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Mass-conserving stochastic partial differential equations and backward doubly stochastic differential equations

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Abstract

In this paper, we first study the connection between mass-conserving SPDEs on a bounded domain and backward doubly stochastic differential equations, which is a new extension of nonlinear Feynman-Kac formula to mass-conserving SPDEs. Then the infinite horizon mass-conserving SPDEs and their stationary solutions are considered without monotonic conditions, while the Poincare inequality plays an important role. Finally, the existence and the stationarity to solutions of non-Lipschitz mass-conserving stochastic Allen-Cahn equations are obtained.

MSC: 60H15; 60H10; 37H10

Keywords: Stochastic partial differential equations; Mass-conservative; Backward doubly stochastic differential equations; Neumann boundary condition; Stationary solution; Stochastic Allen-Cahn equation

1. Introduction

In this paper, we study a mass-conserving stochastic partial differential equations (SPDEs) on the domain $\overline{D}$:

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$$\begin{cases} dv(t,x) = [\frac{1}{2}\triangle v(t,x) + f\big(t,x,v(t,x)\big) - \int_D f\big(t,\xi,v(t,\xi)\big)\rho(d\xi)]dt \\ \quad -g(t,x)dB_t \quad t\in(0,T], \quad x\in D \\ v(0,x) = h(x) \quad x\in\overline{D} \\ \frac{\partial v}{\partial \mathbf{n}}(t,x) = 0 \quad t\in(0,T], \quad x\in\partial D. \end{cases} \tag{1.1}$$

Here D is an open connected bounded subset with C^2 boundary, with normal derivative pointing towards the interior of D, $\overline{D}$ is the closure of D, ρ is the invariant probability measure of Brownian motion in $\overline{D}$ reflected on its boundary ∂D (see (2.2)), $\rho(\partial D) = 0$, and $\{B_t : t \in \mathbb{R}\}$ is a Q-Wiener process with values in a separable Hilbert space U on a probability space $(\Omega^B, \mathscr{F}^B, P^B)$.

Denote by $\{e_i\}_{i=1}^{+\infty}$ the countable basis of U. Then $Q \in L(U)$ is a symmetric nonnegative trace class operator such that $Qe_i = \lambda_i e_i$ and $\sum_{i=1}^{+\infty} \lambda_i < +\infty$. The coefficients $h : \Omega \times \overline{D} \longrightarrow \mathbb{R}$; $f : [0,T] \times \overline{D} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $v \mapsto f(\cdot,\cdot,v)$ is a locally Lipschitz continuous function; $g : [0,T] \times \overline{D} \longrightarrow \mathcal{L}^2_{U_0}(\mathbb{R})$ is a Lipschitz continuous Hilbert-Schmidt operator and satisfies $\int_D g(t,x)\rho(dx) = 0$ for any $t \in [0,T]$. Here $U_0 = Q^{\frac{1}{2}}(U) \subset U$ is a separable Hilbert space with the norm $< u, v >_{U_0} = < Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v >_U$ and the complete orthonormal base $\{\sqrt{\lambda_i} e_i\}_{i=1}^{+\infty}$ and $\mathcal{L}^2_{U_0}(\mathbb{R})$ is the space of all Hilbert-Schmidt operators from U_0 to $\mathbb{R}$ with the Hilbert-Schmidt norm. It is easy to see that under the above setup the solution of SPDE (1.1), if exists, satisfies a mass-conservative condition, i.e., $\int_D v(t,x)\rho(dx) = \int_D h(x)\rho(dx)$ remains as a constant.

Mass-conservation phenomenon occurs often in reality. For example, in physics the concentration of one type of metal in an alloy where the dynamics is constrained to have constant total magnetization or mass, the evolution of the rescaled concentration could be a mass-conservation dynamics. As indicated in Antonopoulou, Bates, Blömker and Karali [1], as the average concentration is close to being a pure state, a phase separation begins by nucleation. In the case of the mass being close to zero, when the two components are roughly equal, as with the Cahn-Hilliard equation, the total mass of each component of the mixture is also conserved but separation from a nearly homogeneous state occurs during spinodal decomposition. Mass-conservation of incompressive flow is another example as described by Navier-Stokes equations.

However, there are only very few mathematical works concerning with mass-conservation phenomena of stochastic Allen-Cahn equation. Recently, deterministic mass-conserving Allen-Cahn equations in global dynamics of boundary droplets was studied in Bates and Jin [5]. For the more complicated case under random forcing, stochastic mass-conserving Allen-Cahn equations were discussed in Antonopoulou, Bates, Blömker and Karali [1] and the metastable dynamics of a discretized version of mass-conserving stochastic Allen-Cahn equations was studied in Berglund and Dutercq [6]. As far as we know, this is a first result for the existence of stationary solutions and invariant measures of stochastic mass-conserving Allen-Cahn equations.

In this paper, we concern with the solvability of a kind of mass-conserving SPDEs with locally Lipschitz coefficients such as mass-conserving stochastic Allen-Cahn equation. We connect mass-conserving SPDEs with backward doubly stochastic differential equations (BDSDEs). The connection is an extension of nonlinear Feynman-Kac formula explored in Peng [17] to the SPDEs case. An extension was initialed by Pardoux and Peng [15] for parabolic SPDEs with Lipschitz coefficients. Then the connection for SPDE with non-smooth or even non-Lipschitz coefficients was further studied in Bally and Matoussi [4], Buckdahn and Ma [9], Zhang and Zhao [18–20], to name but a few. The nonlinear Feynman-Kac formula is first extended to represent viscosity solution of PDE with Neumann condition in Pardoux and Zhang [16] and then to stochastic viscosity solution of SPDEs in Boufoussi, Casteren and Mrhardy [7]. But there is an essential difficulty to represent weak solution of PDE or SPDE with Neumann condition due to the lack of regularities of stochastic flows. To overcome this, we use the invariant measure of the reflected Brownian motion on a bounded domain.

Moreover, $\int_D f\big(t,\xi,v(t,\xi)\big)\rho(d\xi)$ in the drift term of SPDE (1.1) can also be regarded as a mean-filed integral with respect to the spacial variables. There are existing results e.g. Buckdahn, Li and Peng [8] to study the correspondence between mean-field BSDE and PDE, in which the mean-field term is the mean over random samples given by the integration with respect to the probability measure. As far as we know, there do not exist results for BSDEs or BDSDEs with a space variable mean term, which gives the mass-conservation along almost every sample path.

To study the stationary solution of the mass-conserving SPDE, we first investigate the corresponding infinite horizon mass-conserving BDSDE with Lipschitz coefficient. We take the advantage of the Poincare inequality to control the first variable of BDSDE by the second variable. This method also works for non-Lipschitz weakly dissipative mass-conserving stochastic Allen-Cahn equations, where $f\big(t,x,v(t,x)\big)=-v^p(t,x)+v(t,x)$ in (1.1), with $p>1$ being an odd integer. In this way, we remove the assumption of monotonic condition required in all the earlier work of infinite horizon BSDEs/BDSDEs. As far as we know, this is the first time that Poincare inequality is applied in the study of BSDE/BDSDE which allows to deal with weakly dissipative generator. The stationary solution gives the equilibrium of the stochastic systems in terms of SPDEs or BDSDEs which represents the large time limit and infinite horizon mass-conserving dynamics in the pathwise sense. This is of interests to real world physical situations. Needless to say, as the law of stationary solution is an invariant measure which is an equilibrium of the stochastic systems in the statistical sense. So we automatically obtain the existence of the invariant measure. It is worth mentioning here that our result is stronger than the existence of invariant measure result as merely the latter does not imply the result of this paper on the existence of a pathwise stationary path.

The rest of this paper is organized as follows. In Section 2, the existence and uniqueness of finite horizon mass-conserving SPDEs with Lipschitz coefficients and their corresponding BDSDEs are proved. The infinite horizon mass-conserving SPDEs and BDSDEs are then considered in Section 3. In Section 4, the stationary solutions of mass-conserving SPDEs are constructed. Finally, mass-conserving stochastic Allen-Cahn equation is studied in Section 5, where the locally Lipschitz coefficient is approximated by a sequence of Lipschitz coefficients and the crucial analysis of passing limit is carried out.

2. Finite horizon BDSDEs and SPDEs

Let $(\Omega,\mathscr{F},P)$ be a complete probability space, $\{\hat{B}_t: t\in\mathbb{R}\}$ and $\{W_t: t\in\mathbb{R}\}$ be mutually independent a Q-Wiener process valued on U and a standard Brownian motion on $\mathbb{R}^d$ on $(\Omega,\mathscr{F},P)$. Since we will construct a metric dynamical system to study the stationarity of mass-conserving SPDEs and BDSDEs, so both Brownian motions $\hat{B}$ and W used in this paper are defined on the time interval $(-\infty,+\infty)$. See Arnold [2] for more details about the Brownian motion on the negative horizon as an independent copy of a Brownian motion on the positive horizon. Two parameter filtration of probability space with two sided Wiener precess was studied in [2]. We outline here for completeness. Let θ be the shift of Brownian motions as defined in (4.10) later and then $(\Omega,\mathscr{F},P,\{\theta_t: t\in\mathbb{R}\})$ is a metric dynamical system. For $\eta=\hat{B}$ or W, let $\mathscr{F}^{\eta-}$ and $\mathscr{F}^{\eta+}$ be two sub-σ-algebras (representing "past" and "future", respectively) both containing all P-null sets of $\mathscr{F}$ such that

$$\theta_t^{-1}\mathscr{F}^{\eta-}\subset\mathscr{F}^{\eta-}\ \text{ for all } t\le 0 \ \text{ and } \ \theta_t^{-1}\mathscr{F}^{\eta+}\subset\mathscr{F}^{\eta+}\ \text{ for all } t\ge 0.$$

Define

$$\mathscr{F}^{\eta}_{-\infty,s}=\theta_s^{-1}\mathscr{F}^{\eta-},\quad \mathscr{F}^{\eta}_{t,+\infty}=\theta_t^{-1}\mathscr{F}^{\eta+},\quad \mathscr{F}^{\eta}_{t,s}=\mathscr{F}^{\eta}_{-\infty,s}\wedge\mathscr{F}^{\eta}_{t,+\infty}\quad \text{for } t\le s.$$

It is easy to see that $\theta_r^{-1}\mathscr{F}^{\eta}_{t,s}=\mathscr{F}^{\eta}_{t+r,s+r}$. Set now

$$\mathscr{F}_{t,T}=\mathscr{F}^{\hat{B}}_{t,T}\vee\mathscr{F}^{W}_{0,t}\ \text{ for } 0\le t\le T\ \text{ and } \ \mathscr{F}_{t}=\mathscr{F}^{\hat{B}}_{t,+\infty}\vee\mathscr{F}^{W}_{0,t}\ \text{ for } t\ge 0.$$

We first study the following backward SPDE

$$\begin{cases} du(t,x)=-[\frac{1}{2}\Delta u(t,x)+f\big(t,x,u(t,x)\big)-\int_D f\big(t,\xi,u(t,\xi)\big)\rho(d\xi)]dt\\ \quad +g(t,x)d^{\dagger}\hat{B}_t\quad t\in[0,T),\quad x\in D\\ u(T,x)=h(x)\quad x\in\overline{D}\\ \frac{\partial u}{\partial\mathbf{n}}(t,x)=0\quad t\in[0,T)\quad x\in\partial D. \end{cases}\tag{2.1}$$

Here D is an open connected bounded subset in $\mathbb{R}^d$, which is defined as $D=\{\phi>0\}$, $\partial D=\{\phi=0\}$, and normal $|\nabla\phi(x)|=1$ when $x\in\partial D$ for a function $\phi\in C_b^2(\mathbb{R}^d)$, $\overline{D}$ is the closure of D, $h:\Omega\times\overline{D}\longrightarrow\mathbb{R}$, $f:[0,T]\times\overline{D}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $g:[0,T]\times\overline{D}\longrightarrow\mathcal{L}^2_{U_0}(\mathbb{R})$ are measurable. The stochastic integral with $d^{\dagger}\hat{B}_s$ is a backward stochastic integral which is a particular case of Itô-Skorohod integral. See Nualart and Pardoux [14] for more details. If we take Brownian motion $\hat{B}_t=B_{T'-t}-B_{T'}$, where B is as in equation (1.1), then backward and forward stochastic integrals are connected in the following way ([18]): for any stochastic process G with values in $\mathcal{L}^2_{U_0}(\mathbb{R})$ such that $G(s)$ is $\mathscr{F}_s$-measurable for all $s\in[t,T]$ and locally square integrable,

$$\int_t^T G(s)d^{+}\hat{B}_s=-\int_{T'-T}^{T'-t}G(T'-s)dB_s.$$

Define $(X_s^{t,x},K_s^{t,x})$ to be the solution of the following stochastic differential equations for any given $t\ge 0$ and $x\in\mathbb{R}^d$:

$$\begin{cases} X_s^{t,x}=x+W_s-W_t+\int_t^s\nabla\phi(X_r^{t,x})dK_r^{t,x},\quad s\ge t,\\ K_s^{t,x}=\int_t^s I_{\{X_r^{t,x}\in\partial D\}}dK_r^{t,x},\quad K_\cdot^{t,x}\ is\ increasing. \end{cases}\tag{2.2}$$

Here $\{W_t:t\in\mathbb{R}\}$ is a Brownian motion on $\mathbb{R}^d$ on a probability space $(\Omega^W,\mathscr{F}^W,P^W)$, and by E^W we denote the expectation with respect to P^W. For the existence of invariant measure ρ for $X_\cdot^{t,x}$ in $\overline{D}$ we refer the reader to [12].

We will consider the infinite horizon SPDEs and BSDEs and their solutions give the equilibrium of corresponding SPDEs. They are stationary paths and random fixed points which can be regarded as the solutions of the initial /terminal value problems for forward/backward SPDEs (1.1)/(2.1), where the initial/terminal values are in the stationary paths at a different realization. See e.g. (4.10) and (5.20) for details. Thus we study SPDE (1.1) and SPDE (2.1) with h being a random variable. Here h is assumed to be measurable with respect to the filtration generated by driven Brownian motions. The exact measurability will be made in Condition (H.1) later.

Under our assumption on ϕ following equation (2.1) at the beginning of this section, it is well-known that SDE (2.2) has a unique continuous and adapted solution $(X_s^{t,x}, K_s^{t,x})$ with values in $\overline{D}$ for $s \in [t, +\infty)$ (Lions and Sznitman [13]).

Denote by $L^2_\rho(D;\mathbb{R})$ the space of measurable functions $l : \overline{D} \longrightarrow \mathbb{R}$ such that $\int_D l^2(x)\rho(dx) < +\infty$. Define the inner product by

$$\langle l_1, l_2\rangle = \int_D l_1(x) l_2(x)\rho(dx), \quad l_1, l_2 \in L^2_\rho(D;\mathbb{R}),$$

then $L^2_\rho(D;\mathbb{R})$ is a Hilbert space. Similarly, we denote by $L^k_\rho(D;\mathbb{R})$, $k \geq 2$, the weighted L^k space with the norm $||l||_{L^k_\rho(D)} = \Big(\int_D |l(x)|^k \rho(dx)\Big)^{\frac{1}{k}}$.

Definition 2.1. ([18]) Let $\mathbb{S}$ be a separable Banach space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$ and $q \geq 2$, $K > 0$. We denote by $M^{q,-K}([t,+\infty);\mathbb{S})$ the set of $\mathscr{B}([t,+\infty)) \otimes \mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{s\geq t}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s) : \Omega \longrightarrow \mathbb{S}$ is $\mathscr{F}_s$ measurable for $s \geq t$;
(ii) $E[\int_t^{+\infty} \mathrm{e}^{-Ks}\|\psi(s)\|^q_{\mathbb{S}} ds] < +\infty$.

Also we denote by $S^{q,-K}([t,+\infty);\mathbb{S})$ the set of $\mathscr{B}([t,+\infty)) \otimes \mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{s\geq t}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s) : \Omega \longrightarrow \mathbb{S}$ is $\mathscr{F}_s$ measurable for $s \geq t$ and $\psi(\cdot,\omega)$ is continuous a.s.;
(ii) $E[\sup_{s\geq t} \mathrm{e}^{-Ks}\|\psi(s)\|^q_{\mathbb{S}}] < +\infty$.

If we replace time interval $[t,+\infty)$ by $[t,T]$ in the above definition, we denote the spaces by $M^{q,0}([t,T];\mathbb{S})$ and $S^{q,0}([t,T];\mathbb{S})$, respectively. Note that here e^{-Ks} does not play any role as T is finite, so we can always take $K = 0$. When we consider the probability space $(\Omega^{\hat{B}}, \mathscr{F}^{\hat{B}}, P^{\hat{B}})$ and natural filtration $\{\mathscr{F}^{\hat{B}}_t : t \in \mathbb{R}\}$, we denote the above spaces by $M^{\cdot,\cdot}_{\hat{B}}(\cdot;\cdot)$ and $S^{\cdot,\cdot}_{\hat{B}}(\cdot;\cdot)$, respectively.

Definition 2.2. A process u is called a weak solution of SPDE (2.1) if $(u, \nabla u) \in M^{2,0}_{\hat{B}}([0,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,0}_{\hat{B}}([0,T]; L^2_\rho(D;\mathbb{R}^d))$ and for an arbitrary $\varphi \in C^\infty(D;\mathbb{R})$,

$$\int_D u(t,x)\varphi(x)\rho(dx) - \int_D h(x)\varphi(x)\rho(dx) - \frac{1}{2}\int_t^T \int_D \nabla u(s,x)\nabla\varphi(x)\rho(dx)ds$$

$$= \int_t^T \int_D [f\big(s,x,u(s,x)\big) - \int_D f\big(s,\xi,u(s,\xi)\big)\rho(d\xi)]\varphi(x)\rho(dx)ds$$

$$-\int_t^T\int_D g(s,x)\varphi(x)\rho(dx)d^\dagger\hat{B}_s \quad P-\text{a.s.} \tag{2.3}$$

The BDSDE associated with SPDE (2.1) is

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T[f(r,X_r^{t,x},Y_r^{t,x})-\int_D f(r,\xi,Y_r^{r,\xi})\rho(d\xi)]dr$$
$$-\int_s^T g(r,X_r^{t,x})d^\dagger\hat{B}_r-\int_s^T Z_r^{t,x}dW_r,\quad 0\le t\le s\le T. \tag{2.4}$$

Definition 2.3. A pair of processes $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2,0}([t,T);L_\rho^2(D;\mathbb{R}))\times M^{2,0}([t,T);L_\rho^2(D;\mathbb{R}^d))$ is called a solution of BDSDE (2.4) if $(Y_s^{t,x},Z_s^{t,x})$ satisfies (2.4) for all $t\le s\le T$ a.a. $x\in\overline{D}$ a.s.

