A simple formula for the conserved charges of soliton theories

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Abstract

We present a simple formula for all the conserved charges of soliton theories, evaluated on the solutions belonging to the orbit of the vacuum under the group of dressing transformations. For pedagogical reasons we perform the explicit calculations for the case of the sine-Gordon model, taken as a prototype of soliton theories. We show that the energy and momentum are boundary terms for all the solutions on the orbit of the vacuum. That orbit includes practically all the solutions of physical interest, namely solitons, multi-solitons, breathers, and combinations of solitons and breathers. The example of the mKdV equation is also given explicitly.

1 Introduction

In the last years there appeared in the literature several results pointing to the fact that the conserved charges, especially the energy, of $1+1$ dimensional integrable field theories take the form of boundary terms when evaluated on soliton solutions. In other words, the charges seem to depend only on the asymptotic values of the fields at infinity. Using arguments based on conformal symmetry, it was shown in [1] that the energy of one-soliton solutions of all abelian affine Toda theories is determined by the the asymptotic value of a Belinfante type term (which 'improves' the energy momentum tensor). The same result was obtained in [2] by explicit calculations of the energy integral. By considering one of the light cone variables as the time, it was shown in [3] that the corresponding chiral charges are surface terms. All these results relied on the extensions of the Toda field theories proposed in [4, 5, 6], by the addition of extra fields to render them conformally invariant. Similar results were also obtained using Backlund transformations [7].

In order to illustrate the statements above, consider the example of the sine-Gordon model. Its conformal extension carried out along the lines of [4, 5, 6] is defined by the eqs. of motion

$$\begin{aligned} \partial^2\varphi &= -e^{\eta}\sin\varphi, \\ \partial^2\eta &= 0, \\ \partial^2\rho &= e^{\eta}\left(1-\cos\varphi\right). \end{aligned} \tag{1.1}$$

The theory is invariant under the conformal transformations[1] $x_{\pm}\rightarrow f_{\pm}\left(x_{\pm}\right)$ if the sine-Gordon field φ is a scalar under the conformal group and if $e^{-\eta}\rightarrow f_{+}^{\prime}f_{-}^{\prime}e^{-\eta}$. The conformal weights of ρ are arbitrary. The Lagangrean for (1.1) is given by

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\varphi\right)^{2}-\partial_{\mu}\eta\,\partial^{\mu}\rho-e^{\eta}\left(1-\cos\varphi\right) \tag{1.2}$$

and the improved energy-momentum tensor by

$$T_{\mu\nu}=\Theta_{\mu\nu}+2\left(\partial_{\mu}\partial_{\nu}-g_{\mu\nu}\,\partial^{2}\right)\rho, \tag{1.3}$$

where $\Theta_{\mu\nu}$ is the canonical energy-momentum tensor

$$\Theta_{\mu\nu}=\partial_{\mu}\varphi\,\partial_{\nu}\varphi-\partial_{\mu}\rho\,\partial_{\nu}\eta-\partial_{\mu}\eta\,\partial_{\nu}\rho-g_{\mu\nu}\,\mathcal{L}. \tag{1.4}$$

The second term in (3) is the above mentioned Belinfante type term [8].

[1] The light cone coordinates $x_{\pm}$ are defined in (3.2).

As a consequence of the conformal symmetry $T_{\mu\nu}$ is indeed traceless. The Hamiltonian of the pure sine-Gordon theory is obtained by considering the solutions where the free field η is a constant (a spontaneous symmetry breaking of the conformal symmetry) and is given by

$$\mathcal{H}_{SG} = \Theta_{00} \mid_{\eta=0} = \frac{1}{2}\left(\partial_t \varphi\right)^2 + \frac{1}{2}\left(\partial_x \varphi\right)^2 + 1 - \cos\varphi. \tag{1.5}$$

In [1, 2] it was shown that the energy measured by the improved tensor, namely $\int_{-\infty}^{\infty} dx\, T_{00}$, vanishes when evaluated on the soliton solutions. Therefore, the energy measured by the sine-Gordon canonical energy momentum tensor takes the form of a surface term, *i.e.*

$$E = \int dx\, \mathcal{H}_{SG} = -2 \int_{-\infty}^{\infty} dx\, \partial_x^2\, \rho = -2\partial_x \rho \mid_{x=-\infty}^{x=\infty} . \tag{1.6}$$

In this paper we extend this result by showing that not only the soliton solutions but all solutions connected to the vacuum by the so-called dressing transformations, have the energy and momentum given by the boundary terms (1.6). The orbit of solutions obtained this way includes practically all solutions of physical interest like solitons, multi-solitons, breathers, combinations of solitons and breathers, etc. We also give a simple formula for the higher conserved charges for the same orbit of solutions. For instance, for the case of 1-soliton solutions we show that the conserved charges take the form $(n = 0, 1, 2, \ldots)$

$$\Omega_{2n+1}^{(\pm)} = \pm 2 \left[\frac{1+v}{1-v}\right]^{\pm\frac{(2n+1)}{2}} , \tag{1.7}$$

where v is the velocity of the soliton. For the breather solution they are given by

$$\Omega_{2n+1}^{(\pm)} = \pm 4\, \varepsilon \left[\frac{1+v}{1-v}\right]^{\pm\frac{(2n+1)}{2}} \cos\left[(2n+1)\ \theta\right], \tag{1.8}$$

where again v is the velocity of the breather, and the angle θ is related to the breather oscillation frequency ω by $\sin\theta = \omega$. For multi-soliton or multi-breather solutions, and also for solutions that are combinations of solitons and breathers, all the charges simply add up.

We also show that the reasons underlying such results are not really connected to the conformal symmetry. They are a consequence of very special algebraic structures appearing in the construction of the solutions by the dressing transformation method. The first important point is the behavior of the Wilson path ordered integral, used in the construction of the conserved charges, under the dressing transformation. One can write that integral in terms of its vacuum value, which is simple, and the asymptotic values of the group element performing the dressing (gauge) transformation. The second important point relates to special decompositions of those group elements involving oscillators algebras defined by the vacuum solution. We point out however, that in order to prove our results one has to work with the Kac-Moody algebra with a non-trivial central extension, even when the soliton theory needs a zero curvature representation (Lax-Zakharov-Shabat equation) based on a loop algebra only. In many cases, that implies the introduction of an extra field on the lines of [4, 5, 6].

We also extend such results to any integrable hierarchy possessing the basic ingredients for the appearance of soliton solutions. We work out the explicit formulas for all the conserved charges for such theories. The example of the modified Korteweg-de Vries (mKdV) equation is given explicitly to ilustrate that our method also works for non-Lorentz invariant theories appearing in fluid dynamics.

The paper is organized as follows: in section 2 we discuss the construction of the conserved charges of integrable theories using a flat connection satisfying the Lax-Zakharov-Shabat equation, and show how the charges relate to their vacuum value under the dressing transformations. In section 3 we discuss in detail the case of the sine-Gordon model as a prototype of soliton theories, and evaluate the charges explicitly. Section 4 is devoted to the generalization of our results for any soliton theory satisfying the conditions given at the beginning of that section. The example of the mKdV equation is given in section 4.1. In appendix A we give some results about representation theory of Kac-Moody algebras needed in the paper.

2 The conserved charges

A $1+1$ dimensional integrable field theory admits a representation of its equations of motion in terms of the so-called zero curvature condition, or Lax-Zakharov-Shabat equation [9]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [\, A_\mu \, , \, A_\nu \,] = 0 \qquad\qquad \mu, \nu = 0, 1, \tag{2.1}$$

where A_μ is a Lie algebra valued vector field which is a functional of the physical fields of the theory. The vanishing of the curvature $F_{\mu\nu}$ is equivalent to the equations of motion of the underlying field theory. One of the key points of eq. (2.1) is that it constitutes conservation laws in $1+1$ dimensions. The construction of the corresponding conserved charges is obtained as follows: Consider a path $\mathcal{C}$ going from an initial point P_0 to a

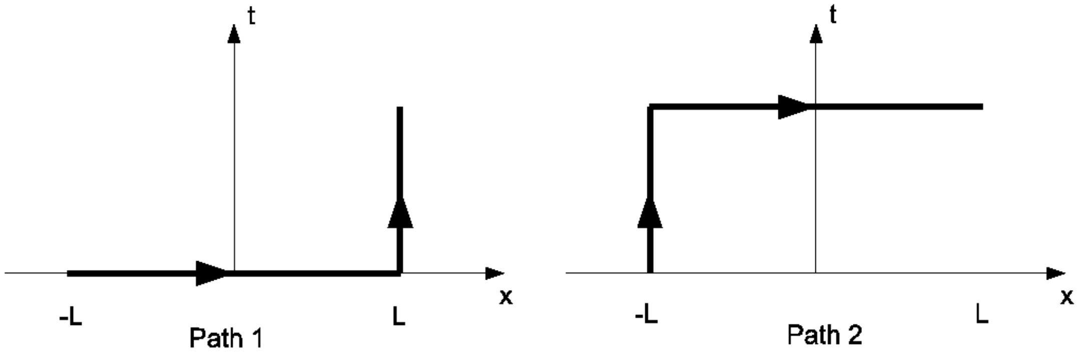

Figure 1:

final point P_1, and let the quantity W be defined on $\mathcal{C}$ through the differential equation

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0, \tag{2.2}$$

where σ parametrizes $\mathcal{C}$. The solution of (2.2) is the path ordered integral

$$\begin{aligned} W &= P\, e^{-\int_C d\sigma\, A_\mu \frac{dx^\mu}{d\sigma}} \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^\sigma d\sigma_1 \int_0^{\sigma_1} d\sigma_2 \ldots \int_0^{\sigma_{n-1}} d\sigma_n\, A_{\mu_1}(\sigma_1) \frac{dx^{\mu_1}}{d\sigma_1} \ldots A_{\mu_n}(\sigma_n) \frac{dx^{\mu_n}}{d\sigma_n}. \end{aligned}$$

Eq. (2.1) is the sufficient condition for W to be path independent as long as the initial and end points of $\mathcal{C}$ are kept fixed [10]. Then if we take the two paths shown in figure 1 we get that ($x^0 \equiv t$, $x^1 \equiv x$)

$$\begin{aligned} & P \exp\left(-\int_0^t dt\, A_t \mid_{x=L}\right) P \exp\left(-\int_{-L}^L dx\, A_x \mid_{t=0}\right) = \\ = & P \exp\left(-\int_{-L}^L dx\, A_x \mid_{t=t}\right) P \exp\left(-\int_0^t dt\, A_t \mid_{x=-L}\right). \end{aligned}$$

Next we impose the boundary condition (for $L \to \infty$)

$$A_t \mid_{x=L} = A_t \mid_{x=-L} + \beta\, C \tag{2.3}$$

where β is some function of t and L and C is the central charge of the algebra. Then one gets the quasi-isospectral evolution equation

$$W_t = e^{\int_0^t dt\, \beta\, C}\, U(t)\, W_0\, U(t)^{-1}, \tag{2.4}$$

where

$$W_{0/t} = P \exp\left(-\int_{-L}^L dx\, A_x \mid_{t=0/t}\right) \; ; \qquad U(t) = P \exp\left(-\int_0^t dt\, A_t \mid_{x=L}\right). \tag{2.5}$$

If β vanishes we have a pure isospectral evolution. Therefore, the eigenvalues of W_t are constant in time, and constitute the conserved charges of the underlying field theory associated with (2.1). However, if β does not vanish we still can have conserved charges in some circumstances. For instance, suppose that the operator Ψ_0 is an eigenstate of W_0 under the adjoint action

$$W_0 \, \Psi_0 \, W_0^{-1} = \lambda \, \Psi_0. \tag{2.6}$$

Then, the operator $\Psi_t = U(t) \; \Psi_0 \, U(t)^{-1}$ is an eigenstate of W_t with the same eigenvalue, since the central term C commutes with every operator.

