where Φ is a conjunction of positive atomic (binary edge) relations. This is a yes-instance to the problem just in case $\mathrm{H} \models \varphi$.

The *canonical query* of G (from [13]) is a primitive positive sentence φ_{G} that has the property that, for all H, G has a homomorphism to H iff $\mathrm{H} \models \varphi_{\mathrm{G}}$. It is built by mapping edges (x, y) from $E(\mathrm{G})$ to atoms $E(x, y)$ is an existentially quantified conjunctive query.

The *direct product* of two digraphs G and H, denoted G×H, is the digraph on vertex set $V(\mathrm{G}) \times V(\mathrm{H})$ with edges $((x, y), (x', y'))$ if and only if $(x, x') \in E(\mathrm{G})$ and $(y, y') \in E(\mathrm{H})$. We denote the direct product of k copies of G by G^k. A k-ary *polymorphism* of G is a homomorphism f from G^k to G; if $k = 1$, then f is also called an *endomorphism*. A k-ary polymorphism f is *essentially unary* if there exists a unary operation g and $i \in [k]$ so that $f(x_1, \ldots, x_k) = g(x_i)$ for every $(x_1, \ldots, x_k) \in \mathrm{G}^k$. Let us say that a k-ary polymorphism f is *uniformly* z for some $z \in V(\mathrm{G})$ if $f(x_1, \ldots, x_k) = z$ for every $(x_1, \ldots, x_k) \in V(\mathrm{G}^k)$. We need the following two lemmas.

LEMMA 2.1. *Let H be a reflexive tournament and f be a k-ary polymorphism of* H. *If $f(x, \ldots, x) = z$ for every $x \in V(\mathrm{H})$, then f is uniformly equal to z.*

PROOF. Consider some tuple $(x_1, \ldots, x_k)$ which has m distinct vertices. We proceed by induction on m, where the base case $m = 1$ is given as an assumption. Suppose we have the result for m vertices and let $(x_1, \ldots, x_k)$ have $m + 1$ distinct entries. For simplicity (and w.l.o.g.) we will consider this reordered and without duplicates as $(y_1, \ldots, y_m, y_{m+1})$. Suppose f maps $(x_1, \ldots, x_k)$ to z'. Assume $(y_m, y_{m+1}) \in E(\mathrm{H})$ (the case (y_{m+1}, y_m) is symmetric). Then consider the tuples $(y_1, \ldots, y_m, y_m)$ and $(y_1, \ldots, y_{m+1}, y_{m+1})$. By the inductive hypothesis, f maps each of these (when reordered and padded appropriately with duplicates) to z. Furthermore, we have co-ordinatewise edges from $(y_1, \ldots, y_m, y_m)$ to $(y_1, \ldots, y_m, y_{m+1})$ and from $(y_1, \ldots, y_m, y_{m+1})$ to $(y_1, \ldots, y_{m+1}, y_{m+1})$. Since we deduce by the definition of polymorphism that both $(z, z'), (z', z) \in E(\mathrm{H})$, it follows that $z' = z$. Thus, f maps also $(y_1, \ldots, y_m, y_{m+1})$ (when reordered and padded appropriately with duplicates) to z. That is, $f(x_1, \ldots, x_k) = z$. □

LEMMA 2.2. *Let* H *be the reflexive tournament* $\mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_i \Rightarrow \cdots \Rightarrow \mathrm{H}_n$. *If f is a k-ary surjective polymorphism of* H, *then f preserves each of $V(\mathrm{H}_1), \ldots, V(\mathrm{H}_n)$; that is, for every i and every tuple of k vertices $x_1, \ldots, x_k \in V(\mathrm{H}_i)$, $f(x_1, \ldots, x_k) \in V(\mathrm{H}_i)$.*

PROOF. Suppose f maps some tuple $(x_1, \ldots, x_m)$ from $V(\mathrm{H}_i)$ to $y \in V(\mathrm{H}_\ell)$. Let $(x'_1, \ldots, x'_m)$ be any tuple from $V(\mathrm{H}_i)$. Since H_i is strongly connected, $f(x'_1, \ldots, x'_m)$ in $V(\mathrm{H}_\ell)$. It follows that if $\ell \neq i$, e.g. w.l.o.g. $\ell < i$, then some component $\ell' \geq i$ can not be in the range of f. □

The relevance of this lemma is in its sequent corollary, which follows according to Proposition 3.15 of [3].

Corollary 2.3. *Let* H *be the reflexive tournament* $\mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_i \Rightarrow \cdots \Rightarrow \mathrm{H}_n$. *Each subset of the domain* $V(\mathrm{H}_i)$ *is definable by a QCSP instance in one free variable.*

An endomorphism e of a digraph G is a *constant map* if there exists a vertex $v \in V(\mathrm{G})$ such that $e(u) = v$ for every $u \in V(\mathrm{G})$, and e is the *identity* if $e(u) = u$ for every $u \in \mathrm{G}$. An *automorphism* is a bijective endomorphism whose inverse is a homomorphism. An endomorphism is *trivial* if it is either an automorphism or a constant map; otherwise it is *non-trivial*. A digraph is *endo-trivial* if all of its endomorphisms are trivial. An endomorphism e of a digraph G *fixes* a subset $S \subseteq V(\mathrm{G})$ if $e(S) = S$, that is, $e(x) \in S$ for every $x \in S$, and e fixes an induced subgraph F of G if it is the identity on $V(\mathrm{F})$. It fixes an induced subgraph F *up to automorphism* if $e(\mathrm{F})$ is an automorphic copy of F. An endomorphism e of G is a *retraction* of G if e is the identity on $e(V(\mathrm{G}))$. A digraph is *retract-trivial* if all of its retractions are the identity or constant maps. Note that endo-triviality implies retract-triviality, but the reverse implication is not necessarily true (see [15]). However, on reflexive tournaments both concepts do coincide [15].

We need a series of results from [15]. The third one follows from the well-known fact that every strongly connected tournament has a directed Hamilton cycle [6].

Lemma 2.4 ([15]). *A reflexive tournament is endo-trivial if and only if it is retract-trivial.*

Lemma 2.5 ([15]). *Let* H *be an endo-trivial reflexive digraph with at least three vertices. Then every polymorphism of* H *is essentially unary.*

Lemma 2.6 ([15]). *If* H *is an endo-trivial reflexive tournament, then* H *contains a directed Hamilton cycle.*

Lemma 2.7 ([15]). *If* H *is an endo-trivial reflexive tournament, then every homomorphic image of* H *of size* $1 < n < |V(\mathrm{H})|$ *has a double edge.*

Corollary 2.8. *If* H *is an endo-trivial reflexive digraph on at least three vertices, then* QCSP(H) *is* NP*-hard (in fact it is even* Pspace*-complete).*

Proof. This follows from Lemma 2.5 and [3]. □

3 THE PROOF OF THE NP-HARD CASES OF THE DICHOTOMY

We commence with the NP-hard cases of the dichotomy. The simpler NL cases will follow, in Section 4. In this section, the central results will appear as Corollaries 3.9 and 3.15. However, each of these proceeds via an induction where there are two base cases and two inductive (general) cases. Thus, there are eight principal propositions. Propositions 3.3, 3.5, 3.7 and 3.8 lead to Corollary 3.9 and Propositions 3.11, 3.12, 3.13 and 3.14 lead to Corollary 3.15. The base cases are the simplest to understand and are given in the most detail. The principal propositions commence in Section 3.2. Before this we introduce our construction with some supporting lemmas.

3.1 The NP-Hardness Gadget

We introduce the gadget Cyl^*_m from [15] drawn in Figure 1. Take m disjoint copies of the (reflexive) directed m-cycle DC^*_m arranged in a cylindrical fashion so that there is an edge from i in the jth copy to i in the $(j+1)$th copy (drawn in red), and an edge from i in the $(j+1)$th copy to $(i+1)$ mod m in the jth copy (drawn in green). We consider DC^*_m to have vertices $\{1,\dots,m\}$. Recall that every strongly connected (reflexive) tournament on m vertices has a Hamilton Cycle HC_m. We label the vertices of HC_m as $1,\dots,m$ in order to attach it to the gadget Cyl^*_m.[3]

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[3] The superscripted $*$ indicates that the corresponding graph is reflexive. This notation is inherited from [15]. It is not significant since we could safely assume every graph we work with is reflexive as the template is a reflexive tournament.

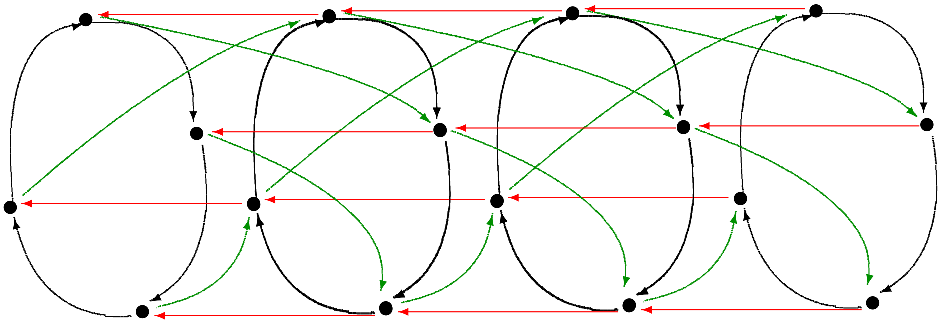

Fig. 1. The gadget Cyl^*_m in the case $m := 4$ (self-loops are not drawn). We usually visualise the right-hand copy of DC^*_4 as the "bottom" copy and then we talk about vertices "above" and "below" according to the red arrows.