Assume the following conditions:

(H.1). The function h is $\mathscr{F}_{T,+\infty}^{\hat{B}}\times\mathscr{B}_{\overline{D}}$ measurable and $E[\int_D|h(x)|^2\rho(dx)]<+\infty$.

(H.2). The functions $f(\cdot,\cdot,0)$ and $g(\cdot,\cdot)$ satisfy $\int_0^T\int_D(|f(s,x,0)|^2+\|g(s,x)\|^2)\rho(dx)ds<+\infty$.

(H.3). There exists a constant $L\ge 0$ such that for any $s\in[0,+\infty)$, $x\in\overline{D}$, $y_1,y_2\in\mathbb{R}$,

$$|f(s,x,y_1)-f(s,x,y_2)|\le L|y_1-y_2|.$$

(H.4). For any given $s\in[0,+\infty)$, $g(s,\cdot)\in C^2(\overline{D};\mathcal{L}_{U_0}^2(\mathbb{R}))$ and $\int_D g(s,x)\rho(dx)=0$.

Remark 2.4. We start from SPDE (2.1) to have

$$\int_D u(t,x)\rho(dx)=\int_D h(x)\rho(dx)+\int_t^T\int_D[\frac{1}{2}\Delta u(s,x)+f\big(s,x,u(s,x)\big)$$
$$-\int_D f\big(s,\xi,u(s,\xi)\big)\rho(d\xi)]\rho(dx)ds-\int_D\int_t^T g(s,x)d^\dagger\hat{B}_s\rho(dx)$$
$$=\int_D h(x)\rho(dx)+\int_t^T\int_D\frac{1}{2}\Delta u(s,x)\rho(dx)ds.$$

That is to say

$$\frac{d}{dt}\int_D u(t,x)\rho(dx)=-\int_D\frac{1}{2}\Delta u(t,x)\rho(dx)=0,$$

by Stoke's theorem. So there exists a $\mathscr{F}^{\hat{B}}_{T,+\infty}$ measurable random variable $c(\omega)$ on $(\Omega^B, \mathscr{F}^B, P^B)$ such that for all $t \in [0, T]$ and $\omega \in \Omega$,

$$\int_D u(t,x)\rho(dx) = \int_D h(x)\rho(dx) = c(\omega). \tag{2.5}$$

Hence, in view of the invariant measure, the solution of finite horizon BDSDE (2.4) also satisfies

$$E^W[\int_D Y_s^{t,x}\rho(dx)] = c(\omega) \quad \text{for all } s \geq t. \tag{2.6}$$

Theorem 2.5. *Under Conditions (H.1)–(H.4), BDSDE (2.4) has a unique solution* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$.

Proof. We define a sequence of BDSDEs,

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T [f(r, X_r^{t,x}, Y_r^{t,x,n-1}) - \int_D f(r,\xi,Y_r^{r,\xi,n-1})\rho(d\xi)]dr \\ &\quad - \int_s^T g(r, X_r^{t,x})d^\dagger \hat{B}_r - \int_s^T Z_r^{t,x,n}dW_r, \quad 0 \leq t \leq s \leq T, \quad n \in \mathbb{N}, \end{aligned} \tag{2.7}$$

and $(Y_\cdot^{t,x,0}, Z_\cdot^{t,x,0}) = (0,0)$.

For a given $Y_\cdot^{t,x,n-1} \in S^{2,0}([t,T]; L^2_\rho(D;\mathbb{R}))$, from Conditions (H.2) and (H.3), we know that

$$E[\int_t^T \int_D |f(r, X_r^{t,x}, Y_r^{t,x,n-1}) - \int_D f(r,\xi,Y_r^{r,\xi,n-1})\rho(d\xi)|^2 \rho(dx)dr] < +\infty.$$

So by applying the result in [18], there exists $(Y_\cdot^{t,x,n}, Z_\cdot^{t,x,n}) \in S^{2,0}([t,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,0}([t,T]; L^2_\rho(D;\mathbb{R}^d))$ for (2.7). We need to prove that the sequence of $(Y_\cdot^{t,x,n}, Z_\cdot^{t,x,n})$ is a Cauchy sequence in this space. For this, we apply Itô's formula to $\mathrm{e}^{Kr}|Y_r^{t,x,n} - Y_r^{t,x,n-1}|^2$ to have

$$\begin{aligned} &\mathrm{e}^{Ks}|Y_r^{t,x,n} - Y_r^{t,x,n-1}|^2 + K\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n} - Y_r^{t,x,n-1}|^2 dr + \int_s^T \mathrm{e}^{Kr}|Z_r^{t,x,n} - Z_r^{t,x,n-1}|^2 dr \\ &= 2\int_s^T \mathrm{e}^{Kr}(Y_r^{t,x,n} - Y_r^{t,x,n-1})[f(r, X_r^{t,x}, Y_r^{t,x,n-1}) - f(r, X_r^{t,x}, Y_r^{t,x,n-2})]dr \\ &\quad -2\int_s^T \mathrm{e}^{Kr}(Y_r^{t,x,n} - Y_r^{t,x,n-1})\int_D [f(r,\xi,Y_r^{r,\xi,n-1}) - f(r,\xi,Y_r^{r,\xi,n-2})]\rho(d\xi)dr \end{aligned}$$

$$
\begin{aligned}
&-2\int_s^T \mathrm{e}^{Kr}(Y_r^{t,x,n}-Y_r^{t,x,n-1})(Z_r^{t,x,n}-Z_r^{t,x,n-1})dW_r\\
&\leq 2L\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}||Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|dr\\
&\quad+2L\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|\int_D |Y_r^{r,\xi,n-1}-Y_r^{r,\xi,n-2}|\rho(d\xi)dr\\
&\quad-2\int_s^T \mathrm{e}^{Kr}(Y_r^{t,x,n}-Y_r^{t,x,n-1})(Z_r^{t,x,n}-Z_r^{t,x,n-1})dW_r\\
&\leq 4L^2\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2dr+\frac{1}{4}\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2dr\\
&\quad+4L^2\int_s^T \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2dr+\frac{1}{4}\int_s^T \mathrm{e}^{Kr}\int_D |Y_r^{r,\xi,n-1}-Y_r^{r,\xi,n-2}|^2\rho(d\xi)dr\\
&\quad-2\int_s^T \mathrm{e}^{Kr}(Y_r^{t,x,n}-Y_r^{t,x,n-1})(Z_r^{t,x,n}-Z_r^{t,x,n-1})dW_r.
\end{aligned}
$$

Taking integration in D, by stochastic Fubini theorem we have

$$
\begin{aligned}
&\int_D \mathrm{e}^{Ks}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)+K\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr\\
&\quad+\int_s^T\int_D \mathrm{e}^{Kr}|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2\rho(dx)dr\\
&\leq 8L^2\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr+\frac{1}{2}\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr\\
&\quad-2\int_s^T\int_D \mathrm{e}^{Kr}(Y_r^{t,x,n}-Y_r^{t,x,n-1})(Z_r^{t,x,n}-Z_r^{t,x,n-1})\rho(dx)dW_r. \qquad (2.8)
\end{aligned}
$$

Next taking expectations leads to

$$E[\int_D \mathrm{e}^{Ks}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)]+(K-8L^2)E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr]$$
$$+E[\int_s^T\int_D \mathrm{e}^{Kr}|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2\rho(dx)dr]$$
$$\leq \frac{1}{2}E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr].$$

Setting $K=8L^2+1$, it follows that

$$E[\int_s^T\int_D \mathrm{e}^{Kr}(|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2+|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2)\rho(dx)dr]$$
$$\leq \frac{1}{2}E[\int_s^T\int_D \mathrm{e}^{Kr}(|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2+|Z_r^{t,x,n-1}-Z_r^{t,x,n-2}|^2)\rho(dx)dr]. \tag{2.9}$$

From the contraction principle, we know that the mapping (2.7) has a pair of fixed point $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})$ which is the limit of the Cauchy sequence $\{(Y_\cdot^{t,\cdot,n},Z_\cdot^{t,\cdot,n})\}_{n=1}^{+\infty}$ in $M^{2,K}([t,T];L^2_\rho(D;\mathbb{R}))\times M^{2,K}([t,T];L^2_\rho(D;\mathbb{R}^d))$. We now prove that $Y_\cdot^{t,\cdot}$ is also the limit of $Y_\cdot^{t,\cdot,n}$ in $S^{2,K}([t,T];L^2_\rho(D;\mathbb{R}))$ as $n\to+\infty$. For this, we only need to prove that $\{Y_\cdot^{t,\cdot,n}\}_{n=1}^{+\infty}$ is a Cauchy sequence in $S^{2,K}([t,T];L^2_\rho(D;\mathbb{R}))$. Noticing (2.8) again, by B-D-G inequality, Cauchy-Schwarz inequality and Young's inequality we have

$$E[\sup_{t\leq s\leq T}\int_D \mathrm{e}^{Ks}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)]$$
$$\leq CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr]+CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr]$$
$$+CE[\sqrt{\int_s^T\left(\int_D \mathrm{e}^{Kr}(Y_r^{t,x,n}-Y_r^{t,x,n-1})(Z_r^{t,x,n}-Z_r^{t,x,n-1})\rho(dx)\right)^2dr}]$$
$$\leq CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr]+CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr]$$
$$+CE[\sqrt{\int_s^T[\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)\int_D \mathrm{e}^{Kr}|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2\rho(dx)]dr}]$$

$$\leq CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr]+CE[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr]$$
$$+\frac{1}{2}E[\sup_{t\leq s\leq T}\int_D \mathrm{e}^{Ks}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)]+CE[\int_s^T\int_D \mathrm{e}^{Kr}|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2\rho(dx)dr].$$

Here and in the rest of this paper, C is a constant whose values depend on given parameters and may change from line by line. Hence

$$E[\sup_{t\leq s\leq T}\int_D \mathrm{e}^{Ks}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)]$$
$$\leq \tilde{C}\big\{E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n}-Y_r^{t,x,n-1}|^2\rho(dx)dr]+E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,n-1}-Y_r^{t,x,n-2}|^2\rho(dx)dr]$$
$$+E[\int_s^T\int_D \mathrm{e}^{Kr}|Z_r^{t,x,n}-Z_r^{t,x,n-1}|^2\rho(dx)dr]\big\}, \tag{2.10}$$

where $\tilde{C}$ is a constant depending only on $|\mu|$, K and the fixed constant in the B-D-G inequality. Thus for any $m,n\in\mathbb{N}$, without loss of any generality assuming $m\geq n$, by (2.9) and (2.10) we have

$$\sqrt{E[\sup_{t\leq s\leq T}\int_D \mathrm{e}^{Ks}|Y_r^{t,x,m}-Y_r^{t,x,n}|^2\rho(dx)]}$$
$$\leq \sum_{i=n+1}^{m}\sqrt{E[\sup_{t\leq s\leq T}\int_D \mathrm{e}^{Ks}|Y_r^{t,x,i}-Y_r^{t,x,i-1}|^2\rho(dx)]}$$
$$\leq \sum_{i=n+1}^{m}\Big(\tilde{C}\big\{E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,i}-Y_r^{t,x,i-1}|^2\rho(dx)dr]$$
$$+E[\int_s^T\int_D \mathrm{e}^{Kr}|Y_r^{t,x,i-1}-Y_r^{t,x,i-2}|^2\rho(dx)dr]$$
$$+E[\int_s^T\int_D \mathrm{e}^{Kr}|Z_r^{t,x,i}-Z_r^{t,x,i-1}|^2\rho(dx)dr]\big\}\Big)^{\frac{1}{2}}$$
$$\leq \sum_{i=n+1}^{m}\sqrt{\frac{3}{2}\tilde{C}E[\int_s^T\int_D \mathrm{e}^{Kr}(|Y_r^{t,x,i-1}-Y_r^{t,x,i-2}|^2+|Z_r^{t,x,i-1}-Z_r^{t,x,i-2}|^2)\rho(dx)dr]}$$

$$\leq \sum_{i=n+1}^{+\infty}\sqrt{(\frac{1}{2})^{i-2}}\sqrt{\frac{3}{2}\tilde{C}E[\int_s^T\int_D \mathrm{e}^{Kr}(|Y_r^{t,x,1}|^2+|Z_r^{t,x,1}|^2)\rho(dx)dr]} \to 0,$$

as $m, n \to +\infty$. So $\{(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})\}_{n=1}^{+\infty}$ converges to $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ in $S^{2,K}([t,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,K}([t,T]; L^2_\rho(D;\mathbb{R}^d))$. Due to the equivalence of the norms in $S^{2,K}([t,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,K}([t,T]; L^2_\rho(D;\mathbb{R}^d))$ and in $S^{2,0}([t,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,0}([t,T]; L^2_\rho(D;\mathbb{R}^d))$, the convergence still holds in the latter space. Then taking the strong limit on both sides of (2.7) in $L^2(\Omega \times \overline{D};\mathbb{R})$, we see that $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is a solution to BDSDE (2.4).

The uniqueness of solution can be proved by Itô's formula, and we leave it to the reader. ◇

Remark 2.6. On the interval $[0, t]$, BDSDE (2.4) has a form below

$$Y_s^x = Y_t^{t,x} + \int_s^t [f(r,x,Y_r^x) - \int_D f(r,x,Y_r^x)\rho(dx)]dr - \int_s^t g(r,x)d^\dagger \hat{B}_r - \int_s^t Z_r^x dW_r. \quad (2.11)$$

Note that $Y_t^{t,x}$ satisfies Condition (H.1). By Theorem 2.5 we can obtain $(Y_\cdot^\cdot, Z_\cdot^\cdot) \in S^{2,0}([0,t]; L^2_\rho(D;\mathbb{R})) \times M^{2,0}([0,t]; L^2_\rho(D;\mathbb{R}^d))$ as the unique solution of BDSDE (2.11). To unify the notation, we define $(Y_s^{t,x}, Z_s^{t,x}) = (Y_s^x, Z_s^x)$ when $s \in [0,t)$. Thus $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,0}([0,T]; L^2_\rho(D;\mathbb{R})) \times M^{2,0}([0,T]; L^2_\rho(D;\mathbb{R}^d))$.

Theorem 2.7. *Assume Conditions (H.1)–(H.4) and let $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ be the solution of BDSDE (2.4). Then $u(t,x) = Y_t^{t,x}$ is the unique weak solution of SPDE (2.1). Moreover, $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $\nabla u(s, X_s^{t,x}) = Z_s^{t,x}$ for a.a. $s \in [t,T]$, a.a. $x \in D$, a.s.*

Proof. We first consider the affine SPDE (2.1) when $f(t,x,y)$ is independent of y. Then SPDE (2.1) becomes to a linear equation:

$$\begin{cases} du(t,x) = -[\frac{1}{2}\Delta u(t,x) + F(t,x)]dt + g(t,x)d^\dagger \hat{B}_t, \quad t \in [0,T), \quad x \in D, \\ u(T,x) = h(x), \quad x \in \overline{D}, \\ \frac{\partial u}{\partial \mathbf{n}}(t,x) = 0, \quad t \in [0,T), \quad x \in \partial D, \end{cases} \quad (2.12)$$

where

$$F(t,x) = f(t,x) - \int_D f(t,\xi)\rho(d\xi).$$

By Theorem 5.4 in [11], the mild solution of SPDE (2.12) exists and can be represented by

$$u(t,x) = S(T-t)h(x) + \int_0^{T-t} S(T-t-s)F(s,x)ds + \int_0^{T-t} S(T-t-s)g(s,x)dB_s, \quad (2.13)$$

where $S(t)$, $0 \leq t \leq T$, is the semigroup generated by $\frac{1}{2}\triangle$ and the stochastic integral is forward Itô's integral. On the other hand, from Theorem 2.10 and Remark 2.11 in [10] we know that a unique classical solution of the linear PDE when $g(t,x)=0$ in (2.12) exists, i.e. the first two terms in (2.13) are $C^{1,2}([0,T]\times D)$. For the solution of SPDE (2.12), since the stochastic integral in (2.13) is additive, by Theorem 5.15 in [11], it has the same regularity with respect to x as the PDE.

Then taking $t\leq s=t_0<t_1<t_2<\cdots<t_m=T$, by Itô's formula, we have

$$\begin{aligned}
&\sum_{i=0}^{m-1}[u(t_i,X_{t_i}^{t,x})-u(t_{i+1},X_{t_{i+1}}^{t,x})]\\
&=\sum_{i=0}^{m-1}[u(t_i,X_{t_i}^{t,x})-u(t_i,X_{t_{i+1}}^{t,x})]+\sum_{i=0}^{m-1}[u(t_i,X_{t_{i+1}}^{t,x})-u(t_{i+1},X_{t_{i+1}}^{t,x})]\\
&=\sum_{i=0}^{m-1}\Big(-\int_{t_i}^{t_{i+1}}\frac{1}{2}\triangle u(t_i,X_r^{t,x})dr-\int_{t_i}^{t_{i+1}}\nabla u(t_i,X_r^{t,x})dW_r-\int_{t_i}^{t_{i+1}}\nabla u(t_i,X_r^{t,x})\nabla\phi(X_r^{t,x})dK_r^{t,x}\Big)\\
&\quad+\sum_{i=0}^{m-1}\Big(\int_{t_i}^{t_{i+1}}\frac{1}{2}\triangle u(r,X_{t_{i+1}}^{t,x})dr+\int_{t_i}^{t_{i+1}}F(r,X_{t_{i+1}}^{t,x})dr-\int_{t_i}^{t_{i+1}}g(r,X_{t_{i+1}}^{t,x})d\hat{B}_r\Big).
\end{aligned}$$

Note that the Neumann condition implies

$$\int_{t_i}^{t_{i+1}}\nabla u(t_i,X_r^{t,x})\nabla\phi(X_r^{t,x})dK_r^{t,x}=\int_{t_i}^{t_{i+1}}\frac{\partial u}{\partial\mathbf{n}}(t_i,X_r^{t,x})dK_r^{t,x}=0.$$

Hence, by a similar argument as in [15], taking $m\to+\infty$ we have

$$u(s,X_s^{t,x})=u(T,X_T^{t,x})+\int_s^T F(r,X_r^{t,x})dr-\int_s^T g(r,X_r^{t,x})d\hat{B}_r-\int_s^T\nabla u(r,X_r^{t,x})dW_r.$$

By the uniqueness of solution of above linear BDSDE, it follows that

$$u(s,X_s^{t,x})=Y_s^{t,x}\ \text{ and }\ \nabla u(s,X_s^{t,x})=Z_s^{t,x}\ \text{ for a.a. } s\in[t,T],\ \text{a.a. } x\in D,\ \text{a.s.}$$

Moreover, the continuity of $u(s,x)$ with respect to (s,x), together with the continuity of $X_s^{t,x}$ with respect to s, leads to $u(s,X_s^{t,x})=Y_s^{t,x}$ for all $s\in[t,T]$, a.a. $x\in D$, a.s.