Another key point of eq. (2.1) is that it is invariant under the gauge transformations

$$A_\mu \to A_\mu^g = g \, A_\mu \, g^{-1} - \partial_\mu g \, g^{-1}, \tag{2.7}$$

where g is an element of the Lie group associated to the Lie algebra of A_μ. Under (2.7) the path ordered integral transforms as

$$W(\mathcal{C}) \to W^g(\mathcal{C}) = g(P_1) \; W(\mathcal{C}) \; g^{-1}(P_0), \tag{2.8}$$

where P_0 and P_1 are the initial and final points of $\mathcal{C}$ respectively. One then observes that the conserved charges are invariant under those gauge transformations for which $g(t, x=-L) = e^{\alpha C} \, g(t, x=L)$, since if Ψ_t is an eigenvector of W_t under the adjoint action, *i.e.* $W_t \, \Psi_t \, W_t^{-1} = \lambda \, \Psi_t$, so is $\Psi_t^g = g(t, x=L) \; \Psi_t \, g(t, x=L)^{-1}$, an eigenvector of $W_t^g = e^{-\alpha C} \, g(t, x=L) \; W_t \, g^{-1}(t, x=L)$. However, we are more interested in the gauge transformations that do change the values of the conserved charges, as we explain below.

The transformations (2.7) constitute the so-called hidden symmetries of the underlying field theory associated with (2.1), in the sense that they are not symmetries of the equations of motion but of the zero curvature condition. Under some circumstances, the transformations (2.7) constitute a map among solutions of the theory. In fact, all the soliton and multi-soliton solutions in $1+1$ dimensions can be constructed using special transformations of the type (2.7), named dressing transformations, starting from a simple vacuum solution. Therefore, if one knows the operator $W_t^{(\text{vac.})}$ associated to a given simple vacuum solution, and knows the dressing transformation that maps that vacuum solution to a non-trivial solution, like a soliton, then the corresponding operator will be given by

$$W_t = g(t, x=L) \; W_t^{(\text{vac.})} \, g^{-1}(t, x=-L). \tag{2.9}$$

Consequently, the conserved charges evaluated on such non-trivial solutions, which are the eigenvalues of W_t, will depend upon the eigenvalues of $W_t^{(\text{vac.})}$, which are trivial, and on the asymptotic values of the group element performing the transformation. In many cases, that will imply that the charges are surface terms as we now explain on some concrete examples.

3 The case of the sine-Gordon model

The standard zero curvature condition (2.1) for the sine-Gordon model involves potentials A_μ which live in a $sl(2)$ loop algebra, *i.e.*, they are 2×2 matrices depending on a so-called spectral parameter. That is an infinite dimensional Lie algebra without a central extension. The potentials are given by

$$\begin{aligned} A_+ &= \frac{1}{2}\begin{pmatrix} 0 & e^{i\varphi} \\ \lambda\, e^{-i\varphi} & 0 \end{pmatrix} = \frac{1}{2}\left(\cos\varphi\, b_1 + i\,\sin\varphi\, F_1\right), \\ A_- &= -\frac{1}{2}\begin{pmatrix} i\,\partial_-\varphi & 1/\lambda \\ 1 & -i\,\partial_-\varphi \end{pmatrix} = -\frac{1}{2}\, b_{-1} - \frac{i}{2}\,\partial_-\varphi\, F_0, \end{aligned} \tag{3.1}$$

where the 2×2 matrix representation for $b_{\pm 1}$, F_0 and F_1 are given in (A.6) and λ is the so-called spectral parameter. Moreover, in (3.1) we have used light cone coordinates

$$x_\pm = \frac{1}{2}\,(t\pm x) \qquad \partial_\pm = \partial_t \pm \partial_x \qquad \partial^2 = \partial_t^2 - \partial_x^2 = \partial_+\partial_-. \tag{3.2}$$

Putting (3.1) into the zero curvature condition (2.1) one finds that the diagonal part of the matrices gives the sine-Gordon equation

$$\partial^2\varphi = -\sin\varphi \tag{3.3}$$

and the off-diagonal part is satisfied trivially.

However, to present our arguments that demonstrate the existence of conserved charges we need to centrally extend the basic algebra. We will then work with the full $sl(2)$ Kac-Moody algebra. In order for the zero curvature to remain valid on such algebra, it is necessary to extend the sine-Gordon model by the addition of an extra scalar field. Furthermore, for the theory to possess a Lagrangian, we need to add a further scalar field, which in fact renders the model conformally invariant. This way we end up with the so-called conformal sine-Gordon model [4, 5, 6] defined by the equations of motion (1.1).

The three equations (1.1) are equivalent to (2.1) with the potentials A_μ given by

$$\begin{aligned} A_+ &= \frac{1}{2}\, e^{\eta}\,\left(\cos\varphi\, b_1 + i\,\sin\varphi\, F_1\right), \\ A_- &= -\frac{1}{2}\, b_{-1} - \frac{i}{2}\,\partial_-\varphi\, F_0 - \partial_-\eta\, Q - \frac{1}{4}\,\partial_-\left(\rho+\gamma\right)\, C, \end{aligned} \tag{3.4}$$

where γ is a function satisfying

$$\partial_+\partial_-\gamma = -e^{\eta} \tag{3.5}$$

and where F_0, F_1, $b_{\pm 1}$ and C are generators of a $sl(2)$ Kac-Moody algebra [11]. Its generators and commutation relations are defined on the appendix A. This time, *i.e.* for $C \neq 0$, we do not have a finite matrix representation of the algebra and we have to proceed using just the commutation relations given in the appendix A.

However, we are really interested only in finite energy solutions of the pure sine-Gordon model ($\eta = 0$). From (1.5) one sees that the finiteness of the energy requires that $\varphi\,(t, x = \pm L) \rightarrow 2\pi n_{\pm}$, with $n_{\pm}$ integers, for $L \rightarrow \infty$, but this does not impose any condition on the behaviour of the ρ field. Therefore, the potentials (3.4) can satisfy the boundary condition (2.3) since

$$\begin{aligned} A_t\,(t, x = L) &= A_t\,(t, x = -L) - \frac{1}{8}\,\partial_-\,(\rho + \gamma)\,|_{x=-L}^{x=L}\; C \\ &= \frac{1}{4}\,(b_1 - b_{-1}) - \frac{1}{8}\,\partial_-\,(\rho + \gamma)\,|_{x=L}\; C. \end{aligned} \tag{3.6}$$

In consequence, the conserved charges can be constructed as explained in (2.3)-(2.6). In order to do this we have to build the operator W_t from its form for a vacuum configuration as explained in (2.9). Note that the conformal sine-Gordon eqs. (1.1) have a vacuum solution given by $\varphi = \eta = \rho = 0$. The potentials $A_{\pm}$ given in (3.4), when evaluated on such a vacuum solution become

$$\begin{aligned} A_+^{(\text{vac.})} &= \frac{1}{2}\,b_1, \\ A_-^{(\text{vac.})} &= -\frac{1}{2}\,b_{-1} - \frac{1}{4}\,\partial_-\gamma^{(\text{vac.})}\,C, \end{aligned} \tag{3.7}$$

where, according to (3.5), $\partial_+\partial_-\gamma^{(\text{vac.})} = -1$, and so $\gamma^{(\text{vac.})} = -x_+\,x_-$. Since these potentials are flat we can write them as

$$A_\mu^{(\text{vac.})} = -\partial_\mu \Psi_{\text{vac}}\,\Psi_{\text{vac}}^{-1} \tag{3.8}$$

with

$$\Psi_{\text{vac}} = e^{-\frac{1}{2}\,x_+\,b_1}\,e^{\frac{1}{2}\,x_-\,b_{-1}}. \tag{3.9}$$

The solutions we are interested are those in the orbit of such a vacuum solution under the group of the so-called dressing transformations [12]. In order to construct such an orbit of solutions we consider a constant group element h, obtained by exponentiating the generators of the $sl(2)$ Kac-Moody algebra, which admit the following Gauss like decomposition

$$\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1} = G_-^{-1}\,G_0^{-1}\,G_+, \tag{3.10}$$

where G_+, G_-, and G_0 are group elements obtained by exponentiating the generators of the positive, negative and zero grades respectively, of the grading operator Q defined in (A.5).

Then we define the potential

$$A^h_\mu = -\partial_\mu \Psi_h \, \Psi_h^{-1} \tag{3.11}$$

with

$$\Psi_h = G_0 \, G_- \, \Psi_{\rm vac} \, h = G_+ \, \Psi_{\rm vac}. \tag{3.12}$$

As a consequence we have

$$A^h_\mu = G_+ \, A^{(\rm vac.)}_\mu \, G_+^{-1} - \partial_\mu G_+ \, G_+^{-1} \tag{3.13}$$

$$= G_0 \left(G_- \, A^{(\rm vac.)}_\mu \, G_-^{-1} - \partial_\mu G_- \, G_-^{-1} \right) G_0^{-1} - \partial_\mu G_0 \, G_0^{-1}. \tag{3.14}$$

The fact that A^h_μ and $A^{(\rm vac.)}_\mu$ are related by two gauge transformations, one involving only positive grade generators and the other only non-positive grade generators, guarantees that A^h_μ has the same grading structure as $A^{(\rm vac.)}_\mu$, and so as A_μ defined in (3.4). Indeed, the x_+-component of (3.13) implies that A^h_+ has components of grades greater than or equal to one, and the x_+-component of (3.14) implies that it has components of grades smaller or equal to one. Thus, A^h_+ must have components of grade one only. The same reasoning applies to A^h_-, using the x_--components of (3.13) and (3.14). Notice that the space-time dependency of A^h_μ is explicit, since it depends on the parameters of $G_{0,\pm}$ which, according to (3.10), are explicit functions of the space-time variables. Therefore A^h_μ corresponds to A_μ evaluated on the solution constructed by the dressing method. By equating A^h_μ to A_μ one then generates an explicit solution for the fields, since A_μ is their functional. Note that the dressing transformation from a vacuum with $\eta = 0$ will never produce a solution with $\eta \neq 0$. The reason for this is that the grading operator Q can never be obtained as a result of any commutator and, consequently, the terms proportional to Q will never appear in (3.13)-(3.14).

