FAULT-TOLERANT EMBEDDINGS OF HAMILTONIAN CIRCUITS IN K-ARY N-CUBES*

YAAGOUB A. ASHIR† AND IAIN A. STEWART†

Abstract. We consider the fault-tolerant capabilities of networks of processors whose underlying topology is that of the k-ary n-cube \( Q_k^n \), where \( k \geq 3 \) and \( n \geq 2 \). In particular, given a copy of \( Q_k^n \) where some of the interprocessor links may be faulty but where every processor is incident with at least two healthy links, we show that if the number of faults is at most \( 4n - 5 \), then \( Q_k^n \) still contains a Hamiltonian circuit, but that there are situations where the number of faults is \( 4n - 4 \) (and every processor is incident with at least two healthy links) and no Hamiltonian circuit exists. We also remark that given a faulty \( Q_k^n \), the problem of deciding whether there exists a Hamiltonian circuit is NP-complete.

Key words. Hamiltonian circuits, embeddings, fault-tolerance, k-ary n-cubes, NP-completeness

AMS subject classifications. 68R10, 05C45

PII. S0895-4801(96)31118-3

1. Introduction. The hypercube or, more precisely, the binary n-cube \( B_n \) (where \( n \geq 2 \)), is a popular interconnection network for parallel processing as it possesses a number of topological properties which are highly desirable in the context of parallel processing: for example, it contains a Hamiltonian circuit; many other networks can be embedded into a binary n-cube; and its symmetry results in rich communication properties (see, for example, [3, 5, 8, 10, 12] and the references therein).

Fault-tolerance in the binary n-cube is an important issue, given that many other networks can be embedded therein, and has been studied in a number of contexts. For example, the ability of the binary n-cube to route and reconfigure itself in spite of faults has been considered (see the references in [8]), as has the embedding of Hamiltonian circuits in binary n-cubes in the presence of faults [8]. In particular, Chan and Lee [8] proved that a binary n-cube where at most \( 2n - 5 \) links are faulty and where every node is incident with at least two healthy links (a natural assumption to make) has a Hamiltonian circuit, but that there exist binary n-cubes with \( 2n - 4 \) faults (and where every node is incident with at least two healthy links) not containing a Hamiltonian circuit. It is with an analogous version of this result that we are concerned in this paper.

One drawback of the binary n-cube is that the number of links incident with each node is logarithmic in the number of nodes, and this causes problems with regard to current VLSI technology when the networks built upon the binary n-cube topology involve a large number of processors. One means proposed to alleviate this problem is to base networks on the topology of the k-ary n-cube \( Q_k^n \) (where \( k \geq 3 \) and \( n \geq 2 \)). A network based on \( Q_k^n \) is such that each node is incident with 2n links, and consequently \( k \) can be increased, in order to incorporate more processors, at the same time keeping \( n \) constant. Moreover, “high-dimensional” networks generally cost more

*Received by the editors October 28, 1996; accepted for publication (in revised form) March 22, 2002; published electronically May 8, 2002. The research of the first author was supported by the University of Bahrain.

†Department of Mathematics and Computer Science, Leicester University, Leicester LE1 7RH, UK (y.ashir@mcs.le.ac.uk, i.a.stewart@mcs.le.ac.uk).
and run more slowly than “low-dimensional” networks, and it has also been shown that low-dimensional networks achieve lower latency and better hot-spot throughput than their high-dimensional counterparts [9, 11].

The properties of the $k$-ary $n$-cube $Q^k_n$ relevant to parallel processing have not been determined to such an extent as those of the binary $n$-cube: however, some work has been done (see, for example, [1, 2, 4, 6, 7]). In particular, it has been shown that $Q^k_n$ has a Hamiltonian circuit [6].