For the semi-linear case that $f(t,x,y)$ depending on y, define a sequence of SPDEs with $u^0=0$, and $u^n(t,x)$ defined iteratively by SPDE (2.1), but with $f\big(t,x,u(t,x)\big)$ replaced by $f\big(t,x,u^{n-1}(t,x)\big)$. By the correspondence in the linear case, we have $u^n(s,X_s^{t,x})=Y_s^{t,x,n}$ for all $s\in[t,T]$, a.a. $x\in D$, a.s. and $\nabla u^n(s,X_s^{t,x})=Z_s^{t,x,n}$ for a.a. $s\in[t,T]$, a.a. $x\in D$, a.s. Furthermore, due to the uniqueness of solution of BDSDE (2.4), $Y_s^{t,x}=Y_s^{s,X_s^{t,x}}$ and $Z_s^{t,x}=Z_s^{s,X_s^{t,x}}$. Define $u(s,x)\triangleq Y_s^{s,x}$, then

$$u(s,X_s^{t,x})=Y_s^{s,X_s^{t,x}}=Y_s^{t,x}. \tag{2.14}$$

By using the invariant measure ρ of $X_s^{0,\cdot}$,

$$\begin{aligned}&\lim_{n\to+\infty} E[\int_0^T\int_D |u^n(s,x)-u(s,x)|^2\rho(dx)ds]\\&=\lim_{n\to+\infty} E[\int_0^T\int_D |u^n(s,X_s^{0,x})-u(s,X_s^{0,x})|^2\rho(dx)ds]\\&=\lim_{n\to+\infty} E[\int_0^T\int_D |Y_s^{0,x,n}-Y_s^{0,x}|^2\rho(dx)ds]=0. \qquad (2.15)\end{aligned}$$

On the other hand, by using the invariant measure ρ again, for $m,n\to+\infty$,

$$\begin{aligned}&\lim_{n\to+\infty} E[\int_0^T\int_D \Big(|u^m(s,x)-u^n(s,x)|^2+|\nabla u^m(s,x)-\nabla u^n(s,x)|^2\Big)\rho(dx)ds]\\&=\lim_{n\to+\infty} E[\int_0^T\int_D \Big(|u^m(s,X_s^{0,x})-u^n(s,X_s^{0,x})|^2+|\nabla u^m(s,X_s^{0,x})-\nabla u^n(s,X_s^{0,x})|^2\Big)\rho(dx)ds]\\&=\lim_{n\to+\infty} E[\int_0^T\int_D \Big(|Y_s^{0,x,m}-Y_s^{0,x,n}|^2+|Z_s^{0,x,m}-Z_s^{0,x,n}|^2\Big)\rho(dx)ds]=0.\end{aligned}$$

That is to say $\{u^n\}_{n=1}^+\infty$ is a Cauchy sequence in $L^2(0,T;W_\rho^{1,2}(D;\mathbb{R}))$ and we denote its limit by $\tilde{u}$. Noticing (2.15), we know that u^n converges strongly to u in $L^2(0,T;L_\rho^2(D;\mathbb{R}))$, which implies $u(s,x)=\tilde{u}(s,x)$ for a.a. $s\in[0,T]$, a.a. $x\in\overline{D}$, a.s. If we regard u as an indistinguishable version of $\tilde{u}$, $u\in L^2(0,T;W_\rho^{1,2}(D;\mathbb{R}))$ and u^n converges to u in $L^2(0,T;W_\rho^{1,2}(D;\mathbb{R}))$. Actually, u is a weak solution of SPDE (2.1). For this, first note that for an arbitrary $\varphi\in C^{1,\infty}([0,T]\times D;\mathbb{R})$,

$$\begin{aligned}&\int_D u^{n+1}(t,x)\varphi(x)\rho(dx)-\int_D h(x)\varphi(x)\rho(dx)-\frac{1}{2}\int_t^T\int_D \nabla u^{n+1}(s,x)\nabla\varphi(x)\rho(dx)ds\\&=\int_t^T\int_D [f\big(s,x,u^n(s,x)\big)-\int_D f\big(s,\xi,u^n(s,\xi)\big)\rho(d\xi)]\varphi(x)\rho(dx)ds\\&\quad-\int_t^T\int_D g(s,x)\varphi(x)\rho(dx)d^\dagger\hat{B}_s \quad P-\text{a.s.} \qquad (2.16)\end{aligned}$$

We need to verify that each term in (2.16) converges to the corresponding term in (2.3) in the space $L^1(\Omega;\mathbb{R})$ as $n\to+\infty$ due to the strong convergence of u^n to u in $L^2(0,T;W^{1,2}_\rho(D;\mathbb{R}))$. We only show the convergence of two terms and the convergence of other terms can be similarly deduced. Firstly,

$$
\begin{aligned}
&\lim_n E[|\int_D \big(u^{n+1}(t,x)-u(t,x)\big)\varphi(x)\rho(dx)|^2]\\
&\le \lim_n CE[\int_D |u^{n+1}(t,X_t^{0,x})-u(t,X_t^{0,x})|^2\rho(dx)]\\
&=\lim_n CE[\int_D |Y_t^{0,x,n+1}-Y_t^{0,x}|^2\rho(dx)]\\
&\le \lim_n CE[\sup_{0\le t\le T}\int_D |Y_t^{0,x,n+1}-Y_t^{0,x}|^2\rho(dx)]=0.
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
&\lim_n E[|\int_t^T\int_D\Big(\int_D f\big(s,\xi,u^n(s,\xi)\big)\rho(d\xi)-f\big(s,\xi,u(s,\xi)\big)\rho(d\xi)\Big)\varphi(x)\rho(dx)ds|^2]\\
&\le \lim_n CE[\int_t^T\int_D\Big|\int_D\Big(f\big(s,\xi,u^n(s,\xi)\big)-f\big(s,\xi,u(s,\xi)\big)\Big)\rho(d\xi)\Big|^2\rho(dx)ds]\\
&\le \lim_n CE[\int_t^T\int_D |f\big(s,x,u^n(s,x)\big)-f\big(s,x,u(s,x)\big)|^2\rho(dx)ds]\\
&\le \lim_n CE[\int_t^T\int_D |u^n(s,x)-u(s,x)|^2\rho(dx)ds]=0.
\end{aligned}
$$

In this way we proved that $u(t,x)$ defined by $Y_t^{t,x}$ is the weak solution of SPDE (2.1). Moreover, due to (2.14), $u(s,X_s^{t,x})=Y_s^{t,x}$ for a.a. $s\in[t,T]$, a.a. $x\in D$, a.s. As for $\nabla u(s,X_s^{t,x})=Z_s^{t,x}$ for a.a. $s\in[t,T]$, a.a. $x\in D$, a.s., we see from

$$
\begin{aligned}
&\lim_{n\to+\infty} E[\int_0^T\int_D |\nabla u(s,X_s^{t,x})-Z_s^{t,x}|^2\rho(dx)ds]\\
&\le \lim_{n\to+\infty} 2E[\int_0^T\int_D |\nabla u(s,X_s^{t,x})-\nabla u^n(s,X_s^{0,x})|^2\rho(dx)ds]
\end{aligned}
$$

$$+\lim_{n\to+\infty} 2E[\int_0^T\int_D |\nabla u^n(s,X_s^{0,x})-Z_s^{t,x}|^2\rho(dx)ds]$$

$$\le \lim_{n\to+\infty} CE[\int_0^T\int_D |\nabla u(s,x)-\nabla u^n(s,x)|^2\rho(dx)ds]$$

$$+\lim_{n\to+\infty} 2E[\int_0^T\int_D |Z_s^{t,x,n}-Z_s^{t,x}|^2\rho(dx)ds]=0.$$

For the uniqueness of SPDE (2.1), the proof is similar to Theorem 3.1 in [4]. ◇

Proposition 2.8. *Assume Conditions (H.1)–(H.4). The solution of SPDE (2.1) has an a.s. continuous version.*

Proof. Noting that the solution of SPDE (2.1) exists in $L^2(0,T;W_\rho^{1,2}(D;\mathbb{R}))$, we regard SPDE (2.1) as a linear SPDE with Neumann condition by setting $F(t,x)\triangleq f\big(t,x,u(t,x)\big)-\int_D f\big(t,\xi,u(t,\xi)\big)\rho(d\xi)$. Since

$$\begin{aligned}
&E[\int_0^T\int_D |F(t,x)|^2\rho(dx)dt]\\
&=E[\int_0^T\int_D |f\big(t,x,u(t,x)\big)-\int_D f\big(t,\xi,u(t,\xi)\big)\rho(d\xi)|^2\rho(dx)dt]\\
&\le 2E[\int_0^T\int_D |f\big(t,x,u(t,x)\big)|^2\rho(dx)dt]+2E[\int_0^T\int_D |\int_D f\big(t,\xi,u(t,\xi)\big)\rho(d\xi)|^2\rho(dx)dt]\\
&\le 2E[\int_0^T\int_D |f\big(t,x,u(t,x)\big)|^2\rho(dx)dt]+2E[\int_0^T\int_D |f\big(t,x,u(t,x)\big)|^2\rho(dx)dt]\\
&\le 8E[\int_0^T\int_D \big(|f\big(t,x,u(t,x)\big)-f(t,x,0)|^2+|f(t,x,0)|^2\big)\rho(dx)dt]\\
&\le 8E[\int_0^T\int_D \big(C^2|u(t,x)|^2+|f(t,x,0)|^2\big)\rho(dx)dt]<+\infty,
\end{aligned}$$

by a similar argument as (2.13) again, we know that $u(t,x)$ has an a.s. continuous version. ◇

Remark 2.9. Since the solution of SPDE (2.1) has an a.s. continuous version by Proposition 2.8, following the result of Theorem 2.7 we further have $u(s, X_s^{t,x}) = Y_s^{t,x}$ for all $s \in [t, T]$, a.a. $x \in D$, a.s.

3. Infinite horizon BDSDEs

For a given $K > 0$, we consider the infinite horizon BDSDE with f and g being independent of the time variable:

$$\begin{aligned} e^{-Ks} Y_s^{t,x} = & \int_s^{+\infty} e^{-Kr} [f(X_r^{t,x}, Y_r^{t,x}) - \int_D f(\xi, Y_r^{r,\xi}) \rho(d\xi)] dr + \int_s^{+\infty} K e^{-Kr} Y_r^{t,x} dr \\ & - \int_s^{+\infty} e^{-Kr} g(X_r^{t,x}) d^\dagger \hat{B}_r - \int_s^{+\infty} e^{-Kr} Z_r^{t,x} dW_r, \end{aligned} \tag{3.1}$$

together with the condition (2.6) satisfied by $Y_\cdot^{\cdot,\cdot}$.

Definition 3.1. A pair of processes $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,-K} \bigcap M^{2,-K}([0, +\infty); L_\rho^2(D; \mathbb{R})) \times M^{2,-K}([0, +\infty); L_\rho^2(D; \mathbb{R}^d))$ is called a solution of BDSDE (3.1) if $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.6) and (3.1) for all $s \geq 0$ a.a. $x \in \overline{D}$ a.s.

In fact, in the next section, we will show that for the infinite horizon BDSDE (3.1), any stationary solution with mass conservation condition (2.6) must imply $c(\omega)$ in condition (2.6) being a constant for almost all $\omega \in \Omega$. As this section is devoted to finding the solution of BDSDE (3.1) with mass conservation condition, in the rest part of this section, we assume $c(\omega) = c$ being a constant a.s.

We need some more conditions for infinite horizon BDSDE.

(H.5). Denote by M the reciprocal of the first eigenvalue of $\triangle$ on D, $16M^2 L < 1$.
(H.6). The functions $f(\cdot, 0)$ and $g(\cdot)$ satisfy $\int_D (|f(x, 0)|^2 + \|g(x)\|^2) \rho(dx) < +\infty$.

Lemma 3.2. *Let u be the solution of SPDE (2.1) and (Y, Z) be the solution of corresponding BDSDE (2.4). Then*

$$E[\int_D |Z_s^{t,x}|^2 \rho(dx)] \geq \frac{1}{2M^2} E[\int_D |Y_s^{t,x}|^2 \rho(dx)] - \frac{E[|\int_D u(s, x) \rho(dx)|^2]}{M^2}. \tag{3.2}$$

Proof. By the Poincare inequality, it follows that

$$\begin{aligned} & \left| \sqrt{\int_D |u(s, x)|^2 \rho(dx)} - \sqrt{\int_D |\int_D u(s, \xi) \rho(d\xi)|^2 \rho(dx)} \right| \\ \leq & \sqrt{\int_D |u(s, x) - \int_D u(s, \xi) \rho(d\xi)|^2 \rho(dx)} \end{aligned}$$

$$\leq M\sqrt{\int_D |\nabla u(s,x)|^2\rho(dx)}.$$

Hence

$$\int_D |u(s,x)|^2\rho(dx) \leq 2M^2\int_D |\nabla u(s,x)|^2\rho(dx) + 2|\int_D u(s,x)\rho(dx)|^2.$$

Noticing that ρ is an invariant measure for $X_\cdot^{t,x}$ in $\overline{D}$, we have

$$E[\int_D |\nabla u(s,X_s^{t,x})|^2\rho(dx)] \geq \frac{1}{2M^2}E[\int_D |u(s,X_s^{t,x})|^2\rho(dx)] - \frac{E[|\int_D u(s,x)\rho(dx)|^2]}{M^2},$$

which implies (3.2). $\diamond$

Theorem 3.3. *Under Conditions (H.3)–(H.6), for a given sufficiently small $K > 0$ and a given constant c, BDSDE (3.1) has a unique solution $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ satisfying*

$$E^W[\int_D Y_s^{t,x}\rho(dx)] = c. \tag{3.3}$$

Proof. For each $n \in \mathbb{N}$, we define a sequence of BDSDEs by setting $h(x) = c$ and $T = n$ in BDSDE (2.4):

$$Y_s^{t,x,n} = c + \int_s^n [f(X_r^{t,x}, Y_r^{t,x,n}) - \int_D f(\xi, Y_r^{r,\xi,n})\rho(d\xi)]dr - \int_s^n g(X_r^{t,x})d^\dagger\hat{B}_r - \int_s^n Z_r^{t,x,n}dW_r. \tag{3.4}$$

Equivalently,

$$\begin{aligned}
e^{-Ks}Y_s^{t,x,n} &= e^{-Kn}c + \int_s^n e^{-Kr}[f(X_r^{t,x}, Y_r^{t,x,n}) - \int_D f(\xi, Y_r^{r,\xi,n})\rho(d\xi)]dr \\
&\quad + \int_s^n Ke^{-Kr}Y_r^{t,x,n}dr - \int_s^n e^{-Kr}g(X_r^{t,x})d^\dagger\hat{B}_r - \int_s^n e^{-Kr}Z_r^{t,x,n}dW_r.
\end{aligned} \tag{3.5}$$

It is easy to verify that BDSDE (3.4) satisfies conditions of Theorem 2.5. Hence, for each n, by Theorem 2.5 and Remark 2.6 there exists $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,-K}([0,n]; L_\rho^2(\overline{D};\mathbb{R})) \times M^{2,-K}([0,n]; L_\rho^2(D;\mathbb{R}^d))$ as the unique solution of BDSDE (3.4). Let $(Y_s^{t,x,n}, Z_s^{t,x,n})_{s>n} = (c,0)$. Then $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,-K}\bigcap M^{2,-K}([0,+\infty); L_\rho^2(D;\mathbb{R})) \times M^{2,-K}([0,+\infty); L_\rho^2(D;\mathbb{R}^d))$.