The following lemma follows from induction on the copies of DC^*_m, since a reflexive tournament has no double edges.

Lemma 3.1 ([15]). *In any homomorphism h from Cyl^*_m, with bottom cycle DC^*_m, to a reflexive tournament, if $|h(\mathrm{DC}^*_m)| = 1$, then $|h(\mathrm{Cyl}^*_m)| = 1$.*

We will use another property, denoted (†), of Cyl^*_m, which is that the retractions from Cyl^*_m to its bottom copy of DC^*_m, once propagated through the intermediate copies, induce on the top copy precisely the set of automorphisms of DC^*_m. That is, the top copy of DC^*_m is mapped isomorphically to the bottom copy, and all such isomorphisms may be realised. The reason is that in such a retraction, the $(j+1)$th copy may either map under the identity to the jth copy, or rotate one edge of the cycle clockwise, and Cyl^*_m consists of sufficiently many (namely m) copies of DC^*_m. Now let H be a reflexive tournament that contains a subtournament H_0 on m vertices that is endo-trivial. By Lemma 2.6, we find that H_0 contains at least one directed Hamilton cycle HC_0. Define $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ as follows. Begin with H and add a copy of the gadget Cyl^*_m, where the bottom copy of DC^*_m is identified with HC_0, to build a digraph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. Now ask, for some $y \in V(\mathrm{H})$ whether there is a retraction r of $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H so that some vertex x (not dependent on y) in the top copy of DC^*_m in Cyl^*_m is such that $r(x) = y$. Such vertices y comprise the set $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$.

Remark 1. If x belongs to some copy of DC^*_m that is not the top copy, we can find a vertex x' in the top copy of DC^*_m and a retraction r' from $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H with $r'(x') = r(x) = y$, namely by letting r' map the vertices of higher copies of DC^*_m to the image of their corresponding vertex in the copy that contains x. In particular this implies that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ contains $V(\mathrm{H}_0)$.

We note that the set $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0])$ is potentially dependent on which Hamilton cycle in H_0 is chosen. We now recall that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ if H retracts to H_0.

Lemma 3.2 ([15]). *If H is a reflexive tournament that retracts to a subtournament H_0 with Hamilton cycle HC_0, then $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$.*

We now review a variant of a construction from [8]. Let G be a graph containing H where $|V(\mathrm{H})|$ is of size n. Consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H})$ (let us write $\lambda \in V(\mathrm{H})^{[n]}$ of cardinality N). For some such λ, let $\mathcal{G}(\lambda)$ be the graph G enriched with constants $c_1, \ldots, c_n$ where these are interpreted over $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts). We use calligraphic notation to remind the reader the signature has changed from $\{E\}$ to $\{E, c_1, \ldots, c_n\}$

but we will still treat these structures as graphs. If we write $\mathrm{G}(\lambda)$ without calligraphic notation we mean we look at only the $\{E\}$-reduct, that is, we drop the constants. Of course, $\mathrm{G}(\lambda)$ will always be G.

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}(\lambda)$. That is, the vertices of $\mathcal{G}$ are N-tuples over $V(\mathrm{G})$ and there is an edge between two such vertices $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ if and only if $(x_1, y_1), \ldots, (x_N, y_N) \in E(\mathrm{G})$. Finally, the constants c_i are interpreted as $(x_1, \ldots, x_N)$ so that $\lambda_1(c_i) = x_1, \ldots, \lambda_N(c_i) = x_N$. An important induced substructure of $\mathcal{G}$ is $\{(x, \ldots, x) : x \in V(\mathrm{G})\}$. It is a copy of G called the *diagonal* copy and will play an important role in the sequel. To comprehend better the construction of $\mathcal{G}$ from the sundry $\mathcal{G}(\lambda)$, confer on Figure 2.

The final ingredient of our fundamental construction involves taking some structure $\mathcal{G}$ and making its canonical query with all vertices other than those corresponding to $c_1, \ldots, c_n$ becoming existentially quantified variables (as usual in this construction). We then turn the $c_1, \ldots, c_n$ to variables $y_1, \ldots, y_n$ to make $\varphi_{\mathcal{G}}(y_1, \ldots, y_n)$. Let $\mathcal{H}$ come from the given construction in which $G = H$. It is proved in [8] that $\mathrm{H}' \models \forall y_1, \ldots, y_n\ \varphi_{\mathcal{H}}(y_1, \ldots, y_n)$ if and only if $\mathrm{QCSP(H)} \subseteq \mathrm{QCSP(H')}$ (here we identify QCSP(H) with the set of sentences that form its yes-instances). By way of a side note, let us consider a k-ary relation R over H with tuples $(x^1_1, \ldots, x^1_k)$, …, $(x^r_1, \ldots, x^r_k)$. For $i \in [r]$, let λ_i map $(c_1, \ldots, c_k)$ to $(x^i_1, \ldots, x^i_k)$. Let $\mathcal{H} = \bigotimes_{\lambda \in \{\lambda_1, \ldots, \lambda_r\}} \mathcal{H}(\lambda)$. Then $\varphi_{\mathcal{H}}(y_1, \ldots, y_n)$ is the closure of R under the polymorphisms of H.

3.2 The strongly connected case: Two Base Cases

Recall that if H is a (reflexive) endo-trivial tournament, then QCSP(H) is NP-hard due to Lemma 2.5 combined with the results from [3]. Indeed, Theorem 5.2 in [3] states that any H with more than one element, such that all surjective polymorphisms of H are essentially unary, satisfies that QCSP(H) is Pspace-complete. However H may not be endo-trivial. We will now show how to deal with the case where H is not endo-trivial but retracts to an endo-trivial subtournament. For doing this we use the NP-hardness gadget, but we need to distinguish between two different cases.

Proposition 3.3 (Base Case I.). *Let* H *be a reflexive tournament that retracts to an endo-trivial subtournament* H_0 *with Hamilton cycle* HC_0. *Assume that* H *retracts to* H'_0 *for every isomorphic copy* $\mathrm{H}'_0 = i(\mathrm{H}_0)$ *of* H_0 *in* H *with* $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}'_0, i(\mathrm{HC}_0)]) = V(\mathrm{H})$. *Then* H_0-Retraction *can be polynomially reduced to* QCSP(H).

Proof. Let m be the size of $|V(\mathrm{H}_0)|$ and n be the size of $|V(\mathrm{H})|$. Let G be an instance of H_0-Retraction. We build an instance φ of QCSP(H) in the following fashion. First, take a copy of H together with G and build G′ by identifying these on the copy of H_0 that they both possess as an induced subgraph. Now, consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d and H^d_0 be the diagonal copies of G′, H and H_0 in $\mathcal{G}'$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}) \times \cdots \times V(\mathrm{H})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of Cyl^*_m for every vertex $v \in V(\mathcal{H}) \setminus V(\mathrm{H}^d_0)$. Vertex v is to be identified with any vertex in the top copy of DC^*_m in Cyl^*_m and the bottom copy of DC^*_m is to be identified with HC_0 in H^d_0 according to the identity function. (Thus, in each case, the new vertices are the middle cycles of Cyl^*_m and all but one of the vertices in the top cycle of Cyl^*_m.)

Finally, build φ from the canonical query of $\mathcal{G}''$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables. The size of φ is doubly exponential in n (the size of H) but this is constant, so still polynomial in the size of G.

We claim that G retracts to H_0 if and only if $\varphi \in \mathrm{QCSP(H)}$.

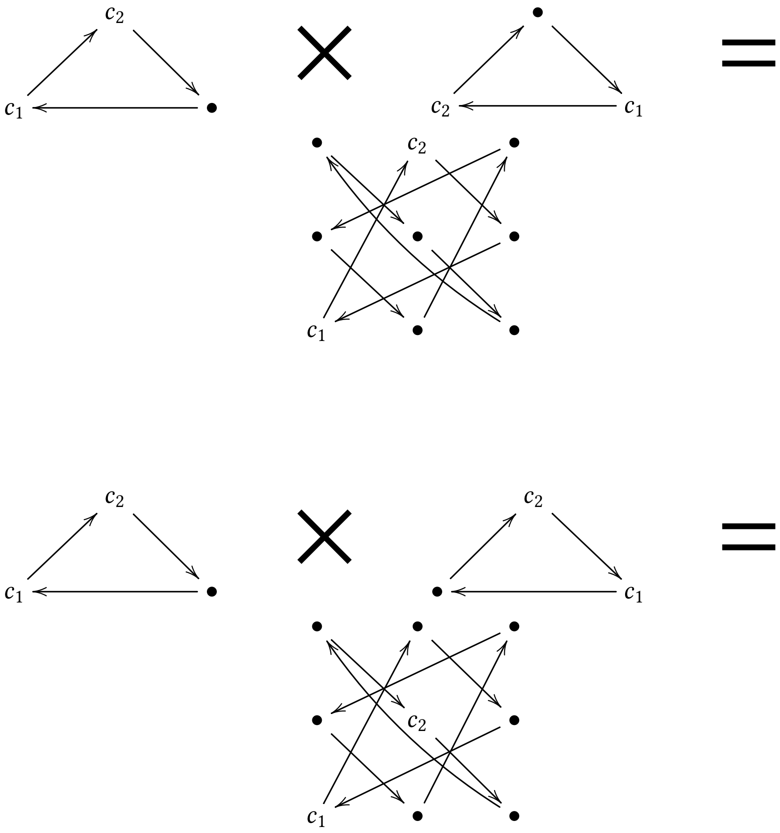

Fig. 2. Illustrations of direct product with constants.