The best way of extracting the solutions for the fields is as follows: first note that the grade zero part of the x_--component of (3.14) is $\left(-\partial_- G_0 \, G_0^{-1} - \frac{1}{4} \, \partial_- \gamma^{(\rm vac.)} \, C \right)$. Comparing this with the grade zero component of A_- in (3.4) (with $\eta = 0$ since the dressing transformation does not excite η) one gets that

$$G_0 = e^{\frac{i}{2} \, \varphi \, F_0 + \frac{1}{4} \rho \, C}. \tag{3.15}$$

The highest weight states of the two fundamental representations of the $sl(2)$ Kac-Moody algebra are annihilated by the positive grade generators and so one has $G_+ \mid$

$\lambda_i \rangle = \mid \lambda_i \rangle$, and $\langle \lambda_i \mid G_- = \langle \lambda_i \mid$, for $i = 0, 1$. Then, using the relations (A.7)-(A.9) one gets from (3.10) that

$$
\begin{aligned}
\tau_0 &\equiv \langle \lambda_0 \mid \Psi_{\rm vac}\, h\, \Psi_{\rm vac}^{-1} \mid \lambda_0 \rangle = \langle \lambda_0 \mid G_0^{-1} \mid \lambda_0 \rangle = e^{\frac{i}{4}\varphi - \frac{1}{4}\rho}, \\
\tau_1 &\equiv \langle \lambda_1 \mid \Psi_{\rm vac}\, h\, \Psi_{\rm vac}^{-1} \mid \lambda_1 \rangle = \langle \lambda_1 \mid G_0^{-1} \mid \lambda_1 \rangle = e^{-\frac{i}{4}\varphi - \frac{1}{4}\rho}
\end{aligned}
\tag{3.16}
$$

and so

$$
\varphi = -2\, i\, \log \frac{\tau_0}{\tau_1}, \qquad \rho = -2\, \log\left(\tau_0\, \tau_1\right). \tag{3.17}
$$

In fact, the highest weight states are eigenvectors of G_0^{-1}

$$
G_0^{-1} \mid \lambda_i \rangle = \tau_i \mid \lambda_i \rangle, \qquad \langle \lambda_i \mid G_0^{-1} = \tau_i \langle \lambda_i \mid, \qquad i = 0, 1. \tag{3.18}
$$

The quantities τ_0 and τ_1 are the so-called Hirota's tau functions. Indeed, substituting (3.17) into (1.1) one finds that they satisfy the Hirota's equations

$$
\begin{aligned}
\tau_0\, \partial_+ \partial_- \tau_0 - \partial_+ \tau_0\, \partial_- \tau_0 &= \frac{1}{4}\left(\tau_0^2 - \tau_1^2\right), \\
\tau_1\, \partial_+ \partial_- \tau_1 - \partial_+ \tau_1\, \partial_- \tau_1 &= \frac{1}{4}\left(\tau_1^2 - \tau_0^2\right).
\end{aligned}
\tag{3.19}
$$

We now write the group elements $G_\pm$ as

$$
G_\pm = g_{\pm,F}^{-1}\, g_{\pm,b}, \qquad g_{\pm,F} = \exp\left(\sum_{n=1}^{\infty} \zeta_n^{(\pm)}\, F_{\pm n}\right), \quad g_{\pm,b} = \exp\left(\sum_{n=0}^{\infty} \xi_{2n+1}^{(\pm)}\, b_{\pm(2n+1)}\right). \tag{3.20}
$$

The relation (3.13) can then be rewritten as

$$
g_{+,F}\, A_\mu^h\, g_{+,F}^{-1} - \partial_\mu g_{+,F}\, g_{+,F}^{-1} = g_{+,b}\, A_\mu^{(\rm vac.)}\, g_{+,b}^{-1} - \partial_\mu g_{+,b}\, g_{+,b}^{-1} \equiv a_\mu^{(+)} \tag{3.21}
$$

and the relation (3.14) as

$$
g_{-,F}\, \bar{A}_\mu^h\, g_{-,F}^{-1} - \partial_\mu g_{-,F}\, g_{-,F}^{-1} = g_{-,b}\, A_\mu^{(\rm vac.)}\, g_{-,b}^{-1} - \partial_\mu g_{-,b}\, g_{-,b}^{-1} \equiv a_\mu^{(-)}, \tag{3.22}
$$

which serve as the definitions of the potentials $a_\mu^{(+)}$ and $a_\mu^{(-)}$, and the potential $\bar{A}_\mu^h$ is defined as

$$
\bar{A}_\mu^h \equiv G_0^{-1}\, A_\mu^h\, G_0 - \partial_\mu G_0^{-1}\, G_0. \tag{3.23}
$$

The gauge transformations relating the various potentials can be summarized in the following diagrams

$$
\begin{tikzcd}
A_\mu^{(\rm vac.)} \arrow[r, "G_+"] \arrow[dr, "g_{+,b}"'] & A_\mu^h \arrow[d, "g_{+,F}"] \\
& a_\mu^{(+)}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_\mu^{(\rm vac.)} \arrow[r, "G_-"] \arrow[dr, "g_{-,b}"'] & \bar{A}_\mu^h \arrow[d, "g_{-,F}"] \\
& a_\mu^{(-)}
\end{tikzcd}
\tag{3.24}
$$

The potentials $a_\mu^{(+)}$ and $a_\mu^{(-)}$ are related to the abelian potentials considered in [13, 14]

Equating A_μ^h to A_μ, given in (3.4), with $\eta=0$, we get that

$$
\begin{aligned}
\bar{A}_+^h &= \frac{1}{2}\, b_1 + \frac{i}{2}\,\partial_+\varphi\, F_0 + \frac{1}{4}\,\partial_+\rho\, C, \\
\bar{A}_-^h &= -\frac{1}{2}\,\left(\cos\varphi\, b_{-1} - i\,\sin\varphi\, F_{-1}\right) - \frac{1}{4}\partial_-\gamma^{(\mathrm{vac.})}\, C.
\end{aligned}
\tag{3.25}
$$

In addition, the x_- component of (3.21) gives

$$
\begin{aligned}
&g_{+,F}\,\left(-\frac{1}{2}\, b_{-1} - \frac{i}{2}\,\partial_-\varphi\, F_0\right)\, g_{+,F}^{-1} - \frac{1}{4}\,\partial_-\,\left(\rho + \gamma^{(\mathrm{vac.})}\right)\, C - \partial_- g_{+,F}\, g_{+,F}^{-1} \\
&\qquad = -\frac{1}{2}\, b_{-1} - \frac{1}{4}\,\left(2\,\xi_1^{(+)} + \partial_-\gamma^{(\mathrm{vac.})}\right)\, C - \sum_{n=0}^{\infty} \partial_-\xi_{2n+1}^{(+)}\, b_{2n+1}
\end{aligned}
\tag{3.26}
$$

and the x_+ component of (3.22) gives

$$
\begin{aligned}
&g_{-,F}\,\left(\frac{1}{2}\, b_1 + \frac{i}{2}\,\partial_+\varphi\, F_0\right)\, g_{-,F}^{-1} + \frac{1}{4}\,\partial_+\rho\, C - \partial_+ g_{-,F}\, g_{-,F}^{-1} \\
&\qquad = \frac{1}{2}\, b_1 - \frac{1}{2}\,\xi_1^{(-)}\, C - \sum_{n=0}^{\infty} \partial_+\xi_{2n+1}^{(-)}\, b_{-2n-1}.
\end{aligned}
\tag{3.27}
$$

Observe that the r.h.s. of (3.26) and (3.27) contain terms proportional to the oscillators b_{2n+1} and to the central term C only. Therefore, the components on the l.h.s. of these equations, which are in the direction of the F_n's must vanish. Splitting the relations (3.26) and (3.27) into eigenvectors of the grading operator Q one can then determine the parameters of $g_{\pm,F}$ recursively. Indeed, one finds that

$$
\begin{aligned}
\zeta_1^{(+)} &= -\frac{i}{2}\,\partial_-\varphi, & \zeta_1^{(-)} &= -\frac{i}{2}\,\partial_+\varphi, \\
\zeta_2^{(+)} &= \frac{i}{2}\,\partial_-^2\varphi, & \zeta_2^{(-)} &= -\frac{i}{2}\,\partial_+^2\varphi, \\
&\vdots & &\vdots
\end{aligned}
\tag{3.28}
$$

So, $\zeta_n^{(\pm)}$ are polynomials in the $x_\mp$ derivatives of the field φ and they do not depend on the field ρ. As discussed above, for finite energy solutions, one needs $\varphi \to 2\pi n_\pm$ as $x \to \pm\infty$, with $n_\pm$ integers, and consequently

$$
g_{\pm,F} \to 1 \qquad \text{for} \quad x \to \pm\infty.
\tag{3.29}
$$

We also get from (3.26) and (3.27) that

$$
\begin{aligned}
\xi_1^{(+)} &= \frac{1}{2}\,\partial_-\rho, & \xi_1^{(-)} &= -\frac{1}{2}\,\partial_+\rho, \\
\partial_-\xi_1^{(+)} &= -\frac{1}{4}\,\left(\partial_-\varphi\right)^2, & \partial_+\xi_1^{(-)} &= \frac{1}{4}\,\left(\partial_+\varphi\right)^2, \\
&\vdots & &\vdots
\end{aligned}
\tag{3.30}
$$

From the relations (3.30) we obtain an important property of the solutions in the orbit of the vacuum, which may not necessarily hold for other solutions of (1.1). To get it we note that (3.30) implies that

$$\partial_-^2 \rho = -\frac{1}{2}\,(\partial_- \varphi)^2\,, \qquad\qquad \partial_+^2 \rho = -\frac{1}{2}\,(\partial_+ \varphi)^2\,. \tag{3.31}$$

Using (3.31), (3.2), and the third eq. of (1.1) we see that the components of the canonical energy momentum tensor (1.4), for $\eta = 0$ (see (1.5)), can be written as

$$\begin{aligned} \Theta_{00}\mid_{\eta=0} &= \frac{1}{2}\,(\partial_t \varphi)^2 + \frac{1}{2}\,(\partial_x \varphi)^2 + 1 - \cos\varphi = -2\,\partial_x^2 \rho, \\ \Theta_{01}\mid_{\eta=0} &= \partial_t \varphi\;\partial_x \varphi = -2\,\partial_t \partial_x \rho. \end{aligned} \tag{3.32}$$

In consequence, the energy and momentum of the solutions on the orbit of the vacuum are surface terms:

$$\begin{aligned} E &\equiv \int_{-\infty}^{\infty} dx\,\Theta_{00}\mid_{\eta=0} = -2\,\partial_x \rho\mid_{x=-\infty}^{x=\infty}, \\ P &\equiv \int_{-\infty}^{\infty} dx\,\Theta_{01}\mid_{\eta=0} = -2\,\partial_t \rho\mid_{x=-\infty}^{x=\infty}\,. \end{aligned} \tag{3.33}$$

Replacing (3.17) into (3.31) one gets that the τ-functions, evaluated on the solutions on the orbit of the vacuum, satisfy, in addition to the Hirota's eqs. (3.19), the relations

$$\tau_1 \partial_\pm^2 \tau_0 + \tau_0 \partial_\pm^2 \tau_1 - 2\,\partial_\pm \tau_0\,\partial_\pm \tau_1 = 0 \tag{3.34}$$

The infinite number of conserved charges for the sine-Gordon model can be easily derived using the arguments of section 2 and the gauge transformations defined in this section. Thus, from the definition (2.5) we see that the W_t operator for the vacuum potential (3.7) is given by

$$W_t^{(\mathrm{vac.})} = e^{-\frac{L}{2}\,b_{-1}}\;e^{-\frac{L}{2}\,b_1}\;e^{\frac{1}{8}\left(L^2 + \int_{-L}^{L} dx\,\partial_- \gamma^{(\mathrm{vac.})}\right)\,C}. \tag{3.35}$$

Next we define the operators:

$$\Psi_{2n+1} = :e^{b_{2n+1} + b_{-2n-1}}: \; = e^{b_{-2n-1}}\;e^{b_{2n+1}} \qquad\qquad n = 0, 1, 2, \ldots \tag{3.36}$$

where :: stands for the normal ordering of the oscillators b_{2n+1}. We see that, under the adjoint action, Ψ_{2n+1} are eigenvectors of $W_t^{(\mathrm{vac.})}$, with unity eigenvalue since

$$W_t^{(\mathrm{vac.})}\,\Psi_{2n+1}\,{W_t^{(\mathrm{vac.})}}^{-1} = \Psi_{2n+1}. \tag{3.37}$$

Therefore, the conserved charges for the vacuum solution are indeed trivial.