We will prove $(Y_s^{t,x,n}, Z_s^{t,x,n})$, $n = 1, 2, \cdots$, is a Cauchy sequence. For this, let $(Y_s^{t,x,m}, Z_s^{t,x,m})$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solutions of Eq. (3.4) with the terminal time m and n, respectively. Without losing any generality, assume that $m \geq n$, and define for $s \geq t$,

$$\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m} - Y_s^{t,x,n}, \quad \bar{Z}_s^{t,x,m,n} = Z_s^{t,x,m} - Z_s^{t,x,n},$$
$$\bar{f}^{t,m,n}(s,x) = f(X_s^{t,x}, Y_s^{t,x,m}) - f(X_s^{t,x}, Y_s^{t,x,n}).$$

Consider two cases:
(i) When $n \leq s \leq m$, $\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m} - c$ and we have for any $m \in \mathbb{N}$,

$$\begin{cases} d\bar{Y}_s^{t,x,m,n} = -[f(X_s^{t,x}, Y_s^{t,x,m}) - \int_D f(\xi, Y_s^{s,\xi,m})\rho(d\xi)]ds + g(X_s^{t,x})d^\dagger \hat{B}_s + Z_s^{t,x,m} dW_s \\ \bar{Y}_m^{t,x,m,n} = 0, \quad \text{for } s \in [n,m), \text{ a.a. } x \in \overline{D}, \text{ a.s.} \end{cases}$$

Applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.a. $x \in D$ and taking integration over D, we have

$$\begin{aligned}
&\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx) - K\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr + \int_s^m\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho(dx)dr \\
&= 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}[f(X_r^{t,x}, Y_r^{t,x,m}) - \int_D f(\xi, Y_r^{r,\xi,m})\rho(d\xi)]\rho(dx)dr \\
&\quad + \int_s^m\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr - 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r \\
&\quad -2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}Z_r^{t,x,m}\rho(dx)dW_r \\
&\leq 2L\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr + \frac{1}{L}\int_s^m\int_D \mathrm{e}^{-Kr}|f(X_r^{t,x}, Y_r^{t,x,m})|^2\rho(dx)dr \\
&\quad + \frac{1}{L}\int_s^m\int_D \mathrm{e}^{-Kr}|f(x, Y_r^{r,x,m})|^2\rho(dx)dr + \int_s^m\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr \\
&\quad -2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r - 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}Z_r^{t,x,m}\rho(dx)dW_r \\
&\leq 2L\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr + 4L\int_s^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr
\end{aligned}$$

$$+\frac{2}{L}\int_s^m\int_D \mathrm{e}^{-Kr}|f(X_r^{t,x},0)|^2\rho(dx)dr+\frac{2}{L}\int_s^m\int_D \mathrm{e}^{-Kr}|f(x,0)|^2\rho(dx)dr$$

$$+\int_s^m\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr-2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r$$

$$-2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}Z_r^{t,x,m}\rho(dx)dW_r. \tag{3.6}$$

Taking expectation, we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)]-(2K+8L)E[\int_s^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr]$$

$$-2\int_s^m \mathrm{e}^{-Kr}c^2]dr+E[\int_s^m\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho(dx)dr] \tag{3.7}$$

$$\leq E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)]-(K+2L)E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr]$$

$$-4LE[\int_s^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr]+E[\int_s^m\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho(dx)dr]$$

$$\leq\frac{4}{L}E[\int_s^m\int_D \mathrm{e}^{-Kr}|f(X_r^{t,x},0)|^2\rho(dx)dr]+E[\int_s^m\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr].$$

In the following, we apply Lemma 3.2 to u^m as the solution of SPDE (2.1) with the terminal time m and the terminal value c. Applying this estimate to (3.7), we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)]+\big(\frac{1}{2M^2}-(2K+8L)\big)E[\int_s^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr]$$

$$\leq\frac{1}{K}(2+\frac{1}{M^2})(\mathrm{e}^{-Ks}-\mathrm{e}^{-Km})c^2+\frac{4}{L}E\int_s^m\int_D \mathrm{e}^{-Kr}|f(X_r^{t,x},0)|^2\rho(dx)dr]$$

$$+E[\int_s^m\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr].$$

Noticing $\frac{1}{2M^2}-(2K+8L)>0$, as $m,n\to+\infty$, we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)] + E[\int_n^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr] \tag{3.8}$$
$$\leq C\big(\mathrm{e}^{-Kn} + \mathrm{e}^{-Kn}\int_D |f(x,0)|^2\rho(dx) + \mathrm{e}^{-Kn}\int_D \|g(x)\|^2\rho(dx)\big) \longrightarrow 0.$$

Then, (3.7) together with (3.8) leads to that as $m, n \to +\infty$,

$$E[\int_n^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] + E[\int_n^m\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr] \tag{3.9}$$
$$\leq 2E[\int_n^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr] + 2\int_n^m \mathrm{e}^{-Kr}c^2dr + E[\int_n^m\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho(dx)dr]$$
$$\leq C\mathrm{e}^{-Kn} + C\mathrm{e}^{-Kn}\int_D |f(x,0)|^2\rho(dx) + C\mathrm{e}^{-Kn}\int_D \|g(x)\|^2\rho(dx) \longrightarrow 0.$$

Using the B-D-G inequality to deal with (3.6) on the interval $[n, m]$, by (3.8) and (3.9), as n, $m \to +\infty$ we have

$$\begin{aligned} & E[\sup_{n\leq s\leq m}\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)] \\ \leq\ & C\mathrm{e}^{-Kn} + C\mathrm{e}^{-Kn}\int_D |f(x,0)|^2\rho(dx) + C\mathrm{e}^{-Kn}\int_D \|g(x)\|^2\rho(dx) \\ & +CE[\int_n^m\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2\rho(dx)dr] \\ & +CE[\int_n^m\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,m}|^2\rho(dx)dr] + CE[\int_n^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] \\ \longrightarrow\ & 0. \end{aligned} \tag{3.10}$$

(ii) When $0 \leq s \leq n \leq m$,

$$\bar{Y}_s^{t,x,m,n} = Y_n^{t,x,m} - c + \int_s^n [\bar{f}^{t,m,n}(r,x) - \int_D \bar{f}^{r,m,n}(r,\xi)\rho(d\xi)]dr - \int_s^n \bar{Z}_r^{t,x,m,n}dW_r.$$

Applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.a. $x \in \mathbb{R}^d$ and using Lipschitz condition and Young inequality, we have

$$\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx) - K\int_s^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dxdr$$

$$+\int_s^n\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr$$

$$= \int_D \mathrm{e}^{-Kn}|Y_n^{t,x,m} - c|^2\rho(dx) + 2\int_s^n\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}[\bar{f}^{t,m,n}(r,x)$$

$$-\int_D \bar{f}^{r,m,n}(r,\xi)\rho(d\xi)]\rho(dx)dr - 2\int_s^n\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}\bar{Z}_r^{t,x,m,n}\rho(dx)dW_r.$$

$$\leq 2\int_D \mathrm{e}^{-Kn}|Y_n^{t,x,m}|^2\rho(dx) + 2\mathrm{e}^{-Kn}|c|^2 + 3L\int_s^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr$$

$$+L\int_s^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{r,x,m,n}|^2\rho(dx)dr - 2\int_s^n\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,m,n}\bar{Z}_r^{t,x,m,n}\rho(dx)dW_r. \quad (3.11)$$

Note

$$E[\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{r,x,m,n}|^2\rho(dx)] = E[\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{r,X_r^{t,x},m,n}|^2\rho(dx)] = E[\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)].$$

Taking expectation, we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)] - (K+4L)E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr]$$

$$+E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr]$$

$$\leq 2E[\int_D \mathrm{e}^{-Kn}|Y_n^{t,x,m}|^2\rho(dx)] + 2\mathrm{e}^{-Kn}c^2. \quad (3.12)$$

For $s \geq 0$, define

$$\bar{u}^{m,n}(s,x) \triangleq u^m(s,x) - u^n(s,x) = Y_s^{s,x,m} - Y_s^{s,x,n}.$$

Obviously, $\bar{u}^{m,n}$ satisfies the following random PDE:

$$\begin{cases} d\bar{u}^{m,n}(t,x) = -[\frac{1}{2}\triangle\bar{u}^{m,n}(t,x) + \Big(f\big(x,u^m(t,x)\big) - f\big(x,u^n(t,x)\big)\Big) \\ \qquad\qquad - \int_D \Big(f\big(\xi,u^m(t,\xi)\big) - f\big(\xi,u^n(t,\xi)\big)\Big)\rho(d\xi)]dt, \quad t\in[0,T), \quad x\in D, \\ \bar{u}^{m,n}(n,x) = u^m(n,x) - c, \quad x\in\overline{D}, \\ \frac{\partial\bar{u}^{m,n}}{\partial\mathbf{n}}(t,x) = 0, \quad t\in[0,n), \quad x\in\partial D. \end{cases}$$

This PDE is also mass-conservative and for all $t\in[0,n]$ and $\omega\in\Omega$,

$$\int_D \bar{u}^{m,n}(t,x)\rho(dx) = \int_D \big(u^m(n,x)-c\big)\rho(dx) = 0.$$

This is due to that $u^m(n,x)$ satisfies SPDE (2.1) with the terminal time m and the terminal value c, i.e. $\int_D u^m(n,x)\rho(dx) = c$. Applying Lemma 3.2 to $\bar{u}^{m,n}$, we have

$$\begin{aligned} E[\int_D |\bar{Z}_s^{t,x,m,n}|^2\rho(dx)] &= E[\int_D |\nabla\bar{u}^{m,n}(s,X_s^{t,x})|^2\rho(dx)] = E[\int_D |\nabla\bar{u}^{m,n}(s,x)|^2\rho(dx)] \\ &\geq \frac{1}{2M^2}E[\int_D |\bar{u}^{m,n}(s,x)|^2\rho(dx)] = \frac{1}{2M^2}E[\int_D |\bar{u}^{m,n}(s,X_s^{t,x})|^2\rho(dx)] \\ &= \frac{1}{2M^2}E[\int_D |\bar{Y}_s^{t,x,m,n}|^2\rho(dx)]. \end{aligned}$$

Inserting above estimate into (3.12), we have

$$\begin{aligned} &E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)] + \big(\frac{1}{2M^2} - (K+4L)\big)E[\int_s^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] \\ &\leq E[\int_D \mathrm{e}^{-Kn}|Y_n^{t,x,m}|^2\rho(dx)] + 2\mathrm{e}^{-Kn}c^2. \end{aligned}$$

As $n,m\to+\infty$, using (3.10) we have

$$E[\int_0^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] \leq CE[\sup_{n\leq s\leq m}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,m}|^2\rho(dx)] + C\mathrm{e}^{-Kn}c^2 \longrightarrow 0. \tag{3.13}$$

Then, as $m,n\to+\infty$ it follows from (3.10), (3.12) and (3.13) that

$$\begin{aligned} &E[\int_0^n\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr] \\ &\leq CE[\sup_{n\leq s\leq m}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,m}|^2\rho(dx)] + C\mathrm{e}^{-Kn}c^2 \end{aligned}$$

$$+CE[\int_0^n \int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] \longrightarrow 0. \tag{3.14}$$

Also by the B-D-G inequality, (3.10)–(3.11) and (3.13)–(3.14), as $n, m \to +\infty$, we have

$$\begin{aligned}
&E[\sup_{0\le s\le n}\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)]\\
&\le CE[\sup_{n\le s\le m}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,m}|^2\rho(dx)] + C\mathrm{e}^{-Kn}c^2\\
&+CE[\int_0^n\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr] + CE[\int_0^n\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr] \longrightarrow 0.
\end{aligned}$$

Therefore taking a combination of cases (i) and (ii), as $n, m \to +\infty$, we have

$$\begin{aligned}
&E[\sup_{s\ge 0}\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2\rho(dx)] + E[\int_0^{+\infty}\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2\rho(dx)dr]\\
&+E[\int_0^{+\infty}\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2\rho(dx)dr] \longrightarrow 0.
\end{aligned}$$

That is to say $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence in the space $S^{2,-K}\bigcap M^{2,-K}([0,+\infty); L^2_\rho(D;\mathbb{R}))\times M^{2,-K}([0,+\infty); L^2_\rho(D;\mathbb{R}^d))$. Take $(Y_s^{t,x}, Z_s^{t,x})$ as the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ and we will show that $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of BDSDE (3.1). For this, we take the strong limit on both sides of (3.5) in $L^2(\Omega\times D;\mathbb{R})$, then the claim that $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is a solution to BDSDE (3.1) follows. We only take for example the convergence of the terms involving f. Firstly,

$$\begin{aligned}
&\lim_{n\to+\infty} E[\int_D |\int_s^n \mathrm{e}^{-Kr} f(X_r^{t,x}, Y_r^{t,x,n})dr - \int_s^{+\infty}\mathrm{e}^{-Kr} f(X_r^{t,x}, Y_r^{t,x})dr|^2\rho(dx)]\\
&\le \lim_{n\to+\infty} CE[\int_s^n\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n} - Y_r^{t,x}|^2\rho(dx)dr]\\
&+\lim_{n\to+\infty} CE[\int_n^{+\infty}\int_D \mathrm{e}^{-Kr}\big(1+|Y_r^{t,x}|^2+|f(x,0)|^2\big)\rho(dx)dr] = 0.
\end{aligned}$$

For the other term, similarly

$$\lim_{n\to+\infty} E[\int_D |\int_s^n \mathrm{e}^{-Kr}\int_D f(\xi, Y_r^{r,\xi,n})\rho(d\xi)dr - \int_s^{+\infty} \mathrm{e}^{-Kr}\int_D f(\xi, Y_r^{r,\xi})\rho(d\xi)dr|^2\rho(dx)]$$
$$\leq \lim_{n\to+\infty} CE[\int_s^n\int_D \mathrm{e}^{-Kr}|Y_r^{r,x,n} - Y_r^{r,x}|^2\rho(dx)dr]$$
$$+\lim_{n\to+\infty} CE[\int_n^{+\infty}\int_D \mathrm{e}^{-Kr}\big(1+|Y_r^{r,x}|^2+|f(x,0)|^2\big)\rho(dx)dr] = 0.$$

After verifying the convergence of each term, the existence of solution to BDSDE (3.1) is proved.

For the uniqueness of solution, let $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ be two solutions of BDSDE (3.1). Define

$$\bar{Y}_s^{t,x} = \hat{Y}_s^{t,x} - Y_s^{t,x}, \quad \bar{Z}_s^{t,x} = \hat{Z}_s^{t,x} - Z_s^{t,x}, \quad \bar{f}^t(s,x) = f(X_s^{t,x}, \hat{Y}_s^{t,x}) - f(X_s^{t,x}, Y_s^{t,x}), \quad s \geq 0.$$

Then for $s \geq 0$ and a.a. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ satisfy

$$\mathrm{e}^{-Ks}\bar{Y}_s^{t,x} = \int_s^{+\infty} \mathrm{e}^{-Kr}[\bar{f}^t(r,x) - \int_D \bar{f}^r(r,\xi)\rho(d\xi)]dr + \int_s^{+\infty} K\mathrm{e}^{-Kr}\bar{Y}_r^{t,x}dr - \int_s^{+\infty} \mathrm{e}^{-Kr}\bar{Z}_r^{t,x}dW_r.$$

For an arbitrary interval $[0,T]$,

$$\bar{Y}_s^{t,x} = \bar{Y}_T^{t,x} + \int_s^T [\bar{f}^t(r,x) - \int_D \bar{f}^r(r,\xi)\rho(d\xi)]dr - \int_s^T \bar{Z}_r^{t,x}dW_r,$$

where $\bar{Y}_T^{t,x} = \hat{Y}_T^{t,x} - Y_T^{t,x} = \hat{Y}_T^{T,X_T^{t,x}} - Y_T^{T,X_T^{t,x}}$ satisfies Condition (H.1) and $\bar{f}^t(s,x) = f(X_s^{t,x}, \hat{Y}_s^{t,x}) - f(X_s^{t,x}, Y_s^{t,x}) = f(X_s^{t,x}, \hat{Y}_s^{s,X_s^{t,x}}) - f(X_s^{t,x}, Y_s^{s,X_s^{t,x}})$ can be regarded as a given function. Moreover,

$$\lim_{T\to+\infty} \mathrm{e}^{-KT}\bar{Y}_T^{t,x} = 0. \tag{3.15}$$

Similar to (3.12), applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x}|^2$ for a.a. $x \in \mathbb{R}^d$, we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x}|^2\rho(dx)] - (K+4L)E[\int_s^T\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x}|^2\rho(dx)dr]$$
$$+E[\int_s^T\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x}|^2\rho(dx)dr] \leq E[\int_D \mathrm{e}^{-KT}|\bar{Y}_T^{t,x}|^2\rho(dx)]. \tag{3.16}$$

On the other hand, by Theorem 2.5, we know that $\bar{u}(s,x) \triangleq \bar{Y}_s^{s,x}$ is a weak solution to the following mass-conservative random PDE:

$$\begin{cases} d\bar{u}(t,x) = -[\frac{1}{2}\triangle\bar{u}(t,x) + \Big(f\big(x,\hat{u}(t,x)\big) - f\big(x,u(t,x)\big)\Big) \\ \qquad\qquad - \int_D \Big(f\big(\xi,\hat{u}(t,\xi)\big) - f\big(\xi,u(t,\xi)\big)\Big)\rho(d\xi)]dt, \quad t\in[0,T), \quad x\in D, \\ \bar{u}(T,x) = \bar{Y}_T^{T,x}, \quad x\in\overline{D}, \\ \frac{\partial\bar{u}}{\partial\mathbf{n}}(t,x) = 0, \quad t\in[0,T), \quad x\in\partial D. \end{cases}$$

So, for all $t\in[0,T]$ and $\omega\in\Omega$, in view of the invariant measure and (2.6),

$$\int_D \bar{u}(t,x)\rho(dx) = \int_D \bar{Y}_t^{t,x}\rho(dx) = E^W\int_D \bar{Y}_s^{t,x}\rho(dx) = 0.$$

Applying Lemma 3.2 to $\bar{u}$, we have

$$E[\int_D |\bar{Z}_s^{t,x}|^2\rho(dx)] \geq \frac{1}{2M^2}E[\int_D |\bar{Y}_s^{t,x}|^2\rho(dx)].$$

Putting above inequality into (3.16), we have

$$\begin{aligned} &E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x}|^2\rho(dx)] + \big(\frac{1}{2M^2} - (K+4L)\big)E[\int_s^T\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x}|^2\rho(dx)dr] \\ &\leq E[\int_D \mathrm{e}^{-KT}|\bar{Y}_T^{t,x}|^2\rho(dx)]. \end{aligned} \tag{3.17}$$

Taking $K' > K$ such that K' satisfies the condition to K as well, we can see that (3.17) remains true with K replaced by K'. In particular,

$$E[\int_{\mathbb{R}^d} \mathrm{e}^{-K's}|\bar{Y}_s^{t,x}|^2\rho(dx)] \leq E[\int_{\mathbb{R}^d} \mathrm{e}^{-K'T}|\bar{Y}_T^{t,x}|^2\rho(dx)].$$

Therefore, we have that

$$E[\int_{\mathbb{R}^d} \mathrm{e}^{-K's}|\bar{Y}_s^{t,x}|^2\rho^{-1}(x)dx] \leq \mathrm{e}^{-(K'-K)T}E[\int_{\mathbb{R}^d} \mathrm{e}^{-KT}|\bar{Y}_T^{t,x}|^2\rho^{-1}(x)dx]. \tag{3.18}$$

Since $\hat{Y}_s^{t,x}, Y_s^{t,x} \in S^{2,-K}\bigcap M^{2,-K}([0,+\infty); L_\rho^2(D;\mathbb{R}))$,

$$\sup_{T\geq 0} E[\int_{\mathbb{R}^d} \mathrm{e}^{-KT}|\bar{Y}_T^{t,x}|^2\rho(dx)] \leq E[\sup_{T\geq 0}\int_{\mathbb{R}^d} \mathrm{e}^{-KT}(2|\hat{Y}_T^{t,x}|^2 + 2|Y_T^{t,x}|^2)\rho(dx)] < +\infty.$$

Therefore, taking the limit as $T \to +\infty$ in (3.18), we have

$$E[\int_{\mathbb{R}^d} e^{-K's} |\bar{Y}_s^{t,x}|^2 \rho(dx)] = 0.$$

The uniqueness follows. $\diamond$

Remark 3.4. If f and g in (3.1) depend on the time variable, we can also prove the existence and uniqueness theorem of SPDE (3.1) by replacing (H.6) with the following condition:

(H.6)*. There exists $K > 0$ such that $\int_0^{+\infty} \int_D e^{-Ks}(|f(s,x,0)|^2 + \|g(s,x)\|^2)\rho(dx)ds < +\infty$ and $4M^2(K + 4L) < 1$.