First suppose that G retracts to H_0. Let λ be some assignment of the universal variables of φ to H. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H})$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H that extends λ. Then for this it suffices to prove that there is a homomorphism h from $\mathcal{G}'$ that extends λ. Let us explain why. Because H retracts to H_0, we have $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ due to Lemma 3.2. Hence, if $h(x) = y$ for two vertices $x \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$ and $y \in V(\mathrm{H})$, we can always find a retraction of the graph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$ to H that maps x to y, and we mimic this retraction on the corresponding subgraph in $\mathcal{G}''$. The crucial observation is that this can be done independently for each vertex in $V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$, as two vertices of different copies of Cyl_m^* are only adjacent if they both belong to $\mathcal{H}$.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H})$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H that extends λ. Note that it will be sufficient to prove that G' retracts to H. Let h be the natural retraction from G' to H that extends the known retraction from G to H_0. We are done.

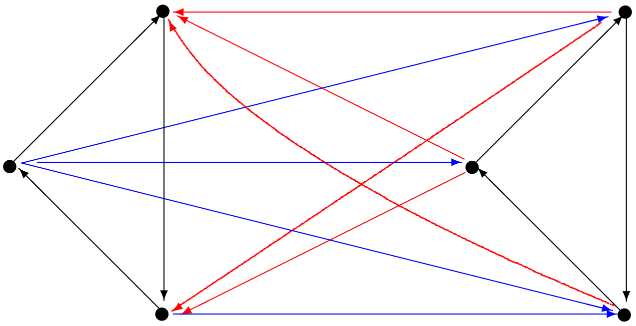

Fig. 3. An interesting tournament H on six vertices (self-loops are not drawn). This tournament does not retract to the DC_3^* on the left-hand side, yet $\mathrm{Spill}_3(\mathrm{H}[\mathrm{DC}_3^*, \mathrm{DC}_3]) = V(\mathrm{H})$.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H})$. Choose some surjection for λ, the assignment of the universal variables of φ to H. Recall $N = |V(\mathrm{H})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H})$ induces a surjective homomorphism s from $\mathcal{G}''$ to H which contains within it a surjective homomorphism s' from $\mathcal{H} = \mathrm{H}^N$ to H. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}^d \subset \mathrm{G}'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$, we have $|s'(\mathrm{H}^d)| = 1$. Indeed, this was the property we noted in Lemma 3.1. By Lemma 2.1, this would mean s' is uniformly mapping $\mathcal{H}$ to one vertex, which is impossible as s' is surjective. Now we will work exclusively in the diagonal copy G'^d. As $1 < |s'(\mathrm{H}_0^d)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(\mathrm{H}_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$ for some isomorphism i.

We claim that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$. In order to see this, consider a vertex $y \in V(\mathrm{H})$. As s' is surjective, there exists a vertex $x \in V(\mathcal{H})$ with $s'(x) = y$. By construction, x belongs to some top copy of DC_m^* in Cyl_m^* in $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. We can extend i^{-1} to an isomorphism from the copy of Cyl_m^* (which has $i(\mathrm{HC}_0^d)$ as its bottom cycle) in the graph $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to the copy of Cyl_m^* (which has HC_0^d as its bottom cycle) in the graph $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$. We define a mapping r^* from $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to H by $r^*(u) = s' \circ i^{-1}(u)$ if u is on the copy of Cyl_m^* in $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ and $r^*(u) = u$ otherwise. We observe that $r^*(u) = u$ if $u \in V(\mathrm{H}_0')$ as s' coincides with i on H_0. As H_0^d separates the other vertices of the copy of Cyl_m^* from $V(\mathrm{H}^d) \setminus V(\mathrm{H}_0^d)$, in the sense that removing H_0^d would disconnect them, this means that r^* is a retraction from $\mathrm{F}(\mathrm{H}_0', i(\mathrm{HC}_0^d))$ to H. We find that r^* maps $i(x)$ to $s' \circ i^{-1}(i(x)) = s'(x) = y$. Moreover, as x is in the top copy of DC_m^* in $\mathrm{F}(\mathrm{H}_0, \mathrm{HC}_0)$, we conclude that y always belongs to $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)])$.

As $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$, we find, by assumption of the lemma, that there exists a retraction r from H to H_0'. Now, recalling that we can view s' acting just on the diagonal copy H^d of H, $i^{-1} \circ r \circ s'$ is the desired retraction of G to H_0. □

We now need to deal with the situation in which we have an isomorphic copy $\mathrm{H}_0' = i(\mathrm{H}_0)$ of H_0 in H with $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) = V(\mathrm{H})$, such that H does not retract to H_0' (see Figure 3 for an example). We cannot deal with this case in a direct manner and first show another base case. For this we need the following lemma and an extension of endo-triviality that we discuss afterwards.

Lemma 3.4 ([15]). *Let* H *be a reflexive tournament, containing a subtournament* H_0 *so that any endomorphism of* H *that fixes* H_0 *as a graph is an automorphism. Then any endomorphism of* H *that maps* H_0 *to an isomorphic copy* $\mathrm{H}_0' = i(\mathrm{H}_0)$ *of itself is an automorphism of* H.

Let H_0 be an induced subgraph of a digraph H. We say that the pair (H, H_0) is *endo-trivial* if all endomorphisms of H that fix H_0 are automorphisms.

PROPOSITION 3.5 (BASE CASE II). *Let* H *be a reflexive tournament with a subtournament* H_0 *with Hamilton cycle* HC_0 *so that* (H, H_0) *and* H_0 *are endo-trivial and* $\mathrm{Spill}_m(H[H_0, HC_0]) = V(H)$*. Then* H-RETRACTION *can be polynomially reduced to* QCSP(H)*.*

PROOF. Let G be an instance of H-RETRACTION. Let m be the size of $|V(H_0)|$ and n be the size of $|V(H)|$. We build an instance φ of QCSP(H) in the following fashion. Consider all possible functions $\lambda : [n] \longrightarrow V(H)$. For some such λ, let $\mathcal{G}(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(H)$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G} = \bigotimes_{\lambda \in V(H)^{[n]}} \mathcal{G}(\lambda)$. Let G^d, H^d and H_0^d be the diagonal copies of G, H and H_0 in $\mathcal{G}$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ induced by $V(H) \times \cdots \times V(H)$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}'$ from $\mathcal{G}$ by augmenting a new copy of Cyl^*_m for every vertex $v \in V(\mathcal{H}) \setminus V(H_0^d)$. Vertex v is to be identified with any vertex in the top copy of DC^*_m in Cyl^*_m and the bottom copy of DC^*_m is to be identified with HC_0 in H_0^d according to the identity function.

Finally, build φ from the canonical query of $\mathcal{G}'$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables.

First suppose that G retracts to H by r. Let λ be some assignment of the universal variables of φ to H. To prove $\varphi \in$ QCSP(H) it suffices to prove that there is a homomorphism from $\mathcal{G}'$ to H that extends λ and for this it suffices to prove that there is a homomorphism from $\mathcal{G}$ that extends λ. This is always possible since we have $\mathrm{Spill}_m(H[H_0, HC_0]) = V(H)$ by assumption.

Henceforth let us consider the homomorphic image of $\mathcal{G}$ that is $\mathcal{G}(\lambda)$. To prove $\varphi \in$ QCSP(H) it suffices to prove that there is a homomorphism from $G(\lambda)$ to H that extends λ. Note that it will be sufficient to prove that G retracts to H. Well this was our original assumption so we are done.

Suppose now $\varphi \in$ QCSP(H). Choose some surjection for λ, the assignment of the universal variables of φ to H. Recall $N = |V(H)^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in$ QCSP(H) induces a surjective homomorphism s from $\mathcal{G}'$ to H which contains within it a surjective homomorphism s' from $\mathcal{H} = H^N$ to H. Consider the diagonal copy of $H_0^d \subset H^d \subset G^d$ in $(G)^N$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(H_0^d)| = 1$, by construction of $\mathcal{G}'$, we have $|s'(H^d)| = 1$. By Lemma 2.1, this would mean s' is uniformly mapping $\mathcal{H}$ to one vertex, which is impossible as s' is surjective. Now we will work exclusively on the diagonal copy G^d. As $1 < |s'(H_0^d)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(H_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $H_0' = i(H_0^d)$ for some isomorphism i.

As (H, H_0) is endo-trivial, Lemma 3.4 tells us that the restriction of s' to H^d is an automorphism of H^d, which we call α. The required retraction from G to H is now given by $\alpha^{-1} \circ s'$. □

3.3 The strongly connected case: Generalising the Base Cases

We now generalise the two base cases to more general cases via some recursive procedure. Afterwards we will show how to combine these two cases to complete our proof. We will first need a slightly generalised version of Lemma 3.4, which nonetheless has virtually the same proof. For completeness of this article we provide this proof from [15].

LEMMA 3.6 ([15]). *Let* $H_2 \supset H_1 \supset H_0$ *be a sequence of strongly connected reflexive tournaments, each one a subtournament of the one before. Suppose that any endomorphism of* H_1 *that fixes* H_0 *is an automorphism. Then any endomorphism* h *of* H_2 *that maps* H_0 *to an isomorphic copy* $H_0' = i(H_0)$ *of itself also gives an isomorphic copy of* H_1 *in* $h(H_1)$*.*

Proof. For contradiction, suppose there is an endomorphism h of H_2 that maps H_0 to an isomorphic copy $\mathrm{H}'_0 = i(\mathrm{H}_0)$ of itself that does not yield an isomorphic copy of H_1. In particular, $|h(\mathrm{H}_1)| < |V(\mathrm{H}_1)|$. We proceed as in the proof of the Lemma 3.4. Choose h^{-1} in the following fashion. We let h^{-1} of $h(\mathrm{H}_0)$ be the natural isomorphism of $h(\mathrm{H}_0)$ to H_0 (that inverts the isomorphism given by h from H_0 to H'_0). Otherwise we choose h^{-1} arbitrarily, such that $h^{-1}(y) = x$ only if $h(x) = y$. Since H_2 is a reflexive tournament, h^{-1} is an isomorphism. And $h^{-1} \circ h$ is an endomorphism of H_2 that fixes H_0 that does not yield an isomorphic copy of H_1 in $h(\mathrm{H}_1)$, a contradiction. □

The following two lemmas generalise Propositions 3.3 and 3.5.