Notice from (3.24) and (3.29) that the gauge potentials A_μ^h and $a_\mu^{(+)}$, and also $\bar{A}_\mu^h$ and $a_\mu^{(-)}$, are connected by gauge transformations involving group elements, namely $g_{\pm,F}$, that go to unity at spatial infinity. Therefore, from the arguments given below (2.8), one concludes that the conserved charges constructed from A_μ^h and $a_\mu^{(+)}$ are the same. For the same reasons, the charges obtained from $\bar{A}_\mu^h$ and $a_\mu^{(-)}$, are also equal. We can then construct the charges from the potentials $a_\mu^{(+)}$ and $a_\mu^{(-)}$, because the calculations are easier since these potentials are related to the vacuum potential via abelian gauge transformations.

The W_t operators for the non-trivial solutions connected to the vacuum by the gauge transformations performed by the group elements $g_{\pm,b}$, according to (2.9), are given by

$$W_t^{(\pm)} = g_{\pm,b}\,(t,x=L)\; W_t^{(\text{vac.})}\; g_{\pm,b}^{-1}\,(t,x=-L)\,. \tag{3.38}$$

Using the definition of $g_{\pm,b}$ in (3.20), one gets that

$$W_t^{(\pm)}\,\Psi_{2n+1}\,W_t^{(\pm)\,-1} = e^{\pm(2n+1)\,C\,\left(\xi_{2n+1}^{(\pm)}(t,x=L)-\xi_{2n+1}^{(\pm)}(t,x=-L)\right)}\;\Psi_{2n+1}. \tag{3.39}$$

Thus, we have two infinite sets of conserved charges given by

$$\Omega_{2n+1}^{(\pm)} \equiv \pm\,(2n+1)\,\left(\xi_{2n+1}^{(\pm)}\,(t,x=L) - \xi_{2n+1}^{(\pm)}\,(t,x=-L)\right) \qquad\qquad n=0,1,2,\ldots \tag{3.40}$$

In order to evaluate those charges we use the expressions for the solutions, on the orbit of the vacuum, given by the dressing transformation method. Using (3.10), (3.20), (3.16), (3.18), and the results of the appendix A we find that (for $n\geq 0$, and $i=0,1$)

$$\begin{aligned}
\langle\,\lambda_i\mid\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1}\,b_{-2n-1}\mid\lambda_i\,\rangle \;\equiv\; &= \langle\,\lambda_i\mid G_-^{-1}\,G_0^{-1}\,G_+\,b_{-2n-1}\mid\lambda_i\,\rangle\\
&= \tau_i\,\langle\,\lambda_i\mid g_{+,F}^{-1}\,g_{+,b}\,b_{-2n-1}\,g_{+,b}^{-1}\mid\lambda_i\,\rangle\\
&= \tau_i\,\langle\,\lambda_i\mid g_{+,F}^{-1}\left(b_{-2n-1}+(2n+1)\,\xi_{2n+1}^{(+)}\,C\right)\mid\lambda_i\,\rangle\\
&= \tau_i\,\left(\langle\,\lambda_i\mid g_{+,F}^{-1}\,b_{-2n-1}\mid\lambda_i\,\rangle+(2n+1)\,\xi_{2n+1}^{(+)}\right).
\end{aligned}$$

Now using (3.29) we get

$$\xi_{2n+1}^{(+)}\,(t,x=\pm L) = \frac{1}{(2n+1)}\,\frac{\langle\,\lambda_i\mid\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1}\,b_{-2n-1}\mid\lambda_i\,\rangle}{\langle\,\lambda_i\mid\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1}\mid\lambda_i\,\rangle}\,\Big|_{x=\pm L}\,. \tag{3.41}$$

Using similar arguments we also see that

$$\xi_{2n+1}^{(-)}\,(t,x=\pm L) = -\,\frac{1}{(2n+1)}\,\frac{\langle\,\lambda_i\mid b_{2n+1}\,\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1}\mid\lambda_i\,\rangle}{\langle\,\lambda_i\mid\Psi_{\text{vac}}\,h\,\Psi_{\text{vac}}^{-1}\mid\lambda_i\,\rangle}\,\Big|_{x=\pm L}\,. \tag{3.42}$$

Therefore, the charges (3.40) become

$$\Omega^{(+)}_{2n+1} = \frac{\langle\, \lambda_i \mid \Psi_{\rm vac}\, h\, \Psi^{-1}_{\rm vac}\, b_{-2n-1} \mid \lambda_i \,\rangle}{\langle\, \lambda_i \mid \Psi_{\rm vac}\, h\, \Psi^{-1}_{\rm vac} \mid \lambda_i \,\rangle} \,|^{x=+L}_{x=-L} \tag{3.43}$$

and

$$\Omega^{(-)}_{2n+1} = \frac{\langle\, \lambda_i \mid b_{2n+1}\, \Psi_{\rm vac}\, h\, \Psi^{-1}_{\rm vac} \mid \lambda_i \,\rangle}{\langle\, \lambda_i \mid \Psi_{\rm vac}\, h\, \Psi^{-1}_{\rm vac} \mid \lambda_i \,\rangle} \,|^{x=+L}_{x=-L}\, . \tag{3.44}$$

In particular, using (3.30), (3.33), and (3.40) we see that the energy and momentum of the solutions on the orbit of the vacuum are given, respectively, by:

$$E = 2\left(\Omega^{(+)}_1 - \Omega^{(-)}_1\right), \qquad\qquad P = -2\left(\Omega^{(+)}_1 + \Omega^{(-)}_1\right). \tag{3.45}$$

3.1 Soliton solutions

The soliton solutions are not only the most important ones in the orbit of the vacuum, but also the simplest ones to construct using the dressing method. The n-soliton solutions are obtained by taking the constant group element h introduced in (3.10) as the product of n exponentials of eigenvectors of the oscillators b_{2n+1} [2, 15, 16], namely the vertex operators defined in (A.11), *i.e.*

$$h = \prod_{i=1}^{n} e^{a_i\, V(z_i)}. \tag{3.46}$$

Therefore, using (3.9) and (A.15) one gets that

$$\Psi_{\rm vac}\, h\, \Psi^{-1}_{\rm vac} = \prod_{i=1}^{n} e^{a_i\, e^{\Gamma(z_i)}\, V(z_i)} = \prod_{i=1}^{n} \left(1 + a_i\, e^{\Gamma(z_i)}\, V\left(z_i\right)\right), \tag{3.47}$$

where we have used the nilpotency property (A.17) of the vertex operator and have introduced

$$\Gamma\left(z_i\right) \equiv z_i\, x_+ - \frac{x_-}{z_i}. \tag{3.48}$$

Using (A.18) and (A.19), one then gets that the tau-functions (3.16) are given by

$$\begin{aligned}
\tau_j \;&=\; 1 + (-1)^j \sum_{l=1}^{n} a_l\, e^{\Gamma(z_l)} + \sum_{l_1<l_2=1}^{n} \left(\frac{z_{l_1}-z_{l_2}}{z_{l_1}+z_{l_2}}\right)^2 a_{l_1}\, a_{l_2}\, e^{\Gamma\left(z_{l_1}\right)+\Gamma\left(z_{l_2}\right)} + \ldots \\
&+\; (-1)^j \sum_{l_1<l_2<l_3=1}^{n} \left(\frac{z_{l_1}-z_{l_2}}{z_{l_1}+z_{l_2}}\right)^2 \left(\frac{z_{l_1}-z_{l_3}}{z_{l_1}+z_{l_3}}\right)^2 \left(\frac{z_{l_2}-z_{l_3}}{z_{l_2}+z_{l_3}}\right)^2 a_{l_1} a_{l_2} a_{l_3} e^{\Gamma\left(z_{l_1}\right)+\Gamma\left(z_{l_2}\right)+\Gamma\left(z_{l_3}\right)} \\
&\quad \ldots + (-1)^{j\,n} \prod_{k_1<k_2=1}^{n} \left(\frac{z_{k_1}-z_{k_2}}{z_{k_1}+z_{k_2}}\right)^2 \prod_{l=1}^{n} a_l\, e^{\Gamma(z_l)} \qquad\qquad \text{for} \quad j = 0, 1.
\end{aligned} \tag{3.49}$$

The solution for the fields are then obtained through (3.17).

Observe that using (A.15) one has that

$$b_{2n+1}\left(1+a\,e^{\Gamma(z)}\,V\left(z\right)\right)=\left(1+a\,e^{\Gamma(z)}\,V\left(z\right)\right)b_{2n+1}-2\,z^{2n+1}a\,e^{\Gamma(z)}\,V\left(z\right).$$

Thus, using (3.47), the charges (3.43) and (3.44) for the n-soliton sector of solutions, become $(l=0,1)$

$$\Omega^{(\pm)}_{2n+1}=\pm 2\sum_{k=1}^{n}z_k^{\mp(2n+1)}a_k\,e^{\Gamma(z_k)}\;\times \tag{3.50}$$
$$\times\frac{\langle\,\lambda_l\mid\left[\prod_{i=1}^{k-1}\left(1+a_i\,e^{\Gamma(z_i)}\,V\left(z_i\right)\right)\right]V\left(z_k\right)\left[\prod_{j=k+1}^{n}\left(1+a_j\,e^{\Gamma(z_j)}\,V\left(z_j\right)\right)\right]\mid\lambda_l\,\rangle}{\langle\,\lambda_l\mid\prod_{i=1}^{n}\left(1+a_i\,e^{\Gamma(z_i)}\,V\left(z_i\right)\right)\mid\lambda_l\,\rangle}\,\Big|_{x=-L}^{x=+L}\,.$$

Let us now parametrize z_i as

$$z_i=e^{-\alpha_i+i\theta_i}\qquad\qquad\text{with } \alpha_i \text{ and } \theta_i \text{ real.} \tag{3.51}$$

Then (3.48) becomes

$$\Gamma\left(z_i\right)=\frac{1}{\sqrt{1-v_i^2}}\,\left[\cos\theta_i\left(x-v_i\,t\right)+i\,\sin\theta_i\,\left(t-v_i\,x\right)\right], \tag{3.52}$$

where v_i are velocities in units of the speed of light and

$$v_i=\tanh\alpha_i,\qquad\qquad\cosh\alpha_i=\frac{1}{\sqrt{1-v_i^2}}. \tag{3.53}$$

Note that the behaviour of $e^{\Gamma(z_i)}$ as $x\rightarrow\pm\infty$ is determined by the sign of $\cos\theta_i$. In addition, if a given combination of exponentials of Γ's dominates the behaviour of the denominator of a given term of the sum in (3.50) for $x\rightarrow\pm\infty$ then the same combination dominates the behaviour of the numerator of that term. Consequently, the corresponding expectation value of the product of V's cancels out and we have that

$$\Omega^{(\pm)}_{2n+1}=\pm 2\sum_{k=1}^{n}\epsilon_k\,z_k^{\mp(2n+1)}, \tag{3.54}$$

where $\epsilon_k=\pm 1$ are signs determined by the dominant combinations of exponentials of Γs. In consequence, the form of the charges for the n-soliton sector of solutions is quite simple.