4. Stationary solutions of SPDEs

In this section we consider the stationary solution of the following SPDE with time variable independent coefficients f and g

$$\begin{cases} dv(t,x) = [\frac{1}{2}\triangle v(t,x) + f\big(x, v(t,x)\big) - \int_D f\big(\xi, v(t,\xi)\big)\rho(d\xi)]dt \\ \quad +g(x)dB_t \quad t \in (0,+\infty), \quad x \in D \\ \frac{\partial v}{\partial \mathbf{n}}(t,x) = 0 \quad t \in (0,+\infty), \quad x \in \partial D \\ \int_D v(t,x)\rho(dx) = c \quad for\ a\ given\ constant\ c. \end{cases} \tag{4.1}$$

For this, we need to study the initial value problem of SPDE (1.1) where f and g are time variable independent. Denote $\big(\Phi(t)h\big)(x) = v(t,x,h)$. We utilize the connection between BDSDEs and SPDEs. In this connection, corresponding to a BDSDE, the SPDE should be of backward type equation (2.1) with backward Itô stochastic integral of $\hat{B}$ which is the time reversal process of Brownian motion B. Thus, as in Section 2, the noise of the corresponding BDSDE should be the time reversal $\hat{B}$. In particular, $\hat{B}_t = B_{T'-t} - B_{T'}$ for a fixed $T' > 0$ and $-\infty < t \leq T'$. In fact, the choice of T' can be arbitrary. It is obvious that $\hat{B}_t$ is a Brownian motion with $\hat{B}_0 = 0$.

We first construct a measurable metric dynamical system through defining a measurable and probability preserving shift operator. Let $\tilde{\theta}_t = \hat{\theta}_t \circ \check{\theta}_t$, $t \geq 0$, where $\hat{\theta}_t, \check{\theta}_t : \Omega \longrightarrow \Omega$ are measurable mappings on $(\Omega, \mathscr{F}, P)$ defined by

$$\hat{\theta}_t \begin{pmatrix} \hat{B} \\ W \end{pmatrix}(s) = \begin{pmatrix} \hat{B}_{s+t} - \hat{B}_t \\ W_s \end{pmatrix}, \quad \check{\theta}_t \begin{pmatrix} \hat{B} \\ W \end{pmatrix}(s) = \begin{pmatrix} \hat{B}_s \\ W_{s+t} - W_t \end{pmatrix}.$$

Then for any $s, t \geq 0$, (i). $P = \tilde{\theta}_t P$; (ii). $\tilde{\theta}_0 = I$, where I is the identity transformation on Ω; (iii). $\tilde{\theta}_s \circ \tilde{\theta}_t = \tilde{\theta}_{s+t}$. Also for an arbitrary $\mathscr{F}$ measurable ϕ and $t \geq 0$, set

$$\tilde{\theta}_t \circ \phi(\omega) = \phi\big(\tilde{\theta}_t(\omega)\big).$$

For any $r \geq 0$, applying $\tilde{\theta}_r$ to SDE (2.2),

$$\begin{cases}\tilde{\theta}_r \circ X_s^{t,x} = x + W_{s+r} - W_{t+r} + \int_{t+r}^{s+r} \nabla\phi(\tilde{\theta}_r \circ X_{u-r}^{t,x}) d(\tilde{\theta}_r \circ K_{u-r}^{t,x}), \quad s \geq t, \\ \tilde{\theta}_r \circ K_s^{t,x} = \int_{t+r}^{s+r} I_{\{\tilde{\theta}_r \circ X_{u-r}^{t,x} \in \partial D\}} d(\tilde{\theta}_r \circ K_{u-r}^{t,x}), \quad \tilde{\theta}_r \circ K_{\cdot}^{t,x} \; is \; increasing.\end{cases}$$

Compare the above equation with the following equation:

$$\begin{cases}X_{s+r}^{t+r,x} = x + W_{s+r} - W_{t+r} + \int_{t+r}^{s+r} \nabla\phi(X_u^{t+r,x}) dK_u^{t+r,x}, \quad s \geq t, \\ K_{s+r}^{t+r,x} = \int_{t+r}^{s+r} I_{\{X_u^{t+r,x} \in \partial D\}} dK_u^{t+r,x}, \quad K_{\cdot}^{t+r,x} \; is \; increasing.\end{cases}$$

By the uniqueness of the solution and a perfection procedure (cf. Arnold [2]), we have

$$\tilde{\theta}_r \circ X_s^{t,\cdot} = \check{\theta}_r \circ X_s^{t,\cdot} = X_{s+r}^{t+r,\cdot} \quad \text{for all } r, s, t \geq 0 \;\text{ a.s.}$$

Firstly, we consider the stationary solution of a time independent version of BDSDE (3.1) with $E^W[\int_D Y_s^{t,x} \rho(dx)] = c(\omega)$, $s \geq t$. So $E^W[\int_D \tilde{\theta}_r Y_s^{t,x} \rho(dx)] = c(\hat{\theta}_r \omega)$ for $r \geq 0$. Then if we look for a stationary solution satisfying $\tilde{\theta}_r Y_s^{t,x} = Y_{s+r}^{t+r,x}$, we need to impose $c(\hat{\theta}_r \omega) = c(\omega)$. But $\hat{\theta}_\cdot$ as a measure preserving dynamical system on $(\Omega^B, \mathscr{F}^B, P^B)$ is ergodic, thus c has to be a constant.

Theorem 4.1. *Under Conditions (H.3)–(H.6), let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of BDSDE (3.1) satisfying $E^W[\int_D Y_s^{t,x} \rho(dx)] = c$, $s \geq t \geq 0$, where c is a given mass-constant. Then $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ satisfies for any $t \geq 0$,*

$$\tilde{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \tilde{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for } s \geq t, \; r \geq 0, \; \text{a.s.}$$

In particular, for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for } r \geq 0, \; \text{a.s.} \tag{4.2}$$

Proof. First note that BDSDE (3.1) is equivalent to the following equation

$$\begin{cases}Y_s^{t,x} = Y_T^{t,x} + \int_s^T [f(X_r^{t,x}, Y_r^{t,x}) - \int_D f(\xi, Y_r^{r,\xi}) \rho(d\xi)] dr - \int_s^T g(X_r^{t,x}) d^\dagger \hat{B}_r - \int_s^T Z_r^{t,x} dW_r \\ \lim_{T \to +\infty} \mathrm{e}^{-KT} Y_T^{t,x} = 0 \quad \text{a.s.}\end{cases} \tag{4.3}$$

For $r \geq 0$, applying $\hat{\theta}_r$ on B_u, we have

$$\begin{aligned}\hat{\theta}_r \circ B_u &= \hat{\theta}_r \circ (\hat{B}_{T'-u} - \hat{B}_{T'}) = \hat{B}_{T'-u+r} - \hat{B}_{T'+r} \\ &= (\hat{B}_{T'-u+r} - \hat{B}_{T'}) - (\hat{B}_{T'+r} - \hat{B}'_T) = B_{u-r} - B_{-r}.\end{aligned}$$

So for $0 \leq s \leq T \leq T'$ and $\{h(u,\cdot)\}_{u \geq 0}$ being a $\mathscr{F}_u$-measurable and locally square integrable stochastic process, we have the relationship between the forward integral and backward Itô integral (cf. [18])

$$\int_s^T h(u,\cdot) d^\dagger B_u = -\int_{T'-T}^{T'-s} h(T'-u,\cdot) dB_u \quad \text{a.s.}$$

and for arbitrary $T \geq 0$, $0 \leq s \leq T$,

$$\tilde{\theta}_r \circ \int_s^T h(u, \cdot) d^\dagger \hat{B}_u = \int_{s+r}^{T+r} \tilde{\theta}_r \circ h(u-r, \cdot) d^\dagger \hat{B}_u. \tag{4.4}$$

Therefore for a.e. $x \in \mathbb{R}^d$,

$$\tilde{\theta}_r \circ \int_s^T h(u, x) d^\dagger \hat{B}_u = \int_{s+r}^{T+r} \tilde{\theta}_r \circ h(u-r, x) d^\dagger \hat{B}_u.$$

Since $g(X_\cdot^{t,\cdot})$ is locally square integrable, by (4.3) and (4.4), for a.e. $x \in \mathbb{R}^d$

$$\tilde{\theta}_r \circ \int_s^T g(X_u^{t,x}) d^\dagger \hat{B}_u = \tilde{\theta}_r \circ \int_{s+r}^{T+r} g(X_{u-r}^{t,x}) d^\dagger \hat{B}_u = \int_{s+r}^{T+r} g(X_u^{t+r,x}) d^\dagger \hat{B}_u. \tag{4.5}$$

Now applying the operator $\tilde{\theta}_r$ on both sides of (4.3), by (4.5) we know that $\tilde{\theta}_r \circ Y_s^{t,x}$ satisfies $E^W[\int_D \tilde{\theta}_r \circ Y_s^{t,x} \rho(dx)] = c$ and the following equation

$$\begin{cases} \tilde{\theta}_r \circ Y_s^{t,x} = \tilde{\theta}_r \circ Y_T^{t,x} + \int_{s+r}^{T+r} [f(X_u^{t+r,x}, \tilde{\theta}_r \circ Y_{u-r}^{t,x}) - \int_D f(\xi, \tilde{\theta}_r \circ Y_{u-r}^{u-r,\xi}) \rho(d\xi)] du \\ \qquad - \int_{s+r}^{T+r} g(X_u^{t+r,x}) d^\dagger \hat{B}_u - \int_{s+r}^{T+r} \tilde{\theta}_r \circ Z_{u-r}^{t,x} dW_u \\ \lim_{T \to +\infty} \mathrm{e}^{-K(T+r)} (\tilde{\theta}_r \circ Y_T^{t,x}) = 0 \quad \text{a.s.} \end{cases} \tag{4.6}$$

On the other hand, from the assumption in Theorem 4.1 it follows that $E^W[\int_D Y_{s+r}^{t+r,x} \rho(dx)] = c$ and

$$\begin{cases} Y_{s+r}^{t+r,x} = Y_{T+r}^{t+r,x} + \int_{s+r}^{T+r} [f(X_u^{t+r,x}, Y_u^{t+r,x}) - \int_D f(\xi, Y_u^{u,\xi}) \rho(d\xi)] du \\ \qquad - \int_{s+r}^{T+r} g(X_u^{t+r,x}) d^\dagger \hat{B}_u - \int_{s+r}^{T+r} Z_u^{t+r,x} dW_u \\ \lim_{T \to +\infty} \mathrm{e}^{-K(T+r)} Y_{T+r}^{t+r,x} = 0 \quad \text{a.s.} \end{cases} \tag{4.7}$$

Let $\tilde{Y}_\cdot^{t,\cdot} = \tilde{\theta}_r \circ Y_{\cdot-r}^{t-r,\cdot}$, $\tilde{Z}_\cdot^{t,\cdot} = \tilde{\theta}_r \circ Z_{\cdot-r}^{t-r,\cdot}$. By the uniqueness of the solution of BDSDE (3.1) with the given c in the space $S^{2,-K} \bigcap M^{2,-K}([0, +\infty); L_\rho^2(D; \mathbb{R})) \times M^{2,-K}([0, +\infty); L_\rho^2(D; \mathbb{R}^d))$, it follows from comparing (4.6) with (4.7) that for any $t \geq 0$, in the space $L_\rho^2(D; \mathbb{R}) \times L_\rho^2(D; \mathbb{R}^d)$,

$$\tilde{\theta}_r \circ Y_s^{t,\cdot} = \tilde{Y}_{s+r}^{t+r,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \tilde{\theta}_r \circ Z_s^{t,\cdot} = \tilde{Z}_{s+r}^{t+r,\cdot} = Z_{s+r}^{t+r,\cdot} \ \text{ for all } s \geq t \text{ a.s.}$$

Then by the perfection procedure ([2], [3]), we can prove above identities are true for all $s \geq t$, but fixed $t \geq 0$ a.s. In particular, for any $t \geq 0$, in the space $L_\rho^2(D; \mathbb{R}) \times L_\rho^2(D; \mathbb{R}^d)$,

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{a.s.} \tag{4.8}$$

The proof is finished. $\diamond$

If we regard $Y_t^{t,\cdot}$ as a function of t, (4.2) gives a "crude" stationary property of $Y_t^{t,\cdot}$. By Theorem 2.7 and Proposition 2.8, $Y_t^{t,\cdot}$ is the unique weak solution of SPDE (2.1) which has a.s. continuous version. Hence it comes without a surprise that

Theorem 4.2. *Under Conditions (H.3)–(H.6), let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of a time independent version of BDSDE (3.1) with the given mass-constant c. Then $Y_t^{t,\cdot}$ satisfies the "perfect" stationary property with respect to $\hat{\theta}$, i.e.*

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } t \geq 0,\ r \geq 0,\ \text{a.s.} \tag{4.9}$$

Consider the equivalent BDSDE (4.3) and its solution $Y_s^{t,\cdot}$ on $[t, T]$. We choose $\hat{B}$ as the time reversal of B from time T, i.e. $\hat{B}_s = B_{T-s} - B_T$ for $s \geq 0$. Note that the random variable $Y_T^{T,\cdot}$ is $\mathscr{F}_{T,+\infty}^{\hat{B}}$ measurable which is independent of $\mathscr{F}_t^W$. Changing variable in SPDE (4.1), we can deduce from the Correspondence Theorem 2.7 that $v(t, \cdot) = u(T-t, \cdot) = Y_{T-t}^{T-t,\cdot}$ is a weak solution of SPDE (4.1) on $[0, T]$ if $Y_T^{T,x}$ satisfies Condition (H.1). Note $Y_T^{t,x} = Y_T^{T,X_T^{t,x}}$, so Condition (H.1) is satisfied.

On the probability space $(\Omega, \mathscr{F}, P)$, we define $\theta_t : \Omega \longrightarrow \Omega$, $t \in \mathbb{R}$, as the shift operator of Brownian motion B:

$$\theta_t \circ B_s = B_{s+t} - B_t,$$

then θ satisfies the usual conditions: (i). $P = P \circ \theta_t$; (ii). $\theta_0 = I$; (iii). $\theta_s \circ \theta_t = \theta_{s+t}$. Noticing that $\hat{B}$ is chosen as the time reversal of B at time T and B, W are independent, we can define $\tilde{\theta}$, served as the shift operator of $\hat{B}$ and W, to be $\tilde{\theta}_t \triangleq (\theta_t)^{-1} \circ \check{\theta}_t$, $t \geq 0$. Actually B is a two-sided Brownian motion, so $(\theta_t)^{-1} = \theta_{-t}$ is well defined (see [2]) and it is easy to see that $\hat{\theta}_t \triangleq (\theta_t)^{-1}$, $t \in \mathbb{R}$, is a shift operator of $\hat{B}$. We can prove a claim that $v(t, \cdot) = Y_{T-t}^{T-t,\cdot}$ does not depend on the choice of T using a similar proof as in [18], [19]. This can be obtained from (4.9) and the fact that $(\hat{\theta}_{T-t}\hat{B})(s) = B_{t-s} - B_t$.

Now since $v(t, \cdot) = u(T-t, \cdot) = Y_{T-t}^{T-t,\cdot}$ a.s., by (4.9),

$$\theta_r v(t, \cdot, \omega) = \hat{\theta}_{-r} u(T-t, \cdot, \hat{\omega}) = \hat{\theta}_{-r}\hat{\theta}_r u(T-t-r, \cdot, \hat{\omega}) = u(T-t-r, \cdot, \hat{\omega}) = v(t+r, \cdot, \omega),$$

for all $r \geq 0$ and $T \geq t+r$ a.s. In particular, let $Y(\cdot, \omega) = v(0, \cdot, \omega) = Y_T^{T,\cdot}(\hat{\omega})$, then the above formula implies:

$$\theta_t Y(\cdot, \omega) = Y(\cdot, \theta_t \omega) = v(t, \cdot, \omega) = v(t, \cdot, \omega, v(0, \cdot, \omega)) = v(t, \cdot, \omega, Y(\cdot, \omega)) \text{ for all } t \geq 0 \text{ a.s.} \tag{4.10}$$

It turns out that $v(t, \cdot, \omega) = Y(\cdot, \theta_t \omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is a stationary solution of SPDE (4.1) with respect to θ. Therefore we obtain

Theorem 4.3. *Under Conditions (H.3)–(H.6), for arbitrary T and $t \in [0, T]$, let $v(t, \cdot) \triangleq Y_{T-t}^{T-t,\cdot}$, where $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is the solution of BDSDE (3.1) with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $v(t, \cdot)$ is a "perfect" stationary solution of SPDE (4.1) independent of the choice of T.*

Remark 4.4. Similar to Theorem 6.8 in [20], it is not difficult to see that in the proof of Theorem 3.3, there is no need to take $h = c$. In fact we can consider BDSDE (3.1) with an arbitrary h satisfying Condition (H.1) and $\int_D h(x)\rho(dx) = c$.