Proposition 3.7 (General Case I). *Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first* k *of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$*, respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$*. Assume that* H_0*,* $(\mathrm{H}_1, \mathrm{H}_0), \ldots, (\mathrm{H}_k, \mathrm{H}_{k-1})$ *are endo-trivial and that*

$$\begin{array}{rcl} \mathrm{Spill}_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\ \mathrm{Spill}_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\ \vdots & \vdots & \vdots \\ \mathrm{Spill}_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k). \end{array}$$

Moreover, assume that H_{k+1} *retracts to* H_k *and also to every isomorphic copy* $\mathrm{H}'_k = i(\mathrm{H}_k)$ *of* H_k *in* H_{k+1} *with* $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}'_k, i(\mathrm{HC}_k)]) = V(\mathrm{H}_{k+1})$*. Then* H_k*-Retraction can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}_{k+1})$*.*

Proof. Let $a_{k+1}, \ldots, a_0$ be the cardinalities of $|V(\mathrm{H}_{k+1})|, \ldots, |V(\mathrm{H}_0|)$, respectively. Let $n = a_{k+1}$. Let G be an instance of H_k-Retraction. We will build an instance φ of $\mathrm{QCSP}(\mathrm{H}_{k+1})$ in the following fashion. First, take a copy of H_{k+1} together with G and build G′ by identifying these on the copy of H_k that they both possess as an induced subgraph.

Consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H}_{k+1})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H}_{k+1})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H}_{k+1})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d_{k+1} and H^d_k etc. be the diagonal copies of G'^d, H_{k+1} and H_k in $\mathcal{G}'$. Let $\mathcal{H}_{k+1}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}_{k+1}) \times \cdots \times V(\mathrm{H}_{k+1})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}_{k+1}$. Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of $\mathrm{Cyl}^*_{a_k}$ for every vertex $v \in V(\mathcal{H}_{k+1}) \setminus V(\mathrm{H}^d_k)$. Vertex v is to be identified with any vertex in the top copy of DC_{a_k} in $\mathrm{Cyl}^*_{a_k}$ and the bottom copy of DC_{a_k} is to be identified with HC_k in H^d_k according to the identity function.

Then, for each $i \in [k]$, and $v \in V(\mathrm{H}^d_i) \setminus V(\mathrm{H}^d_{i-1})$, add a copy of $\mathrm{Cyl}^*_{a_{i-1}}$, where v is identified with any vertex in the top copy of $\mathrm{DC}^*_{a_{i-1}}$ in $\mathrm{Cyl}^*_{a_{i-1}}$ and the bottom copy of DC^*_{i-1} is to be identified with H_{i-1} according to the identity map of $\mathrm{DC}^*_{a_{i-1}}$ to HC_{i-1}.

Finally, build φ from the canonical query of $\mathcal{G}''$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables.

First suppose that G retracts to H_k. Let λ be some assignment of the universal variables of φ to H_{k+1}. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H_{k+1} that extends λ and for this it suffices to prove that there is a homomorphism from $\mathcal{G}'$ that extends λ. Let us explain why. We map the various copies of $\mathrm{Cyl}^*_{a_{i-1}}$ in G″ in any suitable fashion, which will always exist due to our assumptions and the fact that $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) = V(\mathrm{H}_{k+1})$, which follows from our assumption that H_{k+1} retracts to H_k and Lemma 3.2.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H_{k+1} that extends λ. Note that it will be sufficient to prove that G' retracts to H_{k+1}. Let h be the natural retraction from G' to H_{k+1} that extends the known retraction from G to H_k. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$. Choose some surjection for λ, the assignment of the universal variables of φ to H_{k+1}. Let $N = |V(\mathrm{H}_{k+1})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ induces a surjective homomorphism s from $\mathcal{G}'$ to H_{k+1} which contains within it a surjective homomorphism s' from $\mathcal{H} = \mathrm{H}_{k+1}^N$ to H_{k+1}. Consider the diagonal copy of $\mathrm{H}_0^d \subset \cdots \subset \mathrm{H}_k^d \subset \mathrm{H}_{k+1}^d \subset G'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$, we could follow the chain of spills to deduce that $|s'(\mathrm{H}_{k+1}^d)| = 1$, which is not possible by Lemma 2.1. Moreover, $1 < |s'(H_0^d)| < |V(H_0^d)|$ is impossible due to Lemma 2.7. Now we will work exclusively on the diagonal copy G'^d.

Thus, $|s'(\mathrm{H}_0^d)| = |V(\mathrm{H}_0^d)|$ and indeed s' maps H_0^d to an isomorphic copy of itself in H_{k+1} which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$. We now apply Lemma 3.6 as well as our assumed endo-trivialities to derive that s' in fact maps H_k^d by the isomorphism i to a copy of itself in H_{k+1} which we will call H_k'. Since s' is surjective, we can deduce that $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k', i(\mathrm{HC}_k^d)]) = V(\mathrm{H}_{k+1})$ in the same way as in the proof of Proposition 3.3. and so there exists a retraction r from H_{k+1} to H_k'. Now $i^{-1} \circ r \circ s'$ gives the desired retraction of G to H_k. □

Proposition 3.8 (General Case II). *Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first* $k+1$ *of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$*, respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$*. Suppose that* H_0*,* $(\mathrm{H}_1, \mathrm{H}_0), \ldots, (\mathrm{H}_k, \mathrm{H}_{k-1}), (\mathrm{H}_{k+1}, \mathrm{H}_k)$ *are endo-trivial and that*

$$\begin{array}{lcl}
\mathrm{Spill}_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\
\mathrm{Spill}_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\
\quad\vdots & \vdots & \vdots \\
\mathrm{Spill}_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k) \\
\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) & = & V(\mathrm{H}_{k+1})
\end{array}$$

Then H_{k+1}-Retraction *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}_{k+1})$*.*

Proof. Let $n = a_{k+1} = |V(\mathrm{H}_{k+1})|$ and let $a_k, \ldots, a_0$ be the cardinalities of $|V(\mathrm{H}_k)|, \ldots, |V(\mathrm{H}_0)|$, respectively. Let G be an instance of H_{k+1}-Retraction. We build an instance φ of $\mathrm{QCSP}(\mathrm{H}_{k+1})$ in the following fashion. Consider all possible functions $\lambda : [n] \to V(\mathrm{H}_{k+1})$. For some such λ, let $\mathcal{G}(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H}_{k+1})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H}_{k+1})^{[n]}} \mathcal{G}(\lambda)$. Let $\mathrm{G}^d, \mathrm{H}_{k+1}^d, \mathrm{H}_k^d, \ldots, \mathrm{H}_0^d$ be the diagonal copies of $\mathrm{G}, \mathrm{H}_{k+1}, \mathrm{H}_k, \ldots, \mathrm{H}_0$ in $\mathcal{G}$. Let $\mathcal{H}_{k+1}$ be the subgraph of $\mathcal{G}$ induced by $V(\mathrm{H}_{k+1}) \times \cdots \times V(\mathrm{H}_{k+1})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}_{k+1}$.

Build $\mathcal{G}'$ from $\mathcal{G}$ by first augmenting a new copy of $\mathrm{Cyl}_{a_k}^*$ for every vertex $v \in V(\mathcal{H}_{k+1}) \setminus V(\mathrm{H}_k^d)$. Vertex v is to be identified with any vertex in the top copy of DC_{a_k} in $\mathrm{Cyl}_{a_k}^*$ and the bottom copy of DC_{a_k} is to be identified with HC_k in H_k^d according to the identity function. Now, for each $i \in [k]$, and $v \in V(\mathrm{H}_i^d) \setminus V(\mathrm{H}_{i-1}^d)$, we add a copy of $\mathrm{Cyl}_{a_{i-1}}^*$, where v is identified with any vertex in the top copy of $\mathrm{DC}_{a_{i-1}}^*$ in $\mathrm{Cyl}_{a_{i-1}}^*$ and the bottom copy of DC_{i-1}^* is to be identified with H_{i-1}^d according to the identity map of $\mathrm{DC}_{a_{i-1}}^*$ to HC_{i-1}^d.

Finally, build φ from the canonical query of $\mathcal{G}'$ where we additionally turn the constants $c_1, \ldots, c_n$ to outermost universal variables.

First suppose that G retracts to H_{k+1}. Let h be a retraction from G to H_{k+1}. Let λ be some assignment of the universal variables of φ to H_{k+1}. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}'$ to H_{k+1} that extends λ and for this it suffices to prove that there is a homomorphism from $\mathcal{G}$ that extends λ. The extension of the latter to the former will always be possible due to the spill assumptions.