3.1.1 1-soliton sector

In order to have a real solution for the field φ in the 1-soliton sector we need to take either $\theta = 0$ (soliton) or $\theta = \pi$ (anti-soliton). In addition, we need a to be pure imaginary. Then from (3.49) and (3.52) we have $\tau_0 = \tau_1^*$ and from (3.17)

$$\varphi = 4\,\mathrm{ArcTan}\left[\exp\left(\varepsilon \frac{(x - v\,t - x_0)}{\sqrt{1-v^2}}\right)\right], \tag{3.55}$$

where we have taken $a = i\exp\left(-\varepsilon\, x_0/\sqrt{1-v^2}\right)$, and $\varepsilon \equiv e^{i\theta} = \pm 1$, $\theta = 0, \pi$.

Evaluating the charges (3.50) one gets

$$\Omega^{(\pm)}_{2n+1} = \pm 2\,\varepsilon\, z^{\mp(2n+1)} = \pm 2\, e^{\pm(2n+1)\alpha} = \pm 2\left[\frac{1+v}{1-v}\right]^{\pm\frac{(2n+1)}{2}}. \tag{3.56}$$

In particular, the energy and momentum (3.45) become

$$E = \frac{8}{\sqrt{1-v^2}}, \qquad\qquad P = -\frac{8\,v}{\sqrt{1-v^2}}. \tag{3.57}$$

3.1.2 2-soliton sector

In this sector we have two types of real solutions: unbounded 2-solitons and breathers.

Unbounded 2-soliton solutions

In this case we take $\theta_1, \theta_2 = 0, \pi$, corresponding to the choices of solitons or anti-solitons, and also take a_i, $i = 1, 2$ pure imaginary. We then have $\tau_0 = \tau_1^*$ and

$$\tau_0 = 1 + i\, e^{\Gamma_1} + i\, e^{\Gamma_2} - \left(\frac{1 - \varepsilon_1\,\varepsilon_2\, e^{\alpha_1 - \alpha_2}}{1 + \varepsilon_1\,\varepsilon_2\, e^{\alpha_1-\alpha_2}}\right)^2 e^{\Gamma_1 + \Gamma_2} \tag{3.58}$$

with

$$\Gamma_i = \varepsilon_i\, \frac{\left(x - v_i\, t - x_0^{(i)}\right)}{\sqrt{1 - v_i^2}} \tag{3.59}$$

with $a_i = i\exp\left(-\varepsilon_i\, x_0^{(i)}/\sqrt{1-v_i^2}\right)$, $\varepsilon_i \equiv e^{i\theta_i} = \pm 1$, $\theta_i = 0, \pi$, $i = 1, 2$.

Evaluating the charges (3.50) we get

$$\begin{aligned} \Omega^{(\pm)}_{2n+1} &= \pm 2\left(\varepsilon_1\, z_1^{\mp(2n+1)} + \varepsilon_2\, z_2^{\mp(2n+1)}\right) = \pm 2\left(e^{\pm(2n+1)\alpha_1} + e^{\pm(2n+1)\alpha_2}\right) \\ &= \pm 2\left(\left[\frac{1+v_1}{1-v_1}\right]^{\pm\frac{(2n+1)}{2}} + \left[\frac{1+v_2}{1-v_2}\right]^{\pm\frac{(2n+1)}{2}}\right). \end{aligned} \tag{3.60}$$

Therefore

$$E=\frac{8}{\sqrt{1-v_1^2}}+\frac{8}{\sqrt{1-v_2^2}},\qquad\qquad P=-\frac{8\,v_1}{\sqrt{1-v_1^2}}-\frac{8\,v_2}{\sqrt{1-v_2^2}}. \tag{3.61}$$

Breathers

For the breather solutions we take $\alpha_1=\alpha_2\equiv\alpha$, $\theta_1=-\theta_2\equiv\theta$, and so $z_1=z_2^*$. We also take $a_1=-a_2=-\text{cotan}\,\theta$. We then have $\Gamma(z_1)=\Gamma(z_2)^*$ and, again, $\tau_0=\tau_1^*$, with

$$\tau_0=1+e^{2\Gamma_R}+2\,i\,(\text{cotan}\,\theta)\;e^{\Gamma_R}\,\sin\Gamma_I, \tag{3.62}$$

where

$$\Gamma_R=\frac{\cos\theta}{\sqrt{1-v^2}}\,(x-v\,t)\,,\qquad\qquad \Gamma_I=\frac{\sin\theta}{\sqrt{1-v^2}}\,(t-v\,x)\,. \tag{3.63}$$

Therefore

$$\varphi=4\,\text{Arctan}\frac{(\text{cotan}\,\theta)\;\sin\Gamma_I}{\cosh\Gamma_R}. \tag{3.64}$$

Evaluating the charges (3.50) one gets

$$\begin{aligned}\Omega^{(\pm)}_{2n+1} &= \pm 2\,\varepsilon\,\left(e^{\mp(2n+1)(-\alpha+i\theta)}+e^{\mp(2n+1)(-\alpha-i\theta)}\right)\\ &= \pm 4\,\varepsilon\,\left[\frac{1+v}{1-v}\right]^{\pm\frac{(2n+1)}{2}}\cos\left[(2n+1)\;\theta\right],\end{aligned} \tag{3.65}$$

where $\varepsilon\equiv\text{sign}\,(\cos\theta)$. Therefore, the energy and momentum become

$$E=\frac{16\;|\cos\theta\,|}{\sqrt{1-v^2}},\qquad\qquad P=-\frac{16\,v\;|\cos\theta\,|}{\sqrt{1-v^2}}. \tag{3.66}$$

3.1.3 N-soliton sector

As shown in (3.54), the conserved charges evaluated on the solutions coming from the choice (3.46) for the constant group element h of the dressing method have an additive character. Therefore, if one considers a solution with N solitons and M breathers the charges are given by

$$\Omega^{(\pm)}_{2n+1}=\pm 2\sum_{i=1}^{N}\left[\frac{1+v_i}{1-v_i}\right]^{\pm\frac{(2n+1)}{2}}\pm 4\sum_{j=1}^{M}\varepsilon_j\,\left[\frac{1+v_j}{1-v_j}\right]^{\pm\frac{(2n+1)}{2}}\cos\left[(2n+1)\;\theta_j\right] \tag{3.67}$$

with $\varepsilon_j=\text{sign}\,(\cos\theta_j)$. Consequently the energy and momentum are also additive and one has

$$\begin{aligned}E &= \sum_{i=1}^{N}\frac{8}{\sqrt{1-v_i^2}}+\sum_{j=1}^{M}\frac{16\;|\cos\theta_j\,|}{\sqrt{1-v_j^2}},\\ P &= -\sum_{i=1}^{N}\frac{8\,v_i}{\sqrt{1-v_i^2}}-\sum_{j=1}^{M}\frac{16\,v_j\;|\cos\theta_j\,|}{\sqrt{1-v_j^2}}.\end{aligned} \tag{3.68}$$

4 Generalized soliton hierarchies

The results obtained above for the sine-Gordon model can certainly be generalized to other theories possessing soliton solutions. We sketch here how this can be done using the basic structures known to be responsible for the existence of solitons. As explained for instance in [16], practically all two dimensional exact soliton solutions known in the literature belong to a class that can be characterized by the following features:

1. They are solutions of two dimensional theories that admit a zero curvature representation of their equations of motion, *i.e.* there exist potentials (Lax operators) A_μ, which are functionals of the fields of the theory and which belong to a Kac-Moody algebra $\mathcal{G}$ such that the condition

$$[\,\partial_\mu + A_\mu\,,\,\partial_\nu + A_\nu\,] = 0 \tag{4.1}$$

is equivalent to the equations of motion. The indices μ, ν correspond to the two coordinates of space-time, or to the various times t_N of a hierarchy of soliton theories (see [16] for details).

2. There exist an integer gradation of $\mathcal{G}$

$$\mathcal{G} = \oplus_{n\in\mathbb{Z}}\mathcal{G}_n, \qquad\qquad [\,\mathcal{G}_m\,,\,\mathcal{G}_n\,] \subset \mathcal{G}_{m+n} \tag{4.2}$$

such that the potentials can be decomposed as

$$A_\mu = \sum_{n=N_\mu^-}^{N_\mu^+} A_\mu^{(n)}, \qquad\qquad \text{where} \qquad A_\mu^{(n)} \in \mathcal{G}_n \tag{4.3}$$

with N_μ^- and N_μ^+ being non-positive and non-negative integers, respectively.

3. There exist at least one "vacuum solution" of the theory such that the potentials A_μ evaluated on it belong to an abelian subalgebra, up to central term, of $\mathcal{G}$, *i.e.*

$$A_\mu^{(\mathrm{vac})} = \sum_{n=N_\mu^-}^{N_\mu^+}\sum_{a=1}^{r} c_\mu^{a,n}\, b_n^a + \sigma_\mu\, C \equiv E_\mu + \sigma_\mu\, C, \tag{4.4}$$

where $c_\mu^{a,n}$ are constants, C is the central element of $\mathcal{G}$, and b_n^a satisfy an algebra of oscillators (Heisenberg subalgebra)

$$\left[\, b_m^a\,,\, b_n^b \,\right] = \omega^{ab}\, m\, \delta_{m+n,0}\, C \tag{4.5}$$

with ω^{ab} being a symmetric matrix, and $a, b = 1, 2, \ldots r$, labels the number of infinite sets of oscillators. The index n corresponds to the grade of the oscillators, *i.e.* $b_m^a \in \mathcal{G}_n$, and they do not have to exist for all values of n (for instance, in the case of sine-Gordon, as discussed in section 3 and appendix A, they exist only for odd n)

The soliton solutions are then constructed using the dressing method in a manner similar to that explained in section 3 for the sine-Gordon model. Since $A_\mu^{(\mathrm{vac})}$, given in (4.4), satisfy the zero curvature equation (4.1), there exists a group element Ψ_{vac}, which is an exponentiation of E_μ (oscillators) and C (see [16] for details), such that

$$A_\mu^{(\mathrm{vac})} = -\partial_\mu \Psi_{\mathrm{vac}}\, \Psi_{\mathrm{vac}}^{-1}.$$

We then choose a constant group element h such that there exists a Gauss like decomposition

$$\Psi_{\mathrm{vac}}\, h\, \Psi_{\mathrm{vac}}^{-1} = G_-^{-1}\, G_0^{-1}\, G_+ \tag{4.6}$$

with $G_{+,0,-}$ being group elements obtained by the exponentiation of generators of $\mathcal{G}$ with positive, zero, and negative grades, respectively, with respect to (4.2). We now introduce

$$\Psi_h \equiv G_0\, G_-\, \Psi_{\mathrm{vac}}\, h = G_+\, \Psi_{\mathrm{vac}}, \qquad \bar{\Psi}_h \equiv G_-\, \Psi_{\mathrm{vac}}\, h = G_0^{-1}\, G_+\, \Psi_{\mathrm{vac}} \tag{4.7}$$

and the corresponding potentials

$$A_\mu^h \equiv -\partial_\mu \Psi_h\, \Psi_h^{-1}, \qquad \bar{A}_\mu^h \equiv -\partial_\mu \bar{\Psi}_h\, \bar{\Psi}_h^{-1}. \tag{4.8}$$