Remark 4.5. Define $\mu(\Gamma) = P\{\omega : \ v(0,\omega) \in \Gamma\}$ for $\Gamma \in \mathscr{B}(L^2_\rho(D))$, where $v(s,\cdot)$ is the value of SPDE (4.1) at time s. Note $v(s,\omega) = \theta_s \circ v(0,\omega)$ and θ is measure preserving, so $\mu(\Gamma) = P\{\omega : \ v(0,\theta_s\omega) \in \Gamma\} = P\{\omega : \ v(s,\omega) \in \Gamma\}$. Since $v(s,\cdot) = \Phi(s)\big(v(0,\cdot)\big)$, it follows from Markov property and some standard argument that $\int_{L^2_\rho(D)} p(s,\eta,\Gamma)\mu(d\eta) = \mu(\Gamma)$. Here $p(s,\eta,\Gamma) = P\{\omega : \ \Phi(s)\eta \in \Gamma\}$, $\Gamma \in \mathscr{B}(L^2_\rho(D))$, is the transition probability of the homogeneous Markov process $\Phi(s)\eta$ with initial position η. That is to say that μ is an invariant measure with respect to the Markov process.

5. Mass-conserving stochastic Allen-Cahn equations

In this section, for an odd integer $p > 0$, we aim to solve the following infinite horizon SPDE:

$$\begin{cases} dv(t,x) = [\frac{1}{2}\triangle v(t,x) - v^p(t,x) + v(t,x) + \int_D \big(v^p(t,x) - v(t,x)\big)\rho(dx)]dt \\ \quad -g(x)dB_t, \quad x \in D, \\ \frac{\partial v}{\partial \mathbf{n}}(t,x) = 0, \quad x \in \partial D, \\ \int_D v(t,x)\rho(dx) = c \quad for\ a\ given\ constant\ c. \end{cases} \tag{5.1}$$

This is a mass-conserving stochastic Allen-Cahn equation.

For this specific equation, Condition (H.6) is automatically broken. We assume the following Condition (H.6)$'$ to replace Condition (H.6):

(H.6)$'$. The reciprocal of the first eigenvalue of $\triangle$ on D satisfies $10M^2 < 1$.

Remark 5.1. In Condition (H.6)$'$, the condition on the reciprocal M of the first eigenvalue of Laplacian operator is in fact a requirement of the domain D. We can instead consider $\frac{1}{2}\sigma^2\triangle$, SDE (2.2) is accordingly changed to the following form

$$\begin{cases} X_s^{t,x} = x + \sigma W_s - \sigma W_t + \int_t^s \nabla\phi(X_r^{t,x})dK_r^{t,x}, \quad s \ge t, \\ K_s^{t,x} = \int_t^s I_{\{X_r^{t,x} \in \partial D\}} dK_r^{t,x}, \quad K_\cdot^{t,x}\ is\ increasing. \end{cases}$$

In this case, the requirement in (H.6)$'$ is relieved to $10M^2 < \sigma^2$ which is a relationship of σ and D, rather that a sole requirement of the domain. All the arguments in this section still work. More generally we can consider second order differential operator. Without losing generality, we present our results for SPDE (5.1) with Laplacian operator only.

To begin with, a sequence of BDSDEs with Lipschitz coefficients is constructed as follows. For each $n \in N$, define

$$f_n(y) = -\Pi_n^p(y) - pn^{p-1}(y - \frac{n}{|y|}y)I_{\{|y|>n\}} + y = \tilde{f}_n(y) + y,$$

where $\Pi_n(y)=\frac{\inf(n,|y|)}{|y|}y$ and $\tilde{f}_n(y)=-\Pi_n^p(y)-pn^{p-1}(y-\frac{n}{|y|}y)I_{\{|y|>n\}}$. Obviously, for any $y\in\mathbb{R}$,

$$\tilde{f}_n(y)\longrightarrow -y^p,\quad f_n(y)\longrightarrow -y^p+y,\quad \text{as } n\to+\infty.$$

For each n, and any $y_1,y_2\in\mathbb{R}$, $\tilde{f}_n$ satisfies the monotonic condition:

$$(y_1-y_2)\big(\tilde{f}_n(y_1)-\tilde{f}_n(y_2)\big)\le 0, \tag{5.2}$$

and f_n satisfies the Lipschitz condition:

$$|f_n(y_1)-f_n(y_2)|\le (pn^{p-1}+1)|y_1-y_2|. \tag{5.3}$$

We then study the BDSDE with coefficient f_n and integral conserving $\int_D Y_s^{t,x,n}\rho(dx)=c$ for $s\ge t$:

$$\begin{aligned}\mathrm{e}^{-Ks}Y_s^{t,x,n}=&\int_s^{+\infty}\mathrm{e}^{-Kr}[f_n(Y_r^{t,x,n})-\int_D f_n(Y_r^{r,\xi,n})\rho(d\xi)]dr+\int_s^{+\infty}K\mathrm{e}^{-Kr}Y_r^{t,x,n}dr\\&-\int_s^{+\infty}\mathrm{e}^{-Kr}g(X_r^{t,x})d^\dagger\hat{B}_r-\int_s^{+\infty}\mathrm{e}^{-Kr}Z_r^{t,x,n}dW_r.\end{aligned} \tag{5.4}$$

Please note that even for fixed n, when n is reasonably large, Condition (H.6) is not satisfied. First let's see the following approximating finite horizon BDSDE with $\int_D Y_s^{t,x,n,m}\rho(dx)=c$:

$$\begin{aligned}Y_s^{t,x,n,m}=&c+\int_s^{m}[f_n(Y_r^{t,x,n,m})-\int_D f_n(Y_r^{r,\xi,n,m})\rho(d\xi)]dr\\&-\int_s^{m}g(X_r^{t,x})d^\dagger\hat{B}_r-\int_s^{+\infty}Z_r^{t,x,n,m}dW_r.\end{aligned} \tag{5.5}$$

Notice that the coefficients f_n, g satisfy Conditions (H.1)–(H.4). Hence by Theorems 2.5 and 2.7, we have the following proposition.

Proposition 5.2. *Under Conditions (H.6)′, BDSDE (5.5) has a unique solution* $(Y_s^{t,x,n,m},Z_s^{t,x,n,m})\in S^{2,0}([t,m];L_\rho^2(D;\mathbb{R}))\times M^{2,0}([t,m];L_\rho^2(D;\mathbb{R}^d))$. *And* $u_{n,m}(t,x)=Y_t^{t,x,n,m}$ *is the unique weak solution of the following SPDE:*

$$\begin{cases}du_n^m(t,x)=-[\frac12\triangle u_n^m(t,x)+f_n\big(u_n^m(t,x)\big)-\int_D f_n\big(u_n^m(t,\xi)\big)\rho(d\xi)]dt\\ \quad +g(x)d^\dagger\hat{B}_t\quad t\in[0,m),\quad x\in D\\ u_n^m(m,x)=c\quad x\in\overline{D}\\ \frac{\partial u_n^m}{\partial\mathbf{n}}(t,x)=0\quad t\in[0,m)\quad x\in\partial D.\end{cases} \tag{5.6}$$

Moreover, $u_{n,m}(s, X_s^{t,x}) = Y_s^{t,x,n,m}$ *and* $\nabla u_{n,m}(s, X_s^{t,x}) = Z_s^{t,x,n,m}$ *for a.a.* $s \in [t, T]$*, a.a.* $x \in D$*, a.s.*

We first prove the solvability of BDSDE (5.4). Note that for the finite horizon problem, what we developed in Section 2 works for the approximating system. However, the results and even the proof we developed in Section 3 don't work for the approximating system here as the Lipschitz condition in (5.2) does not have a uniform bound. Thus we need to develop some new rools to tackle the solution of BDSDE (5.4). This is achieved with help of the monotonicity of $\tilde{f}_n$ and the mass-conservation property.

Proposition 5.3. *Under Conditions (H.6)′, BDSDE (5.4) has a unique solution* $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,-K} \bigcap M^{2,-K}([t,+\infty); L^2_\rho(D;\mathbb{R})) \times M^{2,-K}([t,+\infty); L^2_\rho(D;\mathbb{R}^d))$*, where* $K > 0$ *is a given sufficiently small constant.*

Proof. We use the solution of BDSDE (5.5) to approximate the desired solution. Let $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})_{s>m} = (c, 0)$. Then $(Y_\cdot^{t,\cdot,n,m}, Z_\cdot^{t,\cdot,n,m}) \in S^{2,-K} \bigcap M^{2,-K}([t,+\infty); L^2_\rho(\overline{D};\mathbb{R})) \times M^{2,-K}([t,+\infty); L^2_\rho(D;\mathbb{R}^d))$.

We will prove $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})$, $m = 1, 2, \cdots$, is a Cauchy sequence. For this, let $(Y_s^{t,x,n,l}, Z_s^{t,x,n,l})$ and $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})$ be the solutions of BDSDE (5.5) with the terminal time l and m, respectively. Without losing any generality, assume that $l \geq m$, and define for $s \geq t$,

$$
\begin{aligned}
&\bar{Y}_s^{t,x,n,l,m} = Y_s^{t,x,n,l} - Y_s^{t,x,n,m}, \quad \bar{Z}_s^{t,x,n,l,m} = Z_s^{t,x,n,l} - Z_s^{t,x,n,m},\\
&\bar{f}_n^{t,l,m}(s,x) = f_n(Y_s^{t,x,n,l}) - f_n(Y_s^{t,x,n,m}).
\end{aligned}
$$

Consider two cases:
(i) When $m \leq s \leq l$, $\bar{Y}_s^{t,x,n,l,m} = Y_s^{t,x,n,l} - c$ and we have for any $m \in \mathbb{N}$,

$$
\begin{cases}
d\bar{Y}_s^{t,x,n,l,m} = -[f_n(Y_s^{t,x,n,l}) - \int_D f_n(Y_s^{s,\xi,n,l})\rho(d\xi)]ds + g(X_s^{t,x})d^\dagger \hat{B}_s + Z_s^{t,x,n,l}dW_s\\
\bar{Y}_l^{t,x,n,l,m} = 0, \quad \text{for } s \in [m, l),\ \text{a.a. } x \in \overline{D},\ \text{a.s.}
\end{cases}
$$

Applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.a. $x \in D$ and taking integration over D, by the monotonicity of $\tilde{f}_n$ we have

$$
\begin{aligned}
&\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx) - K\int_s^l\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr\\
&+\int_s^l\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,n,l}|^2\rho(dx)dr\\
&=2\int_s^l\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n,l,m}[f_n(Y_r^{t,x,n,l}) - \int_D f_n(Y_r^{r,\xi,n,l})\rho(d\xi)]\rho(dx)dr
\end{aligned}
$$

$$
\begin{aligned}
&+\int_s^l\int_D e^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r\\
&-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}Z_r^{t,x,n,l}\rho(dx)dW_r\\
=&\,2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}[\tilde{f}_n(Y_r^{t,x,n,l})-\tilde{f}_n(c)]\rho(dx)dr\\
&+2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}\tilde{f}_n(c)\rho(dx)dr+2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}Y_r^{t,x,n,l}\rho(dx)dr\\
&-2\int_s^l e^{-Kr}[\int_D Y_r^{t,x,n,l}\rho(dx)-\int_D c\rho(dx)]\int_D f_n(Y_r^{r,\xi,n,l})\rho(d\xi)dr\\
&+\int_s^l\int_D e^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr\\
&-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}Z_r^{t,x,n,l}\rho(dx)dW_r\\
\le&\,2\int_s^l\int_D e^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr+\int_s^l\int_D e^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr+\int_s^l\int_D e^{-Kr}c^{2p}\rho(dx)dr\\
&+\int_s^l\int_D e^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr-2\int_s^l e^{-Kr}[\int_D Y_r^{t,x,n,l}\rho(dx)\\
&-\int_D c\rho(dx)]\int_D f_n(Y_r^{r,\xi,n,l})\rho(d\xi)dr\\
&-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}g(X_r^{t,x})\rho(dx)d^\dagger\hat{B}_r-2\int_s^l\int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}Z_r^{t,x,n,l}\rho(dx)dW_r. \quad (5.7)
\end{aligned}
$$

Taking conditional expectations E^W and E^B in turn, by the mass-conservation property (3.3) we have

$$
E[\int_D e^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)]-2(K+\frac{5}{2})E[\int_s^l\int_D e^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr]
$$

$$-2(K+2)\int_s^l \mathrm{e}^{-Kr}c^2dr + E[\int_s^l\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,n,l}|^2\rho(dx)dr] \tag{5.8}$$

$$\leq E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] - (K+2)E[\int_s^l\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr]$$

$$-E[\int_s^l\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr] + E[\int_s^l\int_D \mathrm{e}^{-Kr}|Z_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$\leq \int_s^l \mathrm{e}^{-Kr}c^{2p}dr + E[\int_s^l\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr].$$

On the other hand, consider the solution $u_{n,m}$ of SPDE (5.6), to which the solution $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})$ of BDSDE (5.5) corresponds. Applying Lemma 3.2 to $u_{n,l}$, we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] - 2(K+\frac{5}{2})E[\int_s^l\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$-2(K+2)\int_s^l \mathrm{e}^{-Kr}c^2dr + E[\int_s^l\int_D \mathrm{e}^{-Kr}\frac{1}{2M^2}|Y_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$\leq \int_s^l \mathrm{e}^{-Kr}c^{2p}dr + E[\int_s^l\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr].$$

Hence

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] + \big(\frac{1}{2M^2} - 2(K+\frac{5}{2})\big)E[\int_s^l\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$\leq \frac{2(K+2)}{K}(\mathrm{e}^{-Ks}-\mathrm{e}^{-Kl})c^2 + \frac{1}{K}(\mathrm{e}^{-Ks}-\mathrm{e}^{-Kl})c^{2p} + E[\int_s^l\int_D \mathrm{e}^{-Kr}\|g(X_r^{t,x})\|^2\rho(dx)dr].$$

Noticing $\frac{1}{2M^2} - 2(K+\frac{5}{2}) > 0$, as $l, m \to +\infty$ we have

$$E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] + E[\int_m^l\int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr] \tag{5.9}$$

$$\leq C\big(\mathrm{e}^{-Km} + \mathrm{e}^{-Km}\int_D \|g(x)\|^2\rho(dx)\big) \longrightarrow 0.$$

Then, (5.8) together with (5.9) leads to, as $l, m \to +\infty$,

$$E[\int_m^l \int_D e^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr] + E[\int_m^l \int_D e^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2\rho(dx)dr] \tag{5.10}$$

$$\leq 2E[\int_m^l \int_D e^{-Kr}|Y_r^{t,x,n,l}|^2\rho(dx)dr] + 2\int_m^l e^{-Kr}c^2dr + E[\int_m^l \int_D e^{-Kr}|Z_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$\leq C\big(e^{-Km} + e^{-Km}\int_D \|g(x)\|^2\rho(dx)\big) \longrightarrow 0.$$

Using the B-D-G inequality to deal with (5.7) on the interval $[m, l]$, by (5.9) and (5.10), as m, $l \to +\infty$ we have

$$E[\sup_{m\leq s\leq l}\int_D e^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)]$$

$$\leq Ce^{-Km} + Ce^{-Km}\int_D \|g(x)\|^2\rho(dx) + CE[\int_m^l \int_D e^{-Kr}|Z_r^{t,x,n,l}|^2\rho(dx)dr]$$

$$+CE[\int_m^l \int_D e^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr] \longrightarrow 0. \tag{5.11}$$

(ii) When $t \leq s \leq m$,

$$\bar{Y}_s^{t,x,n,l,m} = Y_m^{t,x,n,l} - c + \int_s^m [\bar{f}^{t,m,n}(r,x) - \int_D \bar{f}^{r,m,n}(r,\xi)\rho(d\xi)]dr - \int_s^m \bar{Z}_r^{t,x,n,l,m}dW_r.$$

Apply Itô's formula to $e^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2$ for a.a. $x \in \mathbb{R}^d$, then

$$\int_D e^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx) - K\int_s^m \int_D e^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dxdr$$

$$+\int_s^m \int_D e^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2\rho(dx)dr$$

$$= \int_D e^{-Km}|Y_m^{t,x,n,l} - c|^2\rho(dx) + 2\int_s^m \int_D e^{-Kr}\bar{Y}_r^{t,x,n,l,m}[\bar{f}_n^{t,l,m}(r,x)$$

$$
\begin{aligned}
&-\int_D \bar{f}_n^{r,l,m}(r,\xi)\rho(d\xi)]\rho(dx)dr - 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n,l,m}\bar{Z}_r^{t,x,n,l,m}\rho(dx)dW_r\\
=&\int_D \mathrm{e}^{-Km}|Y_m^{t,x,n,l}-c|^2\rho(dx) + 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n,l,m}(\tilde{f}_n(Y_r^{t,x,n,l})-\tilde{f}_n(Y_r^{t,x,n,m}))\rho(dx)dr\\
&-2\int_s^m \mathrm{e}^{-Kr}[\int_D Y_r^{t,x,n,l}\rho(dx)-\int_D Y_r^{t,x,n,m}\rho(dx)]\int_D \bar{f}_n^{r,l,m}(r,\xi)\rho(d\xi)dr\\
&+2\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr - 2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n,l,m}\bar{Z}_r^{t,x,n,l,m}\rho(dx)dW_r\\
\le& 2\int_D \mathrm{e}^{-Km}|Y_m^{t,x,n,l}|^2\rho(dx) + 2\mathrm{e}^{-Km}c^2 + 2\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr\\
&-2\int_s^m \mathrm{e}^{-Kr}[\int_D Y_r^{t,x,n,l}\rho(dx)-\int_D Y_r^{t,x,n,m}\rho(dx)]\int_D \bar{f}_n^{r,l,m}(r,\xi)\rho(d\xi)dr\\
&-2\int_s^m\int_D \mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n,l,m}\bar{Z}_r^{t,x,n,l,m}\rho(dx)dW_r. \qquad (5.12)
\end{aligned}
$$

Taking conditional expectations E^W and E^B in turn, by the mass-conservation property (3.3) we have

$$
\begin{aligned}
&E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] - (K+2)E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dxdr]\\
&+E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2\rho(dx)dr]\\
\le& 2E[\int_D \mathrm{e}^{-Km}|Y_m^{t,x,n,l}|^2\rho(dx)] + 2\mathrm{e}^{-Km}c^2. \qquad (5.13)
\end{aligned}
$$

For $s \ge 0$, define

$$
\bar{u}_n^{l,m}(s,x) \triangleq u_n^l(s,x) - u_n^m(s,x) = Y_s^{s,x,n,l} - Y_s^{s,x,n,m}.
$$