Henceforth let us consider the homomorphic image of $\mathcal{G}$ that is $\mathcal{G}(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}(\lambda)$ to H_{k+1} that extends λ. Note that it will be sufficient to prove that G retracts to H_{k+1}. Well this was our original assumption so we are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$. Choose some surjection for λ, the assignment of the universal variables of φ to H_{k+1}. Let $N = |V(\mathrm{H}_{k+1})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H}_{k+1})$ induces a surjective homomorphism s from $\mathcal{G}$ to H_{k+1} which contains within it a surjective homomorphism s' from $\mathcal{H}_{k+1} = \mathrm{H}_{k+1}^N$ to H_{k+1}. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}_1^d \subset \cdots H_{k+1}^d$ in $\mathcal{G}$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}'$, we have $|s'(\mathrm{H}^d)| = 1$. Now we follow the chain of spills to deduce that $|s'(\mathcal{H}_{k+1})| = 1$, a contradiction. We now apply Lemma 3.6 as well as our assumed endo-trivialities to derive that s' in fact maps H_k^d by the isomorphism i to a copy of itself in H_{k+1}, which we will call H_k'. Now we can deduce, via Lemma 3.4, that $s'(\mathrm{H}_{k+1}^d)$ is an automorphism of H_{k+1}, which we call α. The required retraction from G to H_{k+1} is now given by $\alpha^{-1} \circ s'$. □

Corollary 3.9. *Let* H *be a non-trivial strongly connected reflexive tournament. Then* QCSP(H) *is* NP*-hard.*

Proof. As H is a strongly connected reflexive tournament, which has more than one vertex by our assumption, H is not transitive. Note that H-Retraction is NP-complete (see Section 4.5 in [15], using results from [5, 14, 16]). Thus, if H is endo-trivial, the result follows from Proposition 3.3 (note that we could also have used Corollary 2.8).

Suppose H is not endo-trivial. Then, by Lemma 2.4, H is not retract-trivial either. This means that H has a non-trivial retraction to some subtournament H_0. We may assume that H_0 is endo-trivial, as otherwise we will repeat the argument until we find a retraction from H to an endo-trivial (and consequently strongly connected) subtournament.

Suppose that H retracts to all isomorphic copies $\mathrm{H}_0' = i(\mathrm{H}_0)$ of H_0 within it, except possibly those for which $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) \neq V(\mathrm{H})$. Then the result follows from Proposition 3.3. So there is a copy $\mathrm{H}_0' = i(\mathrm{H}_0)$ to which H does not retract for which $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) = V(\mathrm{H})$. If $(\mathrm{H}, \mathrm{H}_0')$ is endo-trivial, the result follows from Proposition 3.5. Thus we assume $(\mathrm{H}, \mathrm{H}_0')$ is not endo-trivial and we deduce the existence of $\mathrm{H}_0' \subset \mathrm{H}_1 \subset \mathrm{H}$ (H_1 is strictly between H and H_0') so that $(\mathrm{H}_1, \mathrm{H}_0')$ and H_0' are endo-trivial and H retracts to H_1. Now we are ready to break out. Either H retracts to all isomorphic copies of $\mathrm{H}_1' = i(\mathrm{H}_1)$ in H, except possibly for those so that $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_1', i(\mathrm{HC}_1)]) \neq V(\mathrm{H})$, and we apply Proposition 3.7, or there exists a copy H_1', with $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_1', i(\mathrm{HC}_1)]) = V(\mathrm{H})$, to which it does not retract. If $(\mathrm{H}, \mathrm{H}_1')$ is endo-trivial, the result follows from Proposition 3.8. Otherwise we iterate the method, which will terminate because our structures are getting strictly smaller. □

3.4 An initial strongly connected component that is non-trivial

Let H^+ denote any reflexive tournament that has an initial strongly connected component H that is non-trivial (not of size 1). Let Cyl_m^{*+} be Cyl_m^* but with a pendant out-edge hanging from the top-most cycle. This edge is directed to the vertex x. Thus, Cyl_m^{*+} contains one additional vertex to Cyl_m^* and this has an incoming edge from some vertex in the top-most cycle DC_m^* (it does not matter which one). Cyl_m^{*+} is drawn in Figure 4.

	Strongly connected	Initial component strongly connected
Graph	H	H^{+}
Gadget	Cyl_m^{*}	Cyl_m^{*+}
Subgraph (strongly connected)	H_0	H_0
Hamilton cycle	HC_0	HC_0
Spill	$\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0,\mathrm{HC}_0))$	$\mathrm{Spill}_m^{+}(\mathrm{H}^{+}[\mathrm{H}_0,\mathrm{HC}_0])$

Table 2. Mapping notation from the strongly connected case to the case in which there is an initial strongly connected component that is non-trivial.

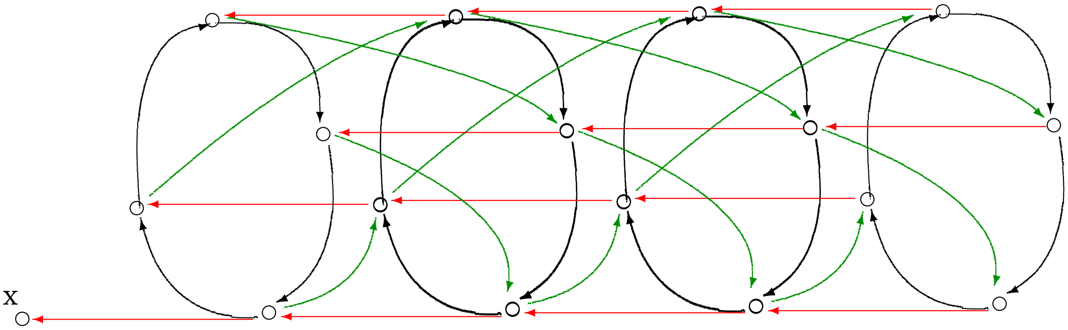

Fig. 4. The gadget Cyl_m^{*+} in the case $m := 4$ (self-loops are not drawn). We usually visualise the right-hand copy of DC_4^{*} as the "bottom" copy and then we talk about vertices "above" and "below" according to the red arrows. The vertex x is depicted at the left-hand extremity.

Define Spill_m^{+} as Spill_m but with respect to Cyl_m^{*+} instead of Cyl_m^{*}. At this point we risk confusion with our overburdened notation. Let us address in Table 2 how our notation maps from the strongly connected case to that in which there is an initial strongly connected component that is non-trivial.

Note that Lemma 3.1, with Cyl_m^{*} replaced by Cyl_m^{*+}, does not hold.

LEMMA 3.10. *Let* H^{+} *be some reflexive tournament that has an initial strongly connected component* H *that is non-trivial and contains endo-trival* H_0 *with Hamilton cycle* HC_0. *Suppose* $\mathrm{Spill}_m^{+}(\mathrm{H}[\mathrm{H}_0,\mathrm{HC}_0]) = V(\mathrm{H})$, *then* $\mathrm{Spill}_m^{+}(\mathrm{H}^{+}[\mathrm{H}_0,\mathrm{HC}_0]) = V(\mathrm{H}^{+})$.

PROOF. We only need to argue for the $x \in \mathrm{H}^{+} \setminus H$. In this case, we may evaluate all the cycles in Cyl_m^{*+} onto HC_0 with each vertex mapping to the one directly beneath it. This works as x is forward-adjacent from every vertex in HC_0. ☐

The condition of endo-triviality of H_0 was not used in the proof of Lemma 3.10.

PROPOSITION 3.11 (BASE CASE A-I.). *Let* H^{+} *be some reflexive tournament that has an initial strongly connected component* H *that is non-trivial and contains endo-trivial* H_0 *with Hamilton cycle* HC_0. *Assume that* H *retracts to* H_0' *for every isomorphic copy* $\mathrm{H}_0' = i(\mathrm{H}_0)$ *of* H_0 *in* H *with* $\mathrm{Spill}_m^{+}(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0)]) = V(\mathrm{H})$. *Then* H_0-RETRACTION *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}^{+})$.

PROOF. Let m be the size of $|V(\mathrm{H}_0)|$ and n be the size of $|V(\mathrm{H})|$. Let G be an instance of H_0-RETRACTION. We build an instance φ of $\mathrm{QCSP}(\mathrm{H}^{+})$ in the following fashion. First, take a copy of H together with G and build G′ by identifying these on the copy of H_0 that they both possess as an induced subgraph.

Now, consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d and H_0^d be the diagonal copies of G′, H and H_0 in $\mathcal{G}'$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}) \times \cdots \times V(\mathrm{H})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of Cyl_m^{*+} for every vertex $v \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$. Vertex v is to be identified with the vertex x that is at the end of the out-edge pendant on the top copy of DC_m^* in Cyl_m^{*+} and the bottom copy of DC_m^* is to be identified with HC_0 in H_0^d according to the identity function. Call these *the* Cyl_m^{*+} *of the second stage.*

Now build $\mathcal{G}'''$ by adding an edge from each vertex c_i to a new vertex d_i (for each $i \in [n]$). Now add a copy of Cyl_m^{*+} for every vertex $v \in \{d_1, \ldots, d_n\}$. Vertex v is to be identified with the vertex x that is at the end of the out-edge pendant on the top copy of DC_m^* in Cyl_m^{*+} and the bottom copy of DC_m^* is to be identified with HC_0 in H_0^d according to the identity function. Call these *the* Cyl_m^{*+} *of the third stage.*

Finally, build φ from the canonical query of $\mathcal{G}'''$, where we additionally turn the vertices $d_1, \ldots, d_n$ to outermost universal variables $z_1, \ldots, z_n$. Then existentially quantify all remaining constants and vertices innermost. Finally, restrict all except the universal variables to be in $V(\mathrm{H})$, appealing to the definition guaranteed by Corollary 2.3.

We claim that G retracts to H_0 if and only if $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$.