Therefore one has

$$\begin{aligned} A_\mu^h &= G_+\, A_\mu^{(\mathrm{vac})}\, G_+^{-1} - \partial_\mu G_+\, G_+^{-1} && (4.9)\\ &= G_0\, G_-\, A_\mu^{(\mathrm{vac})}\, (G_0\, G_-)^{-1} - \partial_\mu\, (G_0\, G_-)\, (G_0\, G_-)^{-1}, && (4.10)\\ \bar{A}_\mu^h &= G_-\, A_\mu^{(\mathrm{vac})}\, G_-^{-1} - \partial_\mu G_-\, G_-^{-1} && (4.11)\\ &= G_0^{-1}\, G_+\, A_\mu^{(\mathrm{vac})} \left(G_0^{-1}\, G_+\right)^{-1} - \partial_\mu \left(G_0^{-1}\, G_+\right) \left(G_0^{-1}\, G_+\right)^{-1}. && (4.12) \end{aligned}$$

Using arguments similar to those given before (3.13)-(3.14) one can then show that A_μ^h and $\bar{A}_\mu^h$ have the same grading structure as A_μ in (4.3). Indeed, using (4.4) one sees that (4.9) implies that A_μ^h has components of grade greater or equal to N_μ^- and (4.10) implies that it has components of grade smaller or equal to N_μ^+. Therefore, A_μ^h must have components of grade varying from N_μ^- to N_μ^+. Using similar arguments for (4.11) and (4.12) one concludes that $\bar{A}_\mu^h$ must also have components of grade varying from N_μ^- to N_μ^+. In fact, from their definition (4.8), one notices that A_μ^h and $\bar{A}_\mu^h$ are related by a gauge transformation with the group element G_0, which involves only zero grade generators, and so they must indeed have the same grading structure. Thus, A_μ^h corresponds to A_μ evaluated on the solution constructed by the dressing method. By equating A_μ^h to A_μ given in (4.3), which is a functional of the underlying fields, one then defines the solution of their equations of motion. Note that we could as well have equated $\bar{A}_\mu^h$ to A_μ, as this corresponds to a gauge choice which we can make in order to get the relation among the parameters of $G_{0,\pm}$ with the fields as simple as possible.

The soliton solutions are obtained by choosing the constant group element h, introduced in (4.6), as

$$h = \prod_{l=1}^{N} \prod_{k=1}^{n} e^{a_{l,k}\, V_l(z_k)}, \tag{4.13}$$

where $V_l\,(z_k)$ are eigenvectors of the operators E_μ, introduced in (4.4),

$$[\, E_\mu \,,\, V_l\,(z_k)\,] = \lambda_\mu^l\,(z_k)\; V_l\,(z_k)$$

and where l labels the species or types of solitons and z_k are parameters that determine the velocities and topological charges of the solitons (see [16] for more details). In the case of the sine-Gordon model discussed in section 3, we have seen that there exist only one species of solitons.

Denoting by $\mathcal{H}$ the (Heisenberg) subalgebra generated by the oscillators b_n^a's and C, and by $\mathcal{F}$ its complement in the Kac-Moody algebra $\mathcal{G}$ we see that

$$\mathcal{G} = \mathcal{H} + \mathcal{F}. \tag{4.14}$$

Then we split the group elements $G_\pm$ according to such a decomposition, *i.e.*

$$G_\pm = g_{\pm,F}^{-1}\, g_{\pm,b} \tag{4.15}$$

with

$$g_{\pm,b} = \exp\left(\sum_{n=1}^{\infty} \sum_{a=1}^{r} \xi_{a,n}^{(\pm)}\, b_{\pm n}^a \right) \tag{4.16}$$

and $g_{\pm,F}$ being group elements obtained by exponentiating the parts $\mathcal{F}_+$ and $\mathcal{F}_-$ of $\mathcal{F}$ containing the generators of positive and negative grades respectively, *i.e.* $g_{\pm,F} = \exp\,(\mathcal{F}_\pm)$. In this case the relations (4.9) and (4.11) can be rewritten, respectively, as

$$g_{+,F}\, A_\mu^h\, g_{+,F}^{-1} - \partial_\mu g_{+,F}\, g_{+,F}^{-1} \;=\; g_{+,b}\, A_\mu^{(\text{vac.})}\, g_{+,b}^{-1} - \partial_\mu g_{+,b}\, g_{+,b}^{-1} \equiv a_\mu^{(+)}, \tag{4.17}$$

$$g_{-,F}\, \bar{A}_\mu^h\, g_{-,F}^{-1} - \partial_\mu g_{-,F}\, g_{-,F}^{-1} \;=\; g_{-,b}\, A_\mu^{(\text{vac.})}\, g_{-,b}^{-1} - \partial_\mu g_{-,b}\, g_{-,b}^{-1} \equiv a_\mu^{(-)}, \tag{4.18}$$

where we have introduced the potentials $a_\mu^{(\pm)}$.

The conserved charges can now be constructed in a manner similar to that of section 3 of the sine-Gordon case. Denote by x and t the space-time coordinates for our generalized soliton theory. Suppose that the time component of the potential introduced in (4.3) satisfies the boundary condition (2.3). Then using (4.17)-(4.18) and the arguments leading to (2.9) one sees that

$$W_t^{(\pm)} = P\, e^{-\int_{x=-L}^{x=L} dx\, a_x^{(\pm)}} = g_{\pm,b}\left(t, x=L\right) \, W_t^{(\text{vac.})} \, g_{\pm,b}^{-1}\left(t, x=-L\right), \tag{4.19}$$

where

$$W_t^{(\text{vac.})} = P\, e^{-\int_{x=-L}^{x=L} dx\, A_x^{(\text{vac})}}. \tag{4.20}$$

For every pair of oscillators, b_n^a and b_{-n}^a, we introduce the operator ($n > 0$)

$$\Psi_{a,n} = :e^{b_n^a + b_{-n}^a}: \; = e^{b_{-n}^a}\, e^{b_n^a} \tag{4.21}$$

where ::, as before, denotes the normal ordering of the oscillators, *i.e.* positive grade oscillators are put to the right of the negative ones. Then, using (4.5) and (4.16) we see that

$$W_t^{(\pm)}\, \Psi_{a,n}\, W_t^{(\pm)\,-1} = e^{\left(\Omega_{a,n}^{(\pm)} + \Omega_{a,n}^{(\text{vac})}\right) C}\, \Psi_{a,n} \tag{4.22}$$

with

$$\Omega_{a,n}^{(\pm)} = \pm\, n \sum_{b=1}^{r} \omega^{ab} \left(\xi_{b,n}^{(\pm)}\left(t, x=L\right) - \xi_{b,n}^{(\pm)}\left(t, x=-L\right)\right) \tag{4.23}$$

and $\Omega_{a,n}^{(\text{vac})}$ are the vacuum values of the charges

$$W_t^{(\text{vac.})}\, \Psi_{a,n}\, W_t^{(\text{vac.})\,-1} = e^{\Omega_{a,n}^{(\text{vac})} C}\, \Psi_{a,n}. \tag{4.24}$$

In a manner similar to that of the sine-Gordon case, the parameters $\xi_{a,n}^{(\pm)}$ are determined for the solution associated to a given constant group element h, by the matrix elements of the form $\langle \lambda \mid \Psi_{\text{vac}}\, h\, \Psi_{\text{vac}}^{-1}\, b_n^a \mid \lambda \rangle$ and $\langle \lambda \mid b_{-n}^a\, \Psi_{\text{vac}}\, h\, \Psi_{\text{vac}}^{-1} \mid \lambda \rangle$ of the operators in (4.6), with $n > 0$, and $\mid \lambda \rangle$ being a highest weight state of a given representation of the Kac-Moody algebra $\mathcal{G}$.

Note that, using (4.4) and (4.16), the r.h.s. of (4.17) and (4.18) give

$$\begin{aligned}
a_\mu^{(+)} &= E_\mu + \sigma_\mu\, C + \sum_{n=1}^{-N_\mu^-} \sum_{a,b=1}^{r} \omega^{ab}\, c_\mu^{a,-n}\, n\, \xi_{b,n}^{(+)}\, C - \sum_{n=1}^{\infty} \sum_{a=1}^{r} \partial_\mu \xi_{a,n}^{(+)}\, b_n^a, \\
a_\mu^{(-)} &= E_\mu + \sigma_\mu\, C + \sum_{n=1}^{N_\mu^+} \sum_{a,b=1}^{r} \omega^{ab}\, c_\mu^{a,n}\, (-n)\, \xi_{b,n}^{(-)}\, C - \sum_{n=1}^{\infty} \sum_{a=1}^{r} \partial_\mu \xi_{a,n}^{(-)}\, b_{-n}^a.
\end{aligned}$$

Equating to the l.h.s. of (4.17) and (4.18) one gets that the C-part gives

$$\begin{aligned}
\sum_{n=1}^{-N_\mu^-} \sum_{a,b=1}^{r} \omega^{ab}\, c_\mu^{a,-n}\, n\, \xi_{b,n}^{(+)} &= \left(g_{+,F}\, A_\mu^h\, g_{+,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu, \\
\sum_{n=1}^{N_\mu^+} \sum_{a,b=1}^{r} \omega^{ab}\, c_\mu^{a,n}\, (-n)\, \xi_{b,n}^{(-)} &= \left(g_{-,F}\, \bar{A}_\mu^h\, g_{-,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu .
\end{aligned}$$

Therefore, from (4.23) one gets

$$\begin{aligned}
\sum_{n=1}^{-N_\mu^-} \sum_{a,b=1}^{r} c_\mu^{a,-n}\, \Omega_{a,n}^{(+)} &= \left[\left(g_{+,F}\, A_\mu^h\, g_{+,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu\right]_{x=-L}^{x=L}, \\
\sum_{n=1}^{N_\mu^+} \sum_{a,b=1}^{r} c_\mu^{a,n}\, \Omega_{a,n}^{(-)} &= \left[\left(g_{-,F}\, \bar{A}_\mu^h\, g_{-,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu\right]_{x=-L}^{x=L} . \qquad (4.25)
\end{aligned}$$

If $\left[\left(g_{+,F}\, A_\mu^h\, g_{+,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu\right]$ and $\left[\left(g_{-,F}\, \bar{A}_\mu^h\, g_{-,F}^{-1}\right)_{\text{coeff.}\, C} - \sigma_\mu\right]$, can be expressed locally in terms of the underlying fields of the theory, we see that the above linear combinations of conserved charges are boundary terms. This happens for instance, in the abelian and non-abelian Toda models [1, 17], where the combinations of charges turn out to be related to the energy and momentum of the solutions.