Obviously, $\bar{u}^{m,n}$ satisfies the following random PDE:

$$\begin{cases} d\bar{u}_n^{l,m}(t,x) = -[\frac{1}{2}\triangle\bar{u}_n^{l,m}(t,x) + \Big(f_n\big(u_n^l(t,x)\big) - f_n\big(u_n^m(t,x)\big)\Big) \\ \qquad\qquad - \int_D \Big(f_n\big(u_n^l(t,\xi)\big) - f_n\big(u_n^m(t,\xi)\big)\Big)\rho(d\xi)]dt, \quad t \in [0,m), \quad x \in D, \\ \bar{u}_n^{l,m}(m,x) = u_n^l(m,x) - c, \quad x \in \overline{D}, \\ \frac{\partial \bar{u}_n^{l,m}}{\partial \mathbf{n}}(t,x) = 0, \quad t \in [0,m), \quad x \in \partial D. \end{cases}$$

This PDE is also mass-conservative and for all $t \in [0,m]$ and $\omega \in \Omega$,

$$\int_D \bar{u}_n^{l,m}(t,x)\rho(dx) = \int_D \big(u_n^l(m,x) - c\big)\rho(dx).$$

Moreover, $u_n^l(m,x)$ satisfies SPDE (2.1) with the terminal time l and the terminal value c. Hence

$$\int_D u_n^l(m,x)\rho(dx) = c.$$

Consequently,

$$\int_D \bar{u}_n^{l,m}(t,x)\rho(dx) = 0.$$

Applying Lemma 3.2 to $\bar{u}_n^{l,m}$, we have

$$\begin{aligned} E[\int_D |\nabla\bar{u}_n^{l,m}(s,x)|^2\rho(dx)] &= E[\int_D |\nabla\bar{u}_n^{l,m}(s,X_s^{t,x})|^2\rho(dx)] = E[\int_D |\bar{Z}_s^{t,x,n,l,m}|^2\rho(dx)] \\ &\geq \frac{1}{2M^2}E[\int_D |\bar{u}_n^{l,m}(s,x)|^2\rho(dx)] = \frac{1}{2M^2}E[\int_D |\bar{u}_n^{l,m}(s,X_s^{t,x})|^2\rho(dx)] \\ &= \frac{1}{2M^2}E[\int_D |\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)]. \end{aligned}$$

Applying above estimate to (5.13), we have

$$\begin{aligned} &E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] + (\frac{1}{2M^2} - K - 2)E[\int_s^m\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2\rho(dx)dr] \\ &\leq E[\int_D \mathrm{e}^{-Km}|Y_m^{t,x,n,l,m}|^2\rho(dx)] + 2\mathrm{e}^{-Km}c^2. \end{aligned}$$

As $l, m \to +\infty$, using (5.11) we have

$$E[\int_0^m \int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \leq CE[\sup_{m\leq s\leq l}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,n,l}|^2\rho(dx)] + C\mathrm{e}^{-Km}c^2 \longrightarrow 0. \tag{5.14}$$

Then, as $l, m \to +\infty$ it follows from (5.11), (5.13) and (5.14) that

$$\begin{aligned}
&E[\int_0^m \int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \\
&\leq CE[\sup_{m\leq s\leq l}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,n,l}|^2\rho(dx)] + C\mathrm{e}^{-Km}c^2 \\
&+CE[\int_0^m \int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \longrightarrow 0.
\end{aligned} \tag{5.15}$$

Also by the B-D-G inequality, (5.11)–(5.12) and (5.14)–(5.15), as l, $m \to +\infty$, we have

$$\begin{aligned}
&E[\sup_{0\leq s\leq m}\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] \\
&\leq CE[\sup_{m\leq s\leq l}\int_D \mathrm{e}^{-Ks}|Y_s^{t,x,n,l}|^2\rho(dx)] + C\mathrm{e}^{-Km}c^2 \\
&+CE[\int_0^m \int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2 \rho(dx)dr] + CE[\int_0^m \int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \longrightarrow 0.
\end{aligned}$$

Therefore taking a combination of cases (i) and (ii), as $l, m \to +\infty$, we have

$$\begin{aligned}
&E[\sup_{s\geq 0}\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n,l,m}|^2\rho(dx)] + E[\int_0^{+\infty} \int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \\
&+E[\int_0^{+\infty} \int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,n,l,m}|^2 \rho(dx)dr] \longrightarrow 0.
\end{aligned}$$

That is to say $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})$ is a Cauchy sequence. Take $(Y_s^{t,x,n}, Z_s^{t,x,n})$ as the limit of $(Y_s^{t,x,n,m}, Z_s^{t,x,n,m})$ in the space $S^{2,-K} \bigcap M^{2,-K}([0, +\infty); L^2_\rho(D; \mathbb{R})) \times M^{2,-K}([0, +\infty); L^2_\rho(D; \mathbb{R}^d))$ and we will show that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the solution of BDSDE (5.4). For this, we take the strong limit on both sides of (5.5) in $L^2(\Omega \times D; \mathbb{R})$, then the claim that $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is a solution to BDSDE (5.4) follows. We only take for example the convergence of the terms involving f. Firstly,

$$
\begin{aligned}
&\lim_{m\to+\infty} E[\int_D |\int_s^m \mathrm{e}^{-Kr} f_n(Y_r^{t,x,n,m})dr - \int_s^{+\infty} \mathrm{e}^{-Kr} f_n(Y_r^{t,x,n})dr|^2\rho(dx)]\\
&\le \lim_{m\to+\infty} 2E[\int_D |\int_s^m \mathrm{e}^{-Kr} f_n(Y_r^{t,x,n,m})dr - \int_s^m \mathrm{e}^{-Kr} f_n(Y_r^{t,x,n})dr|^2\rho(dx)]\\
&\quad + \lim_{m\to+\infty} 2E[\int_D |\int_m^{+\infty} \mathrm{e}^{-Kr} f_n(Y_r^{t,x,n})dr|^2\rho(dx)]\\
&\le \lim_{n\to+\infty} CE[\int_s^m \int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n,m} - Y_r^{t,x,n}|^2\rho(dx)dr]\\
&\quad + \lim_{m\to+\infty} CE[\int_m^{+\infty} \int_D \mathrm{e}^{-Kr}|Y_r^{t,x,n}|^2\rho(dx)dr] = 0.
\end{aligned}
$$

For the other term,

$$
\begin{aligned}
&\lim_{m\to+\infty} E[\int_D |\int_s^m \mathrm{e}^{-Kr} \int_D f_n(Y_r^{r,\xi,n,m})\rho(d\xi)dr - \int_s^{+\infty} \mathrm{e}^{-Kr} \int_D f_n(Y_r^{r,\xi,n})\rho(d\xi)dr|^2\rho(dx)]\\
&\le \lim_{m\to+\infty} 2E[\int_D |\int_s^m \mathrm{e}^{-Kr} \int_D f_n(Y_r^{r,\xi,n,m})\rho(d\xi)dr - \int_s^m \mathrm{e}^{-Kr} \int_D f_n(Y_r^{r,\xi,n})\rho(d\xi)dr|^2\rho(dx)]\\
&\quad + \lim_{m\to+\infty} 2E[\int_D |\int_m^{+\infty} \mathrm{e}^{-Kr} \int_D f_n(Y_r^{r,\xi,n})\rho(d\xi)dr|^2\rho(dx)]\\
&\le \lim_{m\to+\infty} CE[\int_s^m \int_D \mathrm{e}^{-Kr}|Y_r^{r,x,n,m} - Y_r^{r,x,n}|^2\rho(dx)dr]\\
&\quad + \lim_{m\to+\infty} CE[\int_m^{+\infty} \int_D \mathrm{e}^{-Kr}|Y_r^{r,x,n}|^2\rho(dx)dr] = 0.
\end{aligned}
$$

After verifying the convergence of each term, the existence of solution to BDSDE (5.4) is proved.

For the uniqueness of solution, let $(Y_s^{t,x,n}, Z_s^{t,x,n})$ and $(\hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})$ be two solutions of BDSDE (5.4). Define

$$
\begin{aligned}
&\bar{Y}_s^{t,x,n} = \hat{Y}_s^{t,x,n} - Y_s^{t,x,n}, \quad \bar{Z}_s^{t,x,n} = \hat{Z}_s^{t,x,n} - Z_s^{t,x,n},\\
&\bar{f}_n^t(s,x) = f_n(\hat{Y}_s^{t,x,n}) - f_n(Y_s^{t,x,n}), \qquad s \ge 0.
\end{aligned}
$$

Then for $s \ge 0$ and a.a. $x \in \mathbb{R}^d$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ and $(\hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n})$ satisfy

$$\mathrm{e}^{-Ks}\bar{Y}_s^{t,x,n} = \int_s^{+\infty} \mathrm{e}^{-Kr}[\bar{f}_n^t(r,x) - \int_D \bar{f}_n^r(r,\xi)\rho(d\xi)]dr + \int_s^{+\infty} K\mathrm{e}^{-Kr}\bar{Y}_r^{t,x,n}dr - \int_s^{+\infty} \mathrm{e}^{-Kr}\bar{Z}_r^{t,x,n}dW_r.$$

For an arbitrary interval $[0, T]$,

$$\bar{Y}_s^{t,x,n} = \bar{Y}_T^{t,x,n} + \int_s^T [\bar{f}_n^t(r,x) - \int_D \bar{f}_n^r(r,\xi)\rho(d\xi)]dr - \int_s^T \bar{Z}_r^{t,x,n}dW_r,$$

where $\bar{Y}_T^{t,x,n} = \hat{Y}_T^{t,x,n} - Y_T^{t,x,n} = \hat{Y}_T^{T,X_T^{t,x},n} - Y_T^{T,X_T^{t,x},n}$ satisfies Condition (H.1) and $\bar{f}_n^t(r,x) = f_n(\hat{Y}_s^{t,x,n}) - f_n(Y_s^{t,x,n}) = f_n(\hat{Y}_s^{s,X_s^{t,x},n}) - f_n(Y_s^{s,X_s^{t,x},n})$ can be regarded as a given function. Moreover,

$$\lim_{T\to+\infty} \mathrm{e}^{-KT}\bar{Y}_T^{t,x,n} = 0.$$

Similar to (3.12), applying Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n}|^2$ for a.a. $x \in \mathbb{R}^d$, we have

$$\begin{aligned} &E[\int_D \mathrm{e}^{-Ks}|\bar{Y}_s^{t,x,n}|^2\rho(dx)] - KE[\int_s^T\int_D \mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,n}|^2\rho(dx)dr] \\ &+E[\int_s^T\int_D \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,n}|^2\rho(dx)dr] \\ \le\ &E[\int_D \mathrm{e}^{-KT}|Y_T^{t,x,n}|^2\rho(dx)]. \end{aligned} \tag{5.16}$$

On the other hand, by Theorem 2.7, we know that $\bar{u}_n(s,x) \triangleq \bar{Y}_s^{s,x,n}$ is a weak solution to the following mass-conservative random PDE:

$$\begin{cases} d\bar{u}_n(t,x) = -[\frac{1}{2}\triangle\bar{u}_n(t,x) + \big(f_n(\hat{u}_n(t,x)) - f_n(u_n(t,x))\big) \\ \qquad\qquad -\int_D \big(f_n(\hat{u}_n(t,\xi)) - f_n(u_n(t,\xi))\big)\rho(d\xi)]dt, \quad t\in[0,T), \quad x\in D, \\ \bar{u}_n(T,x) = \bar{Y}_T^{T,x,n}, \quad x\in\overline{D}, \\ \frac{\partial\bar{u}_n}{\partial\mathbf{n}}(t,x) = 0, \quad t\in[0,T), \quad x\in\partial D. \end{cases}$$

So, for all $t \in [0, T]$ and $\omega \in \Omega$, in view of the invariant measure and (2.6),

$$\int_D \bar{u}_n(t,x)\rho(dx) = \int_D \bar{Y}_t^{t,x,n}\rho(dx) = E^W\int_D \bar{Y}_s^{t,x,n}\rho(dx) = 0.$$

Applying Lemma 3.2 to $\bar{u}_n$, we have

$$E[\int_D |\bar{Z}_s^{t,x,n}|^2 \rho(dx)] \geq \frac{1}{2M^2} E[\int_D |\bar{Y}_s^{t,x,n}|^2 \rho(dx)].$$

Then putting above inequality into (5.16), we have

$$\begin{aligned} & E[\int_D \mathrm{e}^{-Ks} |\bar{Y}_s^{t,x,n}|^2 \rho(dx)] + (\frac{1}{2M^2} - K) E[\int_s^T \int_D \mathrm{e}^{-Kr} |\bar{Y}_r^{t,x,n}|^2 \rho(dx) dr] \\ \leq & E[\int_D \mathrm{e}^{-KT} |\bar{Y}_T^{t,x,n}|^2 \rho(dx)]. \end{aligned} \tag{5.17}$$

Taking $K' > K$ such that K' satisfies the condition to K as well, we can see that (5.17) remains true with K replaced by K'. In particular,

$$E[\int_D \mathrm{e}^{-K's} |\bar{Y}_s^{t,x,n}|^2 \rho(dx)] \leq E[\int_D \mathrm{e}^{-K'T} |\bar{Y}_T^{t,x,n}|^2 \rho(dx)].$$

Therefore, we have

$$E[\int_D \mathrm{e}^{-K's} |\bar{Y}_s^{t,x,n}|^2 \rho^{-1}(x) dx] \leq \mathrm{e}^{-(K'-K)T} E[\int_D \mathrm{e}^{-KT} |\bar{Y}_T^{t,x,n}|^2 \rho^{-1}(x) dx]. \tag{5.18}$$

Since $\hat{Y}_s^{t,x,n}, Y_s^{t,x,n} \in S^{2,-K} \bigcap M^{2,-K}([0, +\infty); L_\rho^2(D; \mathbb{R}))$,

$$\sup_{T \geq 0} E[\int_D \mathrm{e}^{-KT} |\bar{Y}_T^{t,x,n}|^2 \rho(dx)] \leq E[\sup_{T \geq 0} \int_D \mathrm{e}^{-KT} (2|\hat{Y}_T^{t,x,n}|^2 + 2|Y_T^{t,x,n}|^2) \rho(dx)] < +\infty.$$

Therefore, taking the limit as $T \to +\infty$ in (5.18), we have

$$E[\int_D \mathrm{e}^{-K's} |\bar{Y}_s^{t,x,n}|^2 \rho(dx)] = 0.$$

The uniqueness of the solution is proved. ◇

Proposition 5.4. *Under Conditions (H.6)′, for arbitrary T and $t \in [0, T]$, define $v_n(t, \cdot) \triangleq Y_{T-t}^{T-t,\cdot,n}$, where $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the solution of time independent version of BDSDE (5.4) with the mass-constant c, then $v_n(t, \cdot)$ is a "perfect" stationary weak solution of the following SPDE*

$$\begin{cases} dv_n(t,x) = [\frac{1}{2}\triangle v_n(t,x) + f_n(v_n(t,x)) - \int_D f_n(v_n(t,x))\rho(dx)]dt \\ \quad -g(x)dB_t \quad t \in [0, +\infty), \quad x \in D, \\ \frac{\partial v_n}{\partial \mathbf{n}}(t,x) = 0 \quad t \in [0, +\infty), \quad x \in \partial D, \\ \int_D v_n(t,x)\rho(dx) = c \quad for\ the\ given\ constant\ c. \end{cases} \tag{5.19}$$

Proof. For arbitrary T and $t \in [0, T]$, by Proposition 5.3, $u_n(t,x) \triangleq Y_t^{t,x,n}$ satisfies the following SPDE

$$\begin{cases} du_n(t,x) = -[\frac{1}{2}\triangle u_n(t,x) + f_n(u_n(t,x)) - \int_D f_n(u_n(t,x))\rho(dx)]dt \\ \quad +g(x)d^\dagger \hat{B}_t,\ t \in [0,T),\ x \in D, \\ u_n(T,x) = Y_T^{T,x,n}, \quad x \in \overline{D}, \\ \frac{\partial u_n}{\partial \mathbf{n}}(t,x) = 0, \quad t \in [0,T), \quad x \in \partial D. \end{cases}$$

Moreover, by the uniqueness of the solution of BDSDE (5.4) we can follow Theorem 4.2 to know that $Y_t^{t,x,n}$ satisfies the "perfect" stationary property (4.9) with respect to $\hat{\theta}$.

Define $v_n(t,x) \triangleq Y_{T-t}^{T-t,x,n}$, then, by a time reversal transform it follows that $v_n(t,x)$ is a solution of the following SPDE

$$\begin{cases} dv_n(t,x) = [\frac{1}{2}\triangle v_n(t,x) + f_n(v_n(t,x)) - \int_D f_n(v_n(t,x))\rho(dx)]dt \\ \quad -g(x)dB_t, \quad t \in (0,T], \quad x \in D, \\ v_n(0,x) = Y_T^{T,x,n}, \quad x \in \overline{D}, \\ \frac{\partial v_n}{\partial \mathbf{n}}(t,x) = 0, \quad t \in (0,T], \quad x \in \partial D. \end{cases} \tag{5.20}$$

We can prove a claim that $v_n(t,x)$ does not depend on the choice of T using (4.9) and a similar proof as in [18], [19].

Furthermore, notice

$$\int_D v_n(t,x)\rho(dx) = \int_D Y_T^{T,x,n}\rho(dx) = \int_D Y_T^{t,x,n}\rho(dx) = c.$$

As T tends to $+\infty$, $v_n(t,x)$ satisfies an infinite horizon SPDE (5.19). Due to the "perfect" stationary property (4.9) for $Y_s^{t,x,n}$ with respect to $\hat{\theta}$, we know that $v_n(t,x)$ is a "perfect" stationary solution of SPDE (5.19) with respect to θ. $\diamond$

We turn back to SPDE (5.1). To pass the limits in (5.19) in some desired sense, we need that $(Y_\cdot^{\cdot,\cdot,n}, Z_\cdot^{\cdot,\cdot,n})$ are bounded in some spaces uniformly in n.