First suppose that G retracts to H_0 by r. Let λ' be some assignment of the universal variables $z_1, \ldots, z_n$ of φ to H^+ and choose $y_1, \ldots, y_n$ backwards-adjacent to these in H, mapped by λ. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H^+ that extends λ and for this it suffices to prove that there is a homomorphism h from $\mathcal{G}'$ to H that extends λ. Let us explain why. Because H retracts to H_0, we have $\mathrm{Spill}_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$ due to Lemma 3.2 which implies the weaker $\mathrm{Spill}_m^+(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$. For the Cyl_m^{*+} of the second stage, the weaker statement suffices, but for the Cyl_m^{*+} of the third stage, the stronger statement is needed.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H that extends λ. Note that it will be sufficent to prove that G′ retracts to H. Let h be the natural retraction from G′ to H that extends the known retraction r from G to H_0. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$. Choose some surjection for λ' mapping $z_1, \ldots, z_n$ to H. Choose some $y_1, \ldots, y_n$ backwards-adjacent to these and let this be the map λ. Note that it is not possible for all $y_1, \ldots, y_n$ to be evaluated as a single vertex as the initial strongly connected component is non-trivial.

The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H})$ induces a non-trivial homomorphism s from $\mathcal{G}''$ to H which contains within it a non-trivial homomorphism s' from $\mathcal{H} = \mathrm{H}^N$ to H. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}^d \subset \mathrm{G}'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal.

If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$, we have that $s'(\mathrm{H}^d)$ is an in-star (that is, a single terminal vertex receiving an edge from potentially numerous initial vertices), but this is not possible as H^d is strongly connected. As $1 < |s'(\mathrm{H}_0^d)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(\mathrm{H}_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$ for some isomorphism i.

We claim that $\mathrm{Spill}_m^+(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$. Since λ' is surjective on H^+, this is enforced explicitly by the Cyl_m^{*+} of the third stage. As $\mathrm{Spill}_m^+(\mathrm{H}[\mathrm{H}_0', i(\mathrm{HC}_0^d)]) = V(\mathrm{H})$, we find, by assumption of the lemma, that there exists a retraction r from H^d to H_0'. Now $i^{-1} \circ r \circ s'$ is the desired retraction of G to H_0. □

Proposition 3.12 (Base Case A-II). *Let* H^+ *be some reflexive tournament that has an initial strongly connected component* H *that is non-trivial and contains* H_0 *with Hamilton cycle* HC_0 *so that* $(\mathrm{H}, \mathrm{H}_0)$ *and* H_0 *are endo-trivial and* $\mathrm{Spill}^+_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$. *Then* H-Retraction *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}^+)$.

Proof. Let m be the size of $|V(\mathrm{H}_0)|$ and n be the size of $|V(\mathrm{H})|$. Let G be an instance of H-Retraction. We build an instance φ of $\mathrm{QCSP}(\mathrm{H}^+)$ in the following fashion. Consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H})$. For some such λ, let $\mathcal{G}(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H})^{[n]}} \mathcal{G}(\lambda)$. Let G^d, H^d and H_0^d be the diagonal copies of G, H and H_0 in $\mathcal{G}$. Let $\mathcal{H}$ be the subgraph of $\mathcal{G}$ induced by $V(\mathrm{H}) \times \cdots \times V(\mathrm{H})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}$. Now build $\mathcal{G}'$ from $\mathcal{G}$ by augmenting a new copy of Cyl^{*+}_m for every vertex $v \in V(\mathcal{H}) \setminus V(\mathrm{H}_0^d)$. Vertex v is to be identified with the vertex x that is at the end of the out-edge pendant on the top copy of DC^*_m in Cyl^{*+}_m and the bottom copy of DC^*_m is to be identified with HC_0 in H_0^d according to the identity function.

Now build $\mathcal{G}''$ by adding an edge from each vertex c_i to a new vertex d_i (for each $i \in [n]$).

Finally, build φ from the canonical query of $\mathcal{G}''$, where we additionally turn the vertices $d_1, \ldots, d_n$ to outermost universal variables $z_1, \ldots, z_n$. Then existentially quantify all remaining constants and vertices innermost. Finally, restrict all except the universal variables to be in $V(\mathrm{H})$.

First suppose that G retracts to H by r. Let λ' be some assignment of the universal variables $z_1, \ldots, z_n$ of φ to H^+ and choose $y_1, \ldots, y_n$ backwards-adjacent to these in H, mapped by λ.

To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$ it suffices to prove that there is a homomorphism from $\mathcal{G}'$ to H^+ that extends λ and for this it suffices to prove that there is a homomorphism h from $\mathcal{G}$ to H that extends λ. Let us explain why. By assumption, we have $\mathrm{Spill}^+_m(\mathrm{H}[\mathrm{H}_0, \mathrm{HC}_0]) = V(\mathrm{H})$.

Henceforth let us consider the homomorphic image of $\mathcal{G}$ that is $\mathcal{G}(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$ it suffices to prove that there is a homomorphism from $\mathrm{G}(\lambda)$ to H that extends λ. Note that it will be sufficient to prove that G retracts to H. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}^+)$. Choose some surjection for λ' mapping $z_1, \ldots, z_n$ to H. Choose some $y_1, \ldots, y_n$ backwards-adjacent to these (and therefore in H) and let this be the map λ. Note that it is not possible for all $y_1, \ldots, y_n$ to be evaluated as a single vertex as H is strongly connected. Recall $N = |V(\mathrm{H})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H})$ induces a non-trivial homomorphism s from $\mathcal{G}'$ to H which contains within it a non-trivial homomorphism s' from $\mathcal{H} = \mathrm{H}^N$ to H. Consider the diagonal copy of $\mathrm{H}_0^d \subset \mathrm{H}^d \subset \mathrm{G}^d$ in $\mathcal{G}$. By abuse of notation we will also consider each of s and s' acting just on the diagonal. If $|s'(\mathrm{H}_0^d)| = 1$, by construction of $\mathcal{G}''$ with the Cyl^{*+}_m, we have $s'(\mathrm{H}^d)$ is an in-star, but this is not possible as H^d is strongly connected. As $1 < |s'(\mathrm{H}_0^d)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(\mathrm{H}_0^d)| = m$, and indeed s' maps H_0^d to a copy of itself in H which we will call $\mathrm{H}_0' = i(\mathrm{H}_0^d)$ for some isomorphism i.

As $(\mathrm{H}, \mathrm{H}_0)$ is endo-trivial, Lemma 3.4 tells us that the restriction of s' to H^d is an automorphism of H^d, which we call α. The required retraction from G to H is now given by $\alpha^{-1} \circ s'$. □

It remains to generalise these base cases.

Proposition 3.13 (General Case A-I). *Let* H^+_{k+1} *be some reflexive tournament that has an initial strongly connected component* H_{k+1}. *Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first k of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$, *respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$.

Assume that H_0, $(\mathrm{H}_1, \mathrm{H}_0), \ldots, (\mathrm{H}_k, \mathrm{H}_{k-1})$ *are endo-trivial and that*

$$\begin{array}{lcl} \mathrm{Spill}^+_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\ \mathrm{Spill}^+_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\ \quad\vdots & \vdots & \quad\vdots \\ \mathrm{Spill}^+_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k). \end{array}$$

Moreover, assume that H_{k+1} *retracts to* H_k *and also to every isomorphic copy* $\mathrm{H}'_k = i(\mathrm{H}_k)$ *of* H_k *in* H_{k+1} *with* $\mathrm{Spill}^+_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}'_k, i(\mathrm{HC}_k)]) = V(\mathrm{H}_{k+1})$. *Then* H_k-Retraction *can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}^+_{k+1})$.

Proof. Let $n = a_{k+1} = |V(\mathrm{H}_{k+1})|$ and let $a_k, \ldots, a_0$ be the cardinalities of $|V(\mathrm{H}_k)|, \ldots, |V(\mathrm{H}_0)|$, respectively. Let G be an instance of H_k-Retraction. We will build an instance φ of $\mathrm{QCSP}(\mathrm{H}^+_{k+1})$ in the following fashion. First, take a copy of H_{k+1} together with G and build G′ by identifying these on the copy of H_k that they both possess as an induced subgraph.

Consider all possible functions $\lambda : [n] \longrightarrow V(\mathrm{H}_{k+1})$. For some such λ, let $\mathcal{G}'(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H}_{k+1})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G}' = \bigotimes_{\lambda \in V(\mathrm{H}_{k+1})^{[n]}} \mathcal{G}'(\lambda)$. Let G'^d, H^d_{k+1} and H^d_k etc. be the diagonal copies of G′, H_{k+1} and H_k in $\mathcal{G}'$. Let $\mathcal{H}_{k+1}$ be the subgraph of $\mathcal{G}'$ induced by $V(\mathrm{H}_{k+1}) \times \cdots \times V(\mathrm{H}_{k+1})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}_{k+1}$.

Now build $\mathcal{G}''$ from $\mathcal{G}'$ by augmenting a new copy of $\mathrm{Cyl}^{*+}_{a_k}$ for every vertex $v \in V(\mathcal{H}_{k+1}) \setminus V(\mathrm{H}^d_k)$. Vertex v is to be identified with the vertex x that is at the end of the out-edge pendant on the top copy of DC_{a_k} in $\mathrm{Cyl}^{*+}_{a_k}$ and the bottom copy of DC_{a_k} is to be identified with HC_k in H^d_k according to the identity function. Call these *the* $\mathrm{Cyl}^{*+}_{a_k}$ *of the second stage*. Then, for each $i \in [k]$, and $v \in V(\mathrm{H}^d_i) \setminus V(\mathrm{H}^d_{i-1})$, add a copy of $\mathrm{Cyl}^{*+}_{a_{i-1}}$, where v is identified with the vertex x that is at the end of the out-edge pendant on the top copy of $\mathrm{DC}^*_{a_{i-1}}$ in $\mathrm{Cyl}^{*+}_{a_{i-1}}$ and the bottom copy of DC^*_{i-1} is to be identified with H_{i-1} according to the identity map of $\mathrm{DC}^*_{a_{i-1}}$ to HC_{i-1}.