4.1 The example of the mKdV equation

The modified Korteweg-de Vries equation (mKdV) is an example of a soliton theory that fulfills the requirements described at the beginning of section 4, and so can have its conserved charges calculated as described in this paper. As we pointed out it is important to work with a zero curvature representation based on the Kac-Moody algebra with a non vanishing central term. We use here the zero curvature potentials for the mKdV equation constructed in section IV.A of reference [16]. The potentials are given by

$$\begin{aligned}
A_x &= -b_1 - q\, F_0 - \nu\, C \qquad (4.26) \\
A_t &= -b_3 - q\, F_2 + \frac{1}{2}\, \partial_x q\, F_1 + \frac{1}{2}\, q^2\, b_1 - \frac{1}{2}\left(\frac{1}{2}\, \partial_x^2 q - q^3\right) F_0 - \frac{1}{16} \partial_x q^2\, C
\end{aligned}$$

where C, b_j, $j = 1, 3$, and F_k, $k = 0, 1, 2$, are generators of the $sl(2)$ Kac-Moody algebra defined in appendix A, and which commutation relations are given in (A.3). We have denoted by q the mKdV field, and by ν an extra field associated to the central term C of the algebra. Replacing the potentials (4.26) into the zero curvature condition (4.1) one gets that all components vanish with the exception of those in the direction of F_0 and C which give the equations of motion

$$\partial_t q = \frac{1}{2}\,\partial_x\left(\frac{1}{2}\,\partial_x^2 q - q^3\right) \tag{4.27}$$

$$\partial_t \nu = \frac{1}{16}\,\partial_x^2 q^2 \tag{4.28}$$

and (4.27) is the well known mKdV equation. Notice that the ν field is an expectant since it does affect the equation of motion for the field q. That is similar to the ρ field introduced in the sine-Gordon model in (1.1). However, here in the case of the mKdV equation we would not have to introduce such field to work with a non-vanishing central term C. The reason is that, contrary to the sine-Gordon case, all the generators appearing in the potentials (4.26) have non negative grades w.r.t. the grading operator Q defined in (A.5). Therefore, the commutator term, $[\,A_x\,,\,A_t\,]$, of (4.1) does not produce terms in the direction of C. However as we show below, the introduction of such field is important to make the dressing method consistent with a non vanishing central term. In addition, that field is crucial for the simple formula we obtain for the energy of the solutions.

So, the mKdV theory fulfills the requirement 1 at the beginning of section 4. As for the requirement 2, we have that the potentials (4.26) are decomposed as in (4.3) w.r.t. to the gradation defined by the grading operator Q introduced in (A.5). Indeed, one can check that in this case we have $N_x^- = 0$, $N_x^+ = 1$, $N_t^- = 0$ and $N_t^+ = 3$. The vacuum solution of the requirement 3 can be taken as $q = \nu = 0$, and so the potentials evaluated on it are given by

$$A_x^{(\rm vac)} = -b_1 \qquad\qquad A_t^{(\rm vac)} = -b_3 \tag{4.29}$$

Comparing with (4.4) we have $E_x = -b_1$, $E_t = -b_3$, and $\sigma_\mu = 0$. The relevant oscillator algebra (4.5) in this case is that generated by b_{2n+1} (see (A.3)). The potentials (4.29) can be written as

$$A_\mu^{(\rm vac)} = -\partial_\mu \Psi_{\rm vac}\,\Psi_{\rm vac}^{-1} \qquad \text{with} \qquad \Psi_{\rm vac} = e^{x\,b_1}\,e^{t\,b_3} \tag{4.30}$$

The dressing method can then be applied following the description given from (4.6) to (4.12). With the vacuum potential given by (4.29) it then follows from (4.11)-(4.12) that $\bar{A}_\mu^h$ has the same same grading structure as A_μ given in (4.26). We can then equate those two potentials in order to evaluate the solutions. By comparing the zero grade part of $\bar{A}_x^h$ given in (4.12), with the zero grade part of A_x given in (4.26), one then gets that

$$G_0 = e^{\alpha\,F_0 + \beta\,C} \qquad \text{with} \qquad \partial_x \alpha = -q\;; \qquad \partial_x \beta = -\nu \tag{4.31}$$

Since all the relations on the dressing method are valid on shell, i.e. when the equations of motion hold true, we can use (4.27)-(4.28) to get the time derivatives of the parameters α and β. By taking the integration constants to vanish, one obtains that

$$\partial_t \alpha = -\frac{1}{2}\left(\frac{1}{2}\,\partial_x^2 q - q^3\right) \; ; \qquad\qquad \partial_t \beta = -\frac{1}{16}\,\partial_x q^2 \tag{4.32}$$

Replacing (4.29) into (4.9)-(4.10) one observes that A_μ^h does not have the same grading structure as A_μ given in (4.26). In fact, contrary to A_μ, A_μ^h can not have zero grade components. In fact, A_μ^h corresponds to a potential $\tilde{A}_\mu$ obtained from A_μ of (4.26) by the gauge transformation (see (4.7)-(4.8))

$$\tilde{A}_\mu \equiv G_0\, A_\mu\, G_0^{-1} - \partial_\mu G_0\, G_0^{-1} \tag{4.33}$$

Using (4.31) and (4.32) one gets

$$\begin{aligned} \tilde{A}_x &= -\cosh(2\,\alpha)\; b_1 - \sinh(2\alpha)\; F_1 \qquad (4.34)\\ \tilde{A}_t &= -\cosh(2\,\alpha)\; b_3 - \sinh(2\alpha)\; F_3 - q\, F_2 + \frac{1}{2}\left[\partial_x q\, \cosh(2\,\alpha) + q^2\, \sinh(2\alpha)\right] F_1 \\ &+ \frac{1}{2}\left[q^2\, \cosh(2\,\alpha) + \partial_x q\, \sinh(2\alpha)\right] b_1 \end{aligned}$$

So, $\tilde{A}_\mu$ is local in the parameter α but not on the mKdV field q. In addition, it does not involve the extra filed ν and neither the parameter β. Notice that, the vanishing of the integration constants leading to (4.32) is a requirement of the dressing method, since if those constants were not zero, $\tilde{A}_\mu$ would have zero grade components.

Using (4.6) we now introduce the Hirota's tau functions

$$\begin{aligned} \tau_0 &= \langle\, \lambda_0 \mid \Psi_{\rm vac}\, h\, \Psi_{\rm vac}^{-1} \mid \lambda_0 \,\rangle = \langle\, \lambda_0 \mid G_0^{-1} \mid \lambda_0 \,\rangle = e^{\frac{1}{2}\,\alpha - \beta} \\ \tau_1 &= \langle\, \lambda_1 \mid \Psi_{\rm vac}\, h\, \Psi_{\rm vac}^{-1} \mid \lambda_1 \,\rangle = \langle\, \lambda_1 \mid G_0^{-1} \mid \lambda_1 \,\rangle = e^{-\frac{1}{2}\,\alpha - \beta} \end{aligned} \tag{4.35}$$

where $\mid \lambda_i \,\rangle$, $i = 0, 1$, are the highest weight states of the two fundamental representations of the $sl(2)$ Kac-Moody algebra, and where we have used their properties given in (A.7)-(A.9). Therefore, using (4.31) and (4.35), the relation among the fields and tau functions are given by

$$q = \partial_x \ln \frac{\tau_1}{\tau_0} \qquad\qquad \nu = \frac{1}{2}\,\partial_x \ln(\tau_0\, \tau_1) \tag{4.36}$$

As explained in (4.13) the soliton solutions, on the orbit of the vacumm (4.29), are obtained by taking the constant group element h to be exponentials of the eigenvectors of b_1 and b_3. Evaluating the matrix elements in (4.35) and replacing them into (4.36) one gets the solutions for the mKdV field q.

The decomposition (4.14) in the case of the mKdV is such that $\mathcal{H}$ is generated by the oscillators b_{2n+1}, and the complement $\mathcal{F}$ by the generators F_n, with $n \in \mathbb{Z}$. We then write the group elements introduced in (4.15) as

$$g_{\pm,b} = \exp\left(\sum_{n=0}^{\infty} \xi_{2n+1}^{(\pm)}\, b_{\pm(2n+1)}\right) \qquad\qquad g_{\pm,F} = \exp\left(\sum_{n=1}^{\infty} \zeta_n^{(\pm)}\, F_{\pm n}\right) \tag{4.37}$$

The x-components of the relations (4.17)-(4.18) are then given by

$$\begin{aligned} g_{+,F}\, \tilde{A}_x\, g_{+,F}^{-1} - \partial_x g_{+,F}\, g_{+,F}^{-1} &= -g_{+,b}\, b_1\, g_{+,b}^{-1} - \partial_x g_{+,b}\, g_{+,b}^{-1} \equiv a_x^{(+)} \\ &= -b_1 - \sum_{n=0}^{\infty} \partial_x \xi_{2n+1}^{(+)}\, b_{2n+1} \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} g_{-,F}\, A_x\, g_{-,F}^{-1} - \partial_x g_{-,F}\, g_{-,F}^{-1} &= -g_{-,b}\, b_1\, g_{-,b}^{-1} - \partial_x g_{-,b}\, g_{-,b}^{-1} \equiv a_x^{(-)} \\ &= -b_1 + \xi_1^{(-)}\, C - \sum_{n=0}^{\infty} \partial_x \xi_{2n+1}^{(-)}\, b_{-(2n+1)} \end{aligned} \tag{4.39}$$

with A_x and $\tilde{A}_x$ given by (4.26) and (4.34) respectively. The r.h.s. of (4.38) and (4.39) do not contain terms in the direction of F_n. By imposing the cancellation of the coefficients of F_n on the l.h.s. of those equations one determines the parameters $\zeta_n^{(\pm)}$. The first of them are given by

$$\begin{aligned} \zeta_1^{(-)} &= -\frac{1}{2}\, q & \qquad \partial_x \zeta_1^{(+)} &= -\sinh(2\alpha) \\ \zeta_2^{(-)} &= \frac{1}{4}\, \partial_x q & \qquad \partial_x \zeta_2^{(+)} &= -2\, \zeta_1^{(+)}\, \cosh(2\alpha) \\ &\vdots & &\vdots \end{aligned} \tag{4.40}$$

By equating the coefficients of b_{2n+1} on both sides of (4.38) and (4.39) one determines the parameters $\xi_{2n+1}^{(\pm)}$. The first of them are

$$\begin{aligned} \xi_1^{(-)} &= -\nu & \qquad \partial_x \xi_1^{(+)} &= 2\sinh^2\alpha \\ \partial_x \xi_1^{(-)} &= \frac{1}{2}\, q^2 & \qquad \partial_x \xi_3^{(+)} &= \zeta_2^{(+)}\, \sinh(2\,\alpha) \\ &\vdots & &\vdots \end{aligned} \tag{4.41}$$

In order to construct the conserved charges one needs the time component of the potentials to satisfy the boundary conditions (2.3). If one looks for solutions satisfying the conditions

$$q \to 0\;; \qquad \partial_x^n q \to 0\;; \qquad \text{as} \qquad x \to \pm\infty \tag{4.42}$$

then the potential (4.26) do satisfy (2.3), i.e. $A_t\,(t, x = \infty) = A_t\,(t, x = -\infty) = -b_3$. From (4.40) one observes that the parameters $\zeta_n^{(-)}$ depend locally on q and its derivatives. Therefore

$$\zeta_n^{(-)} \rightarrow 0 \qquad \text{and so} \qquad g_{-,F} \rightarrow 1 \qquad \text{as} \qquad x \rightarrow \pm\infty \tag{4.43}$$

Therefore, according to the discussion below (2.8) one concludes that the charges obtained from the potentials A_μ, given in (4.26), and those from $a_\mu^{(-)}$, defined in (4.39), are the same since they are related by a gauge transformation involving a group element that goes to unity at spatial infinity.