Lemma 5.5. *Under Conditions (H.6)′, if $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ is the solution of BDSDE (5.4), then we have*

$$\sup_n \sup_{s\geq t} \sup_{t\geq 0} E[\int_D |Y_s^{t,x,n}|^2\rho(dx)] + \sup_n \sup_{t\geq 0} E[\int_t^{+\infty}\int_D |Z_s^{t,x,n}|^2\rho(dx)ds] < +\infty.$$

Proof. Define $\psi_M(y) = y^2 I_{\{-M\leq y<M\}} + M(2y-M)I_{\{y\geq M\}} - M(2y+M)I_{\{y<-M\}}$. Obviously, for any $y \in \mathbb{R}$, $\psi_M(y) \longrightarrow y^2$ as $M \to +\infty$. Applying Itô formula to $\mathrm{e}^{-\lambda r}\psi_M(Y_r^{t,x,n})$ we have

$$\int_D \mathrm{e}^{-\lambda s}\psi_M(Y_s^{t,x,n})\rho(dx) + \int_s^T\int_D \mathrm{e}^{-\lambda r} I_{\{-M\leq Y_r^{t,x,n}<M\}}|Z_r^{t,x,n}|^2\rho(dx)dr$$

$$-\lambda \int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M(Y_r^{t,x,n})^2 \rho(dx)dr$$

$$= \int_D \mathrm{e}^{-\lambda T} \psi_M(Y_T^{t,x,n})\rho(dx) + \int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M'(Y_r^{t,x,n})\big[f_n(Y_r^{t,x,n}) - \int_D f_n(Y_r^{r,x,n})\rho(dx)\big]\rho(dx)dr$$

$$+ \int_s^T \int_D \mathrm{e}^{-\lambda r} I_{\{-M\le Y_r^{t,x,n}<M\}} \|g(X_r^{t,x})\|^2 \rho(dx)dr - \int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M'(Y_r^{t,x,n}) g(X_r^{t,x})\rho(dx)d^\dagger \hat{B}_r$$

$$- \int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M'(Y_r^{t,x,n}) Z_r^{t,x,n} \rho(dx) dW_r. \tag{5.21}$$

Since the stochastic integrals are martingales, we have

$$E[\int_D \mathrm{e}^{-\lambda s}\big(\psi_M(Y_s^{t,x,n})\rho(dx)] + E[\int_s^T \int_D \mathrm{e}^{-\lambda r} I_{\{-M\le Y_r^{t,x,n}<M\}} |Z_r^{t,x,n}|^2 \rho(dx)dr]$$

$$-\lambda E[\int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M(Y_r^{t,x,n})^2 \rho(dx)dr]$$

$$= E[\int_D \mathrm{e}^{-\lambda T} \psi_M(Y_T^{t,x,n})\rho(dx)] + E[\int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M'(Y_r^{t,x,n})\big[\tilde{f}_n(Y_r^{t,x,n})$$

$$- \int_D \tilde{f}_n(Y_r^{r,x,n})\rho(dx)\big]\rho(dx)dr] + E[\int_s^T \int_D \mathrm{e}^{-\lambda r} \psi_M'(Y_r^{t,x,n})(Y_r^{t,x,n}$$

$$- \int_D Y_r^{r,x,n}\rho(dx))\rho(dx)dr] + E[\int_s^T \int_D \mathrm{e}^{-\lambda r} I_{\{-M\le Y_r^{t,x,n}<M\}} \|g(X_r^{t,x})\|^2 \rho(dx)dr].$$

Taking $M \to +\infty$ we have

$$E[\int_D \mathrm{e}^{-\lambda s} |Y_s^{t,x,n}|^2 \rho(dx)] + E[\int_s^T \int_D \mathrm{e}^{-\lambda r} |Z_r^{t,x,n}|^2 \rho(dx)dr]$$

$$-\lambda E[\int_s^T \int_D \mathrm{e}^{-\lambda r} |Y_r^{t,x,n}|^2 \rho(dx)dr] - 2E[\int_s^T \int_D \mathrm{e}^{-\lambda r} Y_r^{t,x,n}(Y_r^{t,x,n} - \int_D Y_r^{r,x,n}\rho(dx))\rho(dx)dr]$$

$$= E[\int_D \mathrm{e}^{-\lambda T}|Y_T^{t,x,n}|^2\rho(dx)] + E[\int_s^T\int_D \mathrm{e}^{-\lambda r}\|g(X_r^{t,x})\|^2\rho(dx)dr]$$

$$+2E[\int_s^T\int_D \mathrm{e}^{-\lambda r}Y_r^{t,x,n}\big[\tilde{f}_n(Y_r^{t,x,n}) - \int_D \tilde{f}_n(Y_r^{r,x,n})\rho(dx)\big]\rho(dx)dr]. \tag{5.22}$$

Actually,

$$2E[\int_s^T\int_D \mathrm{e}^{-\lambda r}Y_r^{t,x,n}\big[\tilde{f}_n(Y_r^{t,x,n}) - \int_D \tilde{f}_n(Y_r^{r,x,n})\rho(dx)\big]\rho(dx)dr] \leq 0. \tag{5.23}$$

To see this, first note that due to the invariant measure ρ, we have

$$2E[\int_s^T\int_D \mathrm{e}^{-\lambda r}Y_r^{t,x,n}\big[\tilde{f}_n(Y_r^{t,x,n}) - \int_D \tilde{f}_n(Y_r^{r,x,n})\rho(dx)\big]\rho(dx)dr]$$

$$= 2E^B[E^W[\int_s^T\int_D \mathrm{e}^{-\lambda r}Y_r^{t,x,n}\big[\tilde{f}_n(Y_r^{t,x,n}) - E^W[\int_D \tilde{f}_n(Y_r^{t,x,n})\rho(dx)]\big]\rho(dx)dr]].$$

We only need to prove

$$E^W[\int_D Y_r^{t,x,n}\tilde{f}_n(Y_r^{t,x,n})\rho(dx)] - E^W[\int_D Y_r^{t,x,n}\rho(dx)]E^W[\int_D \tilde{f}_n(Y_r^{t,x,n})\rho(dx)] \leq 0.$$

For this, we divide the left hand side into three cases by the indicator functions.

Case 1: $|Y_r^{t,x,n}| \leq n$.

$$E^W[\int_D Y_r^{t,x,n}\tilde{f}_n(Y_r^{t,x,n})\rho(dx)] - E^W[\int_D Y_r^{t,x,n}\rho(dx)]E^W[\int_D \tilde{f}_n(Y_r^{t,x,n})\rho(dx)]$$

$$\leq -E^W[\int_D |Y_r^{t,x,n}|^{p+1}\rho(dx)] + \big(E^W[\int_D (Y_r^{t,x,n})^{p+1}\rho(dx)]\big)^{\frac{1}{p+1}}\big(E^W[\int_D (Y_r^{t,x,n})^{p+1}\rho(dx)]\big)^{\frac{p}{p+1}}$$

$$= 0.$$

Case 2: $Y_r^{t,x,n} < -n$.

$$E^W[\int_D Y_r^{t,x,n}\tilde{f}_n(Y_r^{t,x,n})\rho(dx)] - E^W[\int_D Y_r^{t,x,n}\rho(dx)]E^W[\int_D \tilde{f}_n(Y_r^{t,x,n})\rho(dx)]$$

$$= E^W[\int_D (n^pY_r^{t,x,n} - pn^{p-1}|Y_r^{t,x,n}|^2 - pn^pY_r^{t,x,n})\rho(dx)]$$

$$
\begin{aligned}
&-E^W[\int_D Y_r^{t,x,n}\rho(dx)]E^W[\int_D (n^p - pn^{p-1}Y_r^{t,x,n} - pn^p)\rho(dx)]\\
&= -pn^{p-1}E^W[\int_D |Y_r^{t,x,n}|^2\rho(dx)] + pn^{p-1}(E^W[\int_D Y_r^{t,x,n}\rho(dx)])^2\\
&\le -pn^{p-1}E^W[\int_D |Y_r^{t,x,n}|^2\rho(dx)] + pn^{p-1}E^W[\int_D |Y_r^{t,x,n}|^2\rho(dx)]\\
&= 0.
\end{aligned}
$$

Case 3: $Y_r^{t,x,n} > n$.
It is similar to Case 2.

On the other hand,

$$
\begin{aligned}
&-2E[\int_s^T\int_D e^{-\lambda r}Y_r^{t,x,n}(Y_r^{t,x,n} - \int_D Y_r^{r,x,n}\rho(dx))\rho(dx)dr]\\
&= -2E[\int_s^T\int_D e^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr]\\
&\quad +2E^B[E^W[\int_s^T\int_D e^{-\lambda r}Y_r^{t,x,n}E^W[\int_D Y_r^{t,x,n}\rho(dx)]\rho(dx)dr]]\\
&= -2E[\int_s^T\int_D e^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr] + 2E^B[\int_s^T e^{-\lambda r}|E^W[\int_D Y_r^{t,x,n}\rho(dx)]|^2dr]\\
&\ge -2E[\int_s^T\int_D e^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr]. \qquad (5.24)
\end{aligned}
$$

Hence (5.23) and (5.24) are true, and putting them into (5.22) we have

$$
\begin{aligned}
&E[\int_D e^{-\lambda s}|Y_s^{t,x,n}|^2\rho(dx)] + E[\int_s^T\int_D e^{-\lambda r}|Z_r^{t,x,n}|^2\rho(dx)dr]\\
&-(\lambda+2)E[\int_s^T\int_D e^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr]\\
&\le E[\int_D e^{-\lambda T}|Y_T^{t,x,n}|^2\rho(dx)] + E[\int_s^T\int_D e^{-\lambda r}\|g(X_r^{t,x})\|^2\rho(dx)dr]. \qquad (5.25)
\end{aligned}
$$

Similar to (3.2), we use the Poincare inequality to have

$$E[\int_D \mathrm{e}^{-\lambda s}|Y_s^{t,x,n}|^2\rho(dx)] - (\lambda+2)E[\int_s^T\int_D \mathrm{e}^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr]$$
$$+\frac{1}{2M^2}E[\int_s^T\int_D \mathrm{e}^{-\lambda r}|Y_r^{t,x,n}|^2\rho(dx)dr]$$
$$\leq E[\int_D \mathrm{e}^{-\lambda T}|Y_T^{t,x,n}|^2\rho(dx)] + \int_s^T\int_D \mathrm{e}^{-\lambda r}\|g(x)\|^2\rho(dx)dr + \frac{Kc^2}{M^2}(\mathrm{e}^{-\lambda s}-\mathrm{e}^{-\lambda T}).$$

Taking $\lambda = \frac{1}{2M^2}+2$, we have

$$E[\int_D |Y_s^{t,x,n}|^2\rho(dx)] \tag{5.26}$$
$$= E[\int_D \mathrm{e}^{-\lambda(T-s)}|Y_T^{T,x,n}|^2\rho(dx)] + \int_s^T\int_D \mathrm{e}^{-\lambda(r-s)}\|g(x)\|^2\rho(dx)dr + \frac{Kc^2}{M^2}(1-\mathrm{e}^{-\lambda(T-s)}).$$

For any T, $Y_T^{T,x,n} = \hat{\theta}_T Y_0^{0,x,n}$. Putting this into (5.26), we have

$$E[\int_D |Y_s^{t,x,n}|^2\rho(dx)]$$
$$= \mathrm{e}^{-\lambda(T-s)}E[\int_D |Y_0^{0,x,n}|^2\rho(dx)] + \int_s^T\int_D \mathrm{e}^{-\lambda(r-s)}\|g(x)\|^2\rho(dx)dr + \frac{Kc^2}{M^2}(1-\mathrm{e}^{-\lambda(T-s)}).$$

Taking $T\to+\infty$ in above, we have for any $s\geq t$,

$$\sup_n\sup_{s\geq t}\sup_{t\geq 0} E[\int_D |Y_s^{t,x,n}|^2\rho(dx)] < +\infty.$$

From (5.25), we also have that

$$\sup_n\sup_{t\geq 0} E[\int_t^{+\infty}\int_D |Z_s^{t,x,n}|^2\rho(dx)ds] < +\infty. \quad \diamond$$

Moreover, we have a bounded estimate for the solution of SPDE (5.19) in the following sense.

Theorem 5.6. *Let v_n be the solution of SPDE (5.19). Under Conditions (H.6)$'$, there exists a full measure set $\widetilde{\Omega \times D} \subset \Omega \times D$ with $(P \times \rho)[\widetilde{\Omega \times D}] = 1$ satisfying for each $(\omega, x) \in \widetilde{\Omega \times D}$, $\exists$ $N(\omega, x) > 0$ such that when $n \geq N(\omega, x)$, $|v_n(t, x, \omega)| \leq n$ for each $t \geq 0$.*

Proof. From Lemma 5.5 we know

$$\sup_n \sup_{t\geq 0} E[\int_D |v_n(t,x)|^2 \rho(dx)] = \sup_n \sup_{t\geq 0} E[\int_D |Y_t^{t,x,n}|^2 \rho(dx)] < +\infty.$$

For $t \geq 0$, set $A_n^t = \{(\omega, x) \in \Omega \times D : |v_n(t, x, \omega)| > n\}$. For arbitrary n, by Chebyshev's inequality, we have

$$(P \times \rho)[A_n^0] \leq \frac{1}{n^2} E[\int_D |v_n(0,x)|^2 \rho(dx)] \leq \frac{1}{n^2} \sup_n \sup_{t\geq 0} E[\int_D |v_n(t,x)|^2 \rho(dx)].$$

Set $(\widetilde{\Omega \times D})_t = \bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} (A_n^t)^c$ for $t \geq 0$. Then $(P \times \rho)[(\widetilde{\Omega \times D})_0] = 1$ by the Borel-Cantelli Lemma. Moreover, it is easy to see from definition of $(\widetilde{\Omega \times D})_0$ that, for each $(\omega, x) \in (\widetilde{\Omega \times D})_0$, $\exists$ $N(\omega, x) > 0$ such that for all $n \geq N(\omega, x)$, $|v_n(0, x)| \leq n$.

Let $\mathbb{Q}$ be the set of rational numbers in $[0, +\infty)$. Note that by Proposition 5.4, $v_n(t, x)$ is a stationary weak solution if the given mass is a constant and the function g is independent of the time variable t. This suggests for any $t' \in \mathbb{Q}$, $v_n(t', x) = \theta_{t'} v_n(0, x)$. Due to the probability-preserving property of $\theta_\cdot$,

$$(\widetilde{\Omega \times D})_{t'} = \theta_{t'}(\widetilde{\Omega \times D})_0 = \bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} \{(\omega, x) \in \Omega \times D : |v_n(0, x, \theta_{t'}\omega)| \leq n\}.$$

Thus $(P \times \rho)[(\widetilde{\Omega \times D})_t] = 1$.

Define now a full-measure set $\widetilde{\Omega \times D} = \bigcap_{t' \in \mathbb{Q}} (\widetilde{\Omega \times D})_{t'} \subset \Omega \times D$. Note

$$\widetilde{\Omega \times D} = \bigcap_{t' \in \mathbb{Q}} \bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} (A_n^{t'})^c = \bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} \bigcap_{t' \in \mathbb{Q}} (A_n^{t'})^c.$$

Then for each $(\omega, x) \in \widetilde{\Omega \times D}$, $\exists$ $N(\omega, x) > 0$ such that for each $n \geq N$, $(\omega, x) \in \bigcap_{t' \in \mathbb{Q}} (A_n^{t'})^c$, i.e. for each $t' \in \mathbb{Q}$, $|v_n(t', \omega, x)| \leq n$. Note that for each $t \geq 0$, there exists a sequence t_m, $m \in \mathbb{Q}$, such that $t = \lim_{m\to+\infty} t_m$. Hence $v_n(t, x) = \lim_{m\to+\infty} v_n(t_m, x)$ by the continuity of v_n in the time variable t proved in Proposition 2.8. Thus for each $(\omega, x) \in \widetilde{\Omega \times D}$, when $n \geq N(\omega, x)$, $|v_n(t, x, \omega)| \leq n$ for each $t \geq 0$. $\diamond$

We denote by A the event that there exists $N(\omega, x) \geq 0$ such that when $n \geq N(\omega, x)$, $|v_n(t, x, \omega)| \leq n$ for each $t \geq 0$. By Theorem 5.6,

$$A \text{ holds on a full} - \text{measure set in } \Omega \times D. \tag{5.27}$$

For any fixed $\omega \in \Omega$, denote by $\tilde{D}(\omega)$ the set in D such that ω happens and A holds. Obviously, for a.s. $\omega \in \Omega$, $\tilde{D}(\omega)$ is a full measure set in D. Otherwise, there is a subset $\Omega^* \subset \Omega$ with $P[\Omega^*] > 0$ such that when $\omega \in \Omega^*$, $\rho[\tilde{D}(\omega)] < 1$. Consider $(\Omega \times D)^* = \{(\omega, x) : \omega \in \Omega^*, x \in \big(\tilde{D}(\omega)\big)^c\}$ Then $(P \times \rho)[(\Omega \times D)^*] = \int_{\Omega^*} \rho[\big(\tilde{D}(\omega)\big)^c] P[d\omega] > 0$ and for $(\omega, x) \in (\Omega \times D)^*$, A does not hold. This contradicts with (5.27). Therefore, for a.s. $\omega \in \Omega$, there exists $N(\omega) \geq 0$ such that when $n \geq N(\omega)$, SPDE (5.19) satisfied by v_n coincides with equation (5.1):

$$\begin{cases} dv_n(t,x) = [\frac{1}{2}\triangle v_n(t,x) - v_n^p(t,x) + v_n(t,x) + \int_D \big(v_n^p(t,x) - v_n(t,x)\big)\rho(dx)]dt \\ \quad -g(x)dB_t, \quad x \in D, \\ \frac{\partial v_n}{\partial \mathbf{n}}(t,x) = 0, \quad x \in \partial D, \\ \int_D v_n(t,x)\rho(dx) = c \quad for\ a\ given\ constant\ c. \end{cases}$$

In this way, we find a stationary weak solution of SPDE (5.1) for a.s. $\omega \in \Omega$.

With the above results and following a similar argument as in Section 4, we obtain the following result.

Theorem 5.7. *Under Conditions (H.6)′, for arbitrary T and $t \in [0,T]$, define $v_n(t,\cdot) \triangleq Y_{T-t}^{T-t,\cdot,n}$, where $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the solution of BDSDE (5.4), then for sufficiently large n, $v_n(t,\cdot)$ is a weak solution of SPDE (5.1). Moreover, $v_n(t,\cdot)$ is a "perfect" stationary solution and its law is the invariant measure of the Markov progress generated by SPDE (5.1).*

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