Now build $\mathcal{G}'''$ by adding an edge from each vertex c_i to a new vertex d_i (for each $i \in [n]$). Now add a copy of $\mathrm{Cyl}^{*+}_{a_k}$ for every vertex $v \in \{d_1, \ldots, d_n\}$. Vertex v is to be identified with the vertex x that is at the end of the out-edge pendant on the top copy of DC_{a_k} in $\mathrm{Cyl}^{*+}_{a_k}$ and the bottom copy of DC_{a_k} is to be identified with HC_k in H^d_k according to the identity function. Call these *the* $\mathrm{Cyl}^{*+}_{a_k}$ *of the third stage.*

Finally, build φ from the canonical query of $\mathcal{G}'''$, where we additionally turn the vertices $d_1, \ldots, d_n$ to outermost universal variables $z_1, \ldots, z_n$. Then existentially quantify all remaining constants and vertices innermost. Finally, restrict all except the universal variables to be in $V(\mathrm{H})$.

First suppose that G retracts to H_k by r. Let λ' be some assignment of the universal variables $z_1, \ldots, z_n$ of φ to H^+_{k+1} and choose $y_1, \ldots, y_n$ backwards-adjacent to these in H_{k+1}, mapped by λ. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}''$ to H^+_{k+1} that extends λ and for this it suffices to prove that there is a homomorphism h from $\mathcal{G}'$ that extends λ. Let us explain why. Because H_{k+1} retracts to H_k, we have $\mathrm{Spill}_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) = V(\mathrm{H}_{k+1})$ due to Lemma 3.2 which implies the weaker $\mathrm{Spill}^+_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) = V(\mathrm{H}_{k+1})$. For the $\mathrm{Cyl}^{*+}_{a_k}$ of the second stage, the weaker statement suffices, but for the $\mathrm{Cyl}^{*+}_{a_k}$ of the third stage, the stronger statement is needed. We continue mapping now the various copies of $\mathrm{Cyl}^{*+}_{a_{i-1}}$ in G″ in any suitable fashion, which will always exist due to our assumptions.

Henceforth let us consider the homomorphic image of $\mathcal{G}'$ that is $\mathcal{G}'(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ it suffices to prove that there is a homomorphism from $\mathrm{G}'(\lambda)$ to H_{k+1} that extends λ. Note that it will be sufficient to prove that G' retracts to H_{k+1}. Let h be the natural retraction from G' to H_{k+1} that extends the known retraction r from G to H_k. We are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$. Choose some surjection for λ, the assignment of the universal variables of φ to H_{k+1}. Choose some $y_1, \ldots, y_n$ backwards-adjacent to these (and therefore in H_{k+1}) and let this be the map λ. Note that it is not possible for all $y_1, \ldots, y_n$ to be evaluated as a single vertex as H_{k+1} is strongly connected. Let $N = |V(\mathrm{H}_{k+1})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ induces a non-trivial homomorphism s from $\mathcal{G}'$ to H_{k+1} which contains within it a non-trivial homomorphism s' from $\mathcal{H} = \mathrm{H}^N_{k+1}$ to H_{k+1}. Consider the diagonal copy of $\mathrm{H}^d_0 \subset \cdots \subset \mathrm{H}^d_k \subset \mathrm{H}^d_{k+1} \subset G'^d$ in $\mathcal{G}'$. By abuse of notation we will also consider each of s and s' acting just on the diagonal.

If $|s'(\mathrm{H}^d_0)| = 1$, by construction of $\mathcal{G}''$, we have that $s'(\mathrm{H}^d_1)$ is either an in-star or a loop, but the former is not possible as H^d_1 is strongly connected. Iterating this argument we find that $|s'(\mathrm{H}^d_{k+1})| = 1$, but this would mean s' is uniformly mapping $\mathcal{H}_{k+1}$ to one vertex, which is impossible as s' is non-trivial. As $1 < |s'(\mathrm{H}^d_0)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(\mathrm{H}^d_0)| = m$, and indeed s' maps H^d_0 to a copy of itself in H which we will call $\mathrm{H}'_0 = i(\mathrm{H}^d_0)$ for some isomorphism i.

We now apply Lemma 3.6 as well as our assumed endo-trivialities to derive that s' in fact maps H^d_k by the isomorphism i to a copy of itself in H_{k+1} which we will call H'_k.

We claim that $\mathrm{Spill}^+_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}'_{k+1}, i(\mathrm{HC}^d_{a_k})]) = V(\mathrm{H}_{k+1})$. Since λ' is surjective on H^+_{k+1}, this is enforced explicitly by the $\mathrm{Cyl}^{*+}_{a_k}$ of the third stage. Thus, there exists a retraction r from H_{k+1} to H'_k. Now $i^{-1} \circ r \circ s'$ gives the desired retraction of G to H_k. ☐

Proposition 3.14 (General Case A-II). *Let* H^+_{k+1} *be some reflexive tournament that has an initial strongly connected component* H_{k+1} *that is non-trivial. Let* $\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_k, \mathrm{H}_{k+1}$ *be reflexive tournaments, the first* $k+1$ *of which have Hamilton cycles* $\mathrm{HC}_0, \mathrm{HC}_1, \ldots, \mathrm{HC}_k$*, respectively, so that* $\mathrm{H}_0 \subseteq H_1 \subseteq \cdots \subseteq \mathrm{H}_k \subseteq \mathrm{H}_{k+1}$*. Suppose that* H_0*,* $(\mathrm{H}_1, \mathrm{H}_0), \ldots, (\mathrm{H}_k, \mathrm{H}_{k-1}), (\mathrm{H}_{k+1}, \mathrm{H}_k)$ *are endo-trivial and that*

$$\begin{array}{lcl}
\mathrm{Spill}^+_{a_0}(\mathrm{H}_1[\mathrm{H}_0, \mathrm{HC}_0]) & = & V(\mathrm{H}_1) \\
\mathrm{Spill}^+_{a_1}(\mathrm{H}_2[\mathrm{H}_1, \mathrm{HC}_1]) & = & V(\mathrm{H}_2) \\
\vdots & \vdots & \vdots \\
\mathrm{Spill}^+_{a_{k-1}}(\mathrm{H}_k[\mathrm{H}_{k-1}, \mathrm{HC}_{k-1}]) & = & V(\mathrm{H}_k) \\
\mathrm{Spill}^+_{a_k}(\mathrm{H}_{k+1}[\mathrm{H}_k, \mathrm{HC}_k]) & = & V(\mathrm{H}_{k+1})
\end{array}$$

Then H_{k+1}*-Retraction can be polynomially reduced to* $\mathrm{QCSP}(\mathrm{H}^+_{k+1})$*.*

Proof. Let $n = a_{k+1} = |V(\mathrm{H}_{k+1})|$ and let $a_k, \ldots, a_0$ be the cardinalities of $|V(\mathrm{H}_k)|, \ldots, |V(\mathrm{H}_0|$, respectively. Let G be an instance of H_{k+1}-Retraction. We build an instance φ of $\mathrm{QCSP}(\mathrm{H}^+_{k+1})$ in the following fashion. Consider all possible functions $\lambda : [n] \to V(\mathrm{H}_{k+1})$. For some such λ, let $\mathcal{G}(\lambda)$ be the graph enriched with constants $c_1, \ldots, c_n$ where these are interpreted over some subset of $V(\mathrm{H}_{k+1})$ according to λ in the natural way (acting on the subscripts).

Let $\mathcal{G} = \bigotimes_{\lambda \in V(\mathrm{H}_{k+1})^{[n]}} \mathcal{G}(\lambda)$. Let $\mathrm{G}^d, \mathrm{H}^d_{k+1}, \mathrm{H}^d_k, \ldots, \mathrm{H}^d_0$ be the diagonal copies of $\mathrm{G}, \mathrm{H}_{k+1}, \mathrm{H}_k, \ldots, \mathrm{H}_0$ in $\mathcal{G}$. Let $\mathcal{H}_{k+1}$ be the subgraph of $\mathcal{G}$ induced by $V(\mathrm{H}_{k+1}) \times \cdots \times V(\mathrm{H}_{k+1})$. Note that the constants $c_1, \ldots, c_n$ live in $\mathcal{H}_{k+1}$.

Now build $\mathcal{G}'$ from $\mathcal{G}$ by the following procedure. For each $i \in [k+1]$, and $v \in V(\mathrm{H}^d_i) \setminus V(\mathrm{H}^d_{i-1})$, add a copy of $\mathrm{Cyl}^{*+}_{a_{i-1}}$, where v is identified with the vertex x that is at the end of the out-edge pendant on the top copy of $\mathrm{DC}^*_{a_{i-1}}$ in $\mathrm{Cyl}^{*+}_{a_{i-1}}$ and the bottom copy of DC^*_{i-1} is to be identified with H_{i-1} according to the identity map of $\mathrm{DC}^*_{a_{i-1}}$ to HC_{i-1}.

Now build $\mathcal{G}''$ by adding an edge from each vertex c_i to a new vertex d_i (for each $i \in [n]$).

Finally, build φ from the canonical query of $\mathcal{G}''$, where we additionally turn the vertices $d_1, \ldots, d_n$ to outermost universal variables $z_1, \ldots, z_n$. Then existentially quantify all remaining constants and vertices innermost. Finally, restrict all except the universal variables to be in $V(\mathrm{H}_{k+1})$.