Assuming the conditions (4.42) one needs in addition that

$$\alpha\,(t, x = \infty) = \alpha\,(t, x = -\infty) \tag{4.44}$$

in order for the potential $\tilde{A}_t$ to satisfy the boundary condition (2.3). However, notice that the mKdV equation (4.27) together with the condition (4.42) constitute a conservation law which leads to the following conserved charge

$$H_1 = \int_{-\infty}^{\infty} dx\, q = -\left[\alpha\,(t, x = \infty) - \alpha\,(t, x = -\infty)\right] = -\ln \frac{\tau_0}{\tau_1}\,|_{x=-\infty}^{x=\infty} \tag{4.45}$$

where we have used (4.31) and (4.35). Therefore, the conditions for the potential $\tilde{A}_\mu$ to give conserved charges imply that H_1 should vanish. In addition, the condition (4.44) is not sufficient for the parameters $\zeta_n^{(+)}$ to vanish at spatial infinity, as seen from (4.40). Consequently, it does not guarantees that the charges coming from $\tilde{A}_\mu$ and $a_\mu^{(+)}$ are the same, as $g_{+,F}$ may not go to unity at spatial infinity. On the other hand, the conditions for the potential $a_\mu^{(+)}$ to satisfy (2.3) and so to lead to conserved charges, independently of what happens to $\tilde{A}_\mu$, is that $\partial_t \xi_{2n+1}^{(+)}\,(t, x = \infty) = \partial_t \xi_{2n+1}^{(+)}\,(t, x = -\infty)$. However, that will involve intricate conditions on α. Therefore, the question if one can construct conserved charges from the potentials $\tilde{A}_\mu$ and $a_\mu^{(+)}$ depends on a very detailed analysis of the boundary conditions satisfied by the solutions obtained by the dressing method.

The conditions (4.42) however, are suffucient to obtain conserved charges from the potentials A_μ and $a_\mu^{(-)}$, as argued above. Those charges are obtained following the discussion given in (4.19)-(4.24), and are given by

$$\Omega_{2n+1}^{(-)} = -\,(2n+1)\left[\xi_{2n+1}^{(-)}\,(t, x = \infty) - \xi_{2n+1}^{(-)}\,(t, x = -\infty)\right] \qquad n = 0, 1, 2, \ldots \tag{4.46}$$

The asymptotic values of $\xi_{2n+1}^{(-)}$ can be evaluated using the highest weight states of the fundamental representations of the $sl(2)$ Kac-Moody algebra in a manner similar to that done for the sine-Gordon case in (3.41)-(3.44).

Using (4.41) one gets that the lowest charge is related to one of the Hamiltonians of the mKdV hierarchy. Indeed, one has from (4.41), (4.46) and (4.36) that

$$\Omega_1^{(-)} = -\frac{1}{2} \int_{-\infty}^{\infty} dx \, q^2 = (\nu \mid_{x=\infty} - \nu \mid_{x=-\infty}) = \frac{1}{2} \partial_x \ln (\tau_0 \, \tau_1) \mid_{x=-\infty}^{x=\infty} \tag{4.47}$$

Therefore, we have here a situation very similar to the sine-Gordon case (see (3.33)) where the energy of the solution is determined by the asymptotic behavior of the extra field associated to the central term of the algebra. Our method therefore gives a very simple formula for the energy, and also for the higher charges, of the mKdV solutions on the orbit of the vacumm, $q^{\rm vac} = \nu^{\rm vac} = 0$, under the dressing transformation group.

A The $sl(2)$ Kac-Moody algebra

The commutation relations of the $sl(2)$ Kac-Moody algebra are given by [11]

$$\begin{aligned} \left[T_3^m \, , \, T_3^n \right] &= \frac{1}{2} m \, \delta_{m+n,0} \, C, \\ \left[T_3^m \, , \, T_\pm^n \right] &= \pm \, T_\pm^{m+n}, \\ \left[T_+^m \, , \, T_-^n \right] &= 2 \, T_3^{m+n} + m \, \delta_{m+n,0} \, C, \end{aligned} \tag{A.1}$$

where C is the central term. The relevant basis for our calculations is given by

$$b_{2m+1} = T_+^m + T_-^{m+1} \; ; \qquad F_{2m+1} = T_+^m - T_-^{m+1} \; ; \qquad F_{2m} = 2 \, T_3^m - \frac{1}{2} \delta_{m,0} \, C, \tag{A.2}$$

which satisfy

$$\begin{aligned} \left[b_{2m+1} \, , \, b_{2n+1} \right] &= (2m+1) \; \delta_{m+n+1,0} \, C, \\ \left[b_{2m+1} \, , \, F_n \right] &= -2 \, F_{n+2m+1}, \\ \left[F_{2m+1} \, , \, F_{2n} \right] &= -2 \, b_{2(m+n)+1}, \\ \left[F_{2m+1} \, , \, F_{2n+1} \right] &= -(2m+1) \, \delta_{m+n+1,0} \, C, \\ \left[F_{2m} \, , \, F_{2n} \right] &= 2m \, \delta_{m+n,0} \, C. \end{aligned} \tag{A.3}$$

The indices of the generators correspond to the grades under

$$\left[Q \, , \, b_{2m+1} \right] = (2m+1) \; b_{2m+1} \; ; \qquad \left[Q \, , \, F_n \right] = n \, F_n, \tag{A.4}$$

where

$$Q = T_3^0 + 2d \qquad \text{with} \qquad \left[d \, , \, T_i^m \right] = m \, T_i^m \qquad i = 3, +, -. \tag{A.5}$$

In the case when the central term vanishes, *i.e.* $C = 0$, the algebra is called the $sl(2)$ loop algebra, and it admits finite matrix representations. In the case of 2×2 matrices on has

$$b_{2m+1} = \begin{pmatrix} 0 & \lambda^m \\ \lambda^{m+1} & 0 \end{pmatrix}, \quad F_{2m+1} = \begin{pmatrix} 0 & \lambda^m \\ -\lambda^{m+1} & 0 \end{pmatrix}, \quad F_{2m} = \lambda^m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.6}$$

with $m = 0, \pm 1, \pm 2 \ldots$. In that case, the operator d takes the form $d \equiv \lambda \frac{d}{d\lambda}$.

For $C \neq 0$ all the representations of the $sl(2)$ Kac-Moody algebra are infinite dimensional. The methods of constructing these representations involve field theory techniques [11], like the vertex operator given below. However, having $C \neq 0$ leads to a very desirable property, namely the existence of the highest weight state representations, *i.e.* representations that contain states that are annihilated by positive root step operators (the generalization of T_+ in the algebra of angular momentum). Indeed, among the highest weight state representations of the $sl(2)$ Kac-Moody algebra there are two that play a very important role. They are the two fundamental representations, with highest weight states $\mid \lambda_i \rangle$, $i = 0, 1$, satisfying

$$\begin{aligned} T_3^0 \mid \lambda_0 \rangle &= 0, & T_3^0 \mid \lambda_1 \rangle &= \frac{1}{2} \mid \lambda_1 \rangle, \\ C \mid \lambda_0 \rangle &= \mid \lambda_0 \rangle, & C \mid \lambda_1 \rangle &= \mid \lambda_1 \rangle \end{aligned} \tag{A.7}$$

and

$$T_3^n \mid \lambda_i \rangle = T_\pm^n \mid \lambda_i \rangle = T_+^0 \mid \lambda_i \rangle = 0, \qquad n > 0 \, ; \quad i = 0, 1. \tag{A.8}$$

From (A.7) one gets

$$F_0 \mid \lambda_0 \rangle = -\frac{1}{2} \mid \lambda_0 \rangle, \qquad F_0 \mid \lambda_1 \rangle = \frac{1}{2} \mid \lambda_1 \rangle. \tag{A.9}$$

An important mathematical tool in the study of solitons is the use of the so-called vertex operator representations of the Kac-Moody algebras. In the case of the sine-Gordon model the relevant representation is the one involving the principal vertex operators. It is based on the Fock space of the oscillators b_{2n+1} satisfying the first relation in (A.3) with $C = 1$, *i.e.*

$$[\, b_{2m+1} \, , \, b_{2n+1} \,] = (2m+1) \; \delta_{m+n+1,0}. \tag{A.10}$$

The vertex operator is defined as [11]:

$$V(z) \equiv : e^{\mathcal{Q}(z)} := e^{\mathcal{Q}_<(z)} \, e^{\mathcal{Q}_>(z)}, \tag{A.11}$$

where, as usual, : :, denotes the normal ordering of the oscillators (b_{2n+1} with $n \geq 0$ are the annihilation operators, and the negative ones the creation operators), and where

$$\mathcal{Q}\left(z\right) \equiv \mathcal{Q}_{<}\left(z\right) + \mathcal{Q}_{>}\left(z\right) \tag{A.12}$$

and

$$\mathcal{Q}_{>}\left(z\right) = \sum_{n=0}^{\infty} \frac{2\, z^{-2n-1}}{2n+1}\, b_{2n+1}, \qquad \mathcal{Q}_{<}\left(z\right) = -\sum_{n=0}^{\infty} \frac{2\, z^{2n+1}}{2n+1}\, b_{-2n-1} \tag{A.13}$$

with z being an arbitrary (complex) parameter.

One can then show that in such a representation the generators F_n are given by [11]:

$$F_n = \oint \frac{dz}{2\pi i z}\, z^n\, V\left(z\right). \tag{A.14}$$

There are two important properties of the vertex operators which are relevant for the solitons. First, the vertex operators are eigenstates of the oscillators

$$\left[\, b_{2n+1}\,,\, V\left(z\right) \right] = -2\, z^{2n+1}\, V\left(z\right), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{A.15}$$

The second property is its operator product expansion

$$V\left(z_1\right)\, V\left(z_2\right) = :V\left(z_1\right)\, V\left(z_2\right): \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2 \tag{A.16}$$

so that $V\left(z\right)$ is nilpotent

$$V\left(z\right)^2 = 0. \tag{A.17}$$

One can also show that

$$V\left(z_1\right)\, V\left(z_2\right)\, V\left(z_3\right) = :V\left(z_1\right)\, V\left(z_2\right)\, V\left(z_3\right): \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2 \left(\frac{z_1 - z_3}{z_1 + z_3}\right)^2 \left(\frac{z_2 - z_3}{z_2 + z_3}\right)^2$$

and in general that

$$\prod_{i=1}^{n} V\left(z_i\right) = :\prod_{i=1}^{n} V\left(z_i\right): \prod_{i<j=1}^{n} \left(\frac{z_i - z_j}{z_i + z_j}\right)^2. \tag{A.18}$$

We also have that

$$\langle\, \lambda_0 \;|: \prod_{i=1}^{n} V\left(z_i\right) :|\; \lambda_0\,\rangle = 1, \qquad \text{and} \qquad \langle\, \lambda_1 \;|: \prod_{i=1}^{n} V\left(z_i\right) :|\; \lambda_1\,\rangle = (-1)^n. \tag{A.19}$$

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