First suppose that G retracts to H_{k+1} by r. Let λ' be some assignment of the universal variables $z_1, \ldots, z_n$ of φ to H^+_{k+1} and choose $y_1, \ldots, y_n$ backwards-adjacent to these in H_{k+1}, mapped by λ. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}'$ to H^+_{k+1} that extends λ and for this it suffices to prove that there is a homomorphism h from $\mathcal{G}$ that extends λ. The extension of the latter to the former will always be possible due to the spill assumptions.

Henceforth let us consider the homomorphic image of $\mathcal{G}$ that is $\mathcal{G}(\lambda)$. To prove $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ it suffices to prove that there is a homomorphism from $\mathcal{G}(\lambda)$ to H_{k+1} that extends λ. Note that it will be sufficient to prove that G retracts to H_{k+1}. Well this was our original assumption so we are done.

Suppose now $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$. Choose some surjection for λ' mapping $z_1, \ldots, z_n$ to H_{k+1}. Choose some $y_1, \ldots, y_n$ backwards-adjacent to these (and therefore in H_{k+1}) and let this be the map λ. Note that it is not possible for all $y_1, \ldots, y_n$ to be evaluated as a single vertex as H_{k+1} is strongly connected. Recall $N = |V(\mathrm{H})^{[n]}|$. The evaluation of the existential variables that witness $\varphi \in \mathrm{QCSP}(\mathrm{H}^+_{k+1})$ induces a non-trivial homomorphism s from $\mathcal{G}$ to H_{k+1} which contains within it a non-trivial homomorphism s' from $\mathcal{H}_{k+1} = \mathrm{H}^N_{k+1}$ to H_{k+1}. Consider the diagonal copy of $\mathrm{H}^d_0 \subset \mathrm{H}^d_1 \subset \cdots H^d_{k+1}$ in $\mathcal{G}$. By abuse of notation we will also consider each of s and s' acting just on the diagonal.

If $|s'(\mathrm{H}^d_0)| = 1$ we deduce that $s'(\mathrm{H}^d_1)$ is either an in-star or a loop, but the former is not possible as H^d_1 is strongly connected. Iterating this argument we find that $|s'(\mathrm{H}^d_{k+1})| = 1$, but this would mean s' is uniformly mapping to one vertex, which is impossible as s' is non-trivial. As $1 < |s'(\mathrm{H}^d_0)| < m$ is not possible either due to Lemma 2.7, we find that $|s'(\mathrm{H}^d_0)| = m$, and indeed s' maps H^d_0 to a copy of itself in H which we will call $\mathrm{H}'_0 = i(\mathrm{H}^d_0)$ for some isomorphism i.

We now apply Lemma 3.6 as well as our assumed endo-trivialities to derive that s' in fact maps H^d_k by the isomorphism i to a copy of itself in H_{k+1}, which we will call H'_k. Now we can deduce, via Lemma 3.4, that $h(\mathrm{H}^d_{k+1})$ is an automorphism of H_{k+1}, which we call α. The required retraction from G to H_{k+1} is now given by $\alpha^{-1} \circ s'$. □

The proof of the following is exactly as that for Corollary 3.9 modulo Spill becoming Spill$^+$.

Corollary 3.15. *Let* H *be a reflexive tournament with an initial strongly connected component that is non-trivial. Then* QCSP(H) *is* NP*-hard.*

4 THE PROOF OF THE NL CASES OF THE DICHOTOMY

A particular role in the tractable part of our dichotomy will be played by TT^*_2, the reflexive transitive 2-tournament, which has vertex set $\{0, 1\}$ and edge set $\{(0, 0), (0, 1), (1, 1)\}$.

Lemma 4.1. *Let* $\mathrm{H} = \mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ *be a reflexive tournament on* $m+2$ *vertices with* $V(\mathrm{H}_1) = \{s\}$ *and* $V(\mathrm{H}_n) = \{t\}$. *Then there exists a surjective homomorphism from* $(\mathrm{TT}^*_2)^m$ *to* H.

Proof. Build a surjective homomorphism f from $(\mathrm{TT}^*_2)^m$ to H in the following fashion. Let $\overline{x}_i$ be the m-tuple which has 1 in the ith position and 0 in all other positions. For $i \in [m]$, let f map $\overline{x}_i$ to i. Let f map $(0, \ldots, 0)$ to s and everything remaining to t.

By construction, f is surjective. To see that f is a homomorphism, let $((y_1, \ldots, y_m), (z_1, \ldots, z_m)) \in E((\mathrm{TT}^*_2)^m)$, which is the case exactly when $y_i \leq z_i$ for all $i \in [m]$. Let $f(y_1, \ldots, y_m) = u$ and $f(z_1, \ldots, z_m) = v$. First suppose that $y_1, \ldots, y_m$ are all 0. Then $u = s$. As s has an out-edge to every vertex of H, we find that $(u, v) \in E(\mathrm{H})$. Now suppose that $y_1, \ldots, y_m$ contains a single 1. If $(y_1, \ldots, y_m) = (z_1, \ldots, z_m)$, then $u = v$. As H is reflexive, we find that $(u, v) \in \mathrm{H}$. If $(y_1, \ldots, y_m) \neq$

$(z_1, \ldots, z_m)$, then $v = t$. As t has an in-edge from every vertex of H, we find that $(u, v) \in E(\mathrm{H})$. Finally suppose that $y_1, \ldots, y_m$ contains more than one 1. Then $u = v = t$. As H is reflexive, we find that $(u, v) \in E(\mathrm{H})$. □

We also need the following lemma, which follows from combining some known results.

Lemma 4.2. *If* H *is a transitive reflexive tournament then* QCSP(H) *is in* NL.

Proof. It is noted in [15] that H has the ternary median operation as a polymorphism. It follows from well-known results (e.g. in [7, 9]) that QCSP(H) is in NL. Specifically, one can apply Theorem 5.16 from [7] to reduce QCSP(H) to an ensemble of instances of CSP(H), which may also reference constants, each of which can be solved in NL by Corollary 4 from [9]. Each of these instances may be solved independently and the ensemble is polynomial in number, hence the whole procedure can be accomplished in NL. □

The other tractable cases are more interesting.

We are now ready to prove the main result of this section.

Theorem 4.3. *Let* $\mathrm{H} = \mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ *be a reflexive tournament. If* $|V(H_1)| = |V(H_n)| = 1$, *then* QCSP(H) *is in* NL.

Proof. Let $|V(\mathrm{H})| = m + 2$ for some $m \geq 0$. By Lemma 4.1, there exists a surjective homomorphism from $(\mathrm{TT}_2^*)^m$ to H. There exists also a surjective homomorphism from H to TT_2^*; we map s to 0 and all other vertices of H to 1. It follows from Theorem 3.4 in [8] that $\mathrm{QCSP(H)} = \mathrm{QCSP}(\mathrm{TT}_2^*)$ meaning we may consider the latter problem. We note that TT_2^* is a transitive reflexive tournament. Hence, we may appply Lemma 4.2. □

5 FINAL RESULT AND REMARKS

We are now in a position to prove our main dichotomy theorem.

Theorem 5.1. *Let* $\mathrm{H} = \mathrm{H}_1 \Rightarrow \cdots \Rightarrow \mathrm{H}_n$ *be a reflexive tournament. If* $|V(H_1)| = |V(H_n)| = 1$, *then* QCSP(H) *is in* NL; *otherwise it is* NP*-hard.*

Proof. The NL case follow from Theorem 4.3. The NP-hard cases follow from Corollary 3.9 and Corollary 3.15, bearing in mind the case with a non-trivial final strongly connected component is dual to the case with a non-trivial initial strongly connected component (map edges (x, y) to (y, x)). □

Theorem 5.1 resolved the open case in Table 1. It is difficult to position this result in the overall classification program for finite-domain QCSPs save to say that our methods are tailored, indeed specialised, to reflexive tournaments. It is not clear that they can be applied easily to different or wider classes (in this vein we return to mixed-type tournaments below). Since complexities outside of P, NP-complete and Pspace-complete were discovered for QCSPs in [25], for example co-NP-complete, DP-complete and Θ_2^{P}, the whole classification task has been thrown wide open. Classes such as that of reflexive tournaments might provide comfort, as it is doubtful such monstrous complexities could be found here. Though, we cannot be sure, with our lacuna between NP-hard and Pspace-complete.

Recall that the results for the irreflexive tournaments in this table were all proven in a more general setting, namely for irreflexive semicomplete graphs. One natural direction for future research is to determine a complexity dichotomy for QCSP and SCSP for reflexive semicomplete graphs. We leave this as an interesting open direction.

The task of promoting our NP-hardness results to Pspace-complete, while using the same method, seems to require corresponding Pspace-hardness results for reflexive tournaments with constants.

If QCSP^c(H) were Pspace-complete, for H a non-trivial reflexive strongly connected tournament, then likely our NP-hardness results, for the similar class of graphs, would easily rise to Pspace-complete. The cases that are not strongly connected require additional arguments, and perhaps even a different method.

Mixed-type tournaments, where some vertices are reflexive and others irreflexive, are well-understood algebraically [21]. Indeed, from this paper there follows a complexity dichotomy for CSP^c(H) where H is a mixed-type tournament. Furthermore, CSP(H) is either trivial or H is an irreflexive tournament, so the complexity dichotomy for CSP(H) is also known. Though many of our supporting lemmas hold for mixed-type tournaments, some do not. For example, Lemma 2.1 fails for the transitive 2-tournament TT_2 in which one vertex is a self-loop and the other is not. To extend our classification to mixed-type tournaments thus requires still some work.

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