In this paper, we examine the number of link faults that a $k$-ary $n$-cube $Q^k_n$ can tolerate so that there is still a Hamiltonian circuit. (Of course, we assume that every node is incident with at least two healthy links.) In particular, we show that a $k$-ary $n$-cube $Q^k_n$ where at most $4n - 5$ links are faulty and where every node is incident with at least two healthy links has a Hamiltonian circuit, but that there exist $k$-ary $n$-cubes with $4n - 4$ faults (and where every node is incident with at least two healthy links) not containing a Hamiltonian circuit. We also remark that the general problem of deciding whether a faulty $k$-ary $n$-cube contains a Hamiltonian circuit is NP-complete for all (fixed) $k \geq 3$. Our results can be regarded as direct analogues of those in [8] for $k$-ary $n$-cubes as opposed to binary $n$-cubes.

2. Tolerating faults. Throughout this paper, we prefer to use the terminology “nodes” and “links” as opposed to “vertices” and “edges,” for whilst the results in this paper are entirely graph-theoretic, the use of “nodes” and “links” accentuates the motivational source of our research, i.e., the fault-tolerating capabilities of networks.

The binary $n$-cube, for $n \geq 2$, can be represented as the set of $2^n$ nodes $\{0, 1\}^n$ where there is a link joining nodes $u$ and $v$ if and only if $u$ and $v$ agree on all components except one. Note that each node has degree $n$. The $k$-ary $n$-cube $Q^k_n$, for $k \geq 2$ and $n \geq 2$, can be represented as the set of $k^n$ nodes $\{0, 1, \ldots, k - 1\}^n$ where there is a link joining nodes $u$ and $v$ if and only if $u$ and $v$ agree on all components except one, and on that component they differ by $1$ modulo $k$. Note that each node has degree $2n$, when $k \geq 3$, and $n$ when $k = 2$. In particular, $Q^2_n$ is simply $B_n$.

For each $i \in \{1, 2, \ldots, n\}$, we refer to all links whose incident nodes differ in the $i$th component as lying in dimension $i$. Note that for any $i \in \{1, 2, \ldots, n\}$, $Q^k_n$ consists of $k$ disjoint copies of $Q^{k-1}_n$ where corresponding nodes are joined in circuits of length $k$ using links in dimension $i$. When we consider $Q^k_n$ in this way, with the disjoint copies joined by links lying in dimension $i$, we say that we have partitioned $Q^k_n$ over dimension $i$.

Let us now proceed to the proof of our main theorem. This proof is by induction. We begin by proving the inductive step, and then we return to the base cases of the induction.

**Theorem 2.1.** Let $k \geq 4$ and $n \geq 2$, or let $k = 3$ and $n \geq 3$. If $Q^k_n$ has at most $4n - 5$ faulty links and is such that every node is incident with at least $2$ healthy links, then $Q^k_n$ has a Hamiltonian circuit.

**Proof.** The proof proceeds by induction on $n$. We handle the base cases, when $n = 2$ and $k \geq 4$ and when $n = 3$ and $k = 3$, later. As our induction hypothesis, assume that the result holds for $Q^k_{n-1}$, for some $n \geq 2$ and for all $k \geq 4$, or for some $n \geq 3$ and $k = 3$. Let $Q^k_{n+1}$ have $4n - 1$ faults and be such that every node is incident with at least two healthy links. Then there exists some dimension, say dimension $1$, which contains at least three faults. We can partition $Q^k_{n+1}$ over dimension $1$ and consider $Q^k_{n+1}$ to consist of $k$ disjoint copies $Q_1, Q_2, \ldots, Q_k$ of $Q^k_n$ with corresponding...
nodes joined in circuits of length $k$, where the faults contained in $Q_1, Q_2, \ldots, Q_k$ total at most $4n - 4$ (see Figure 2.1). Throughout this proof, if $u$ is a node of $Q_i$, say, then we often denote it by $u_i$, and we refer to its corresponding node in $Q_j$ as $u_j$. Our general aim below is to argue, using induction, that Hamiltonian circuits exist in each of $Q_1, Q_2, \ldots, Q_k$ and that we can “join” these circuits together using links in dimension 1 to obtain a Hamiltonian circuit in $Q_k^{n+1}$. (What we mean by “join” will become clear later: also, the general aim of connecting together circuits in $Q_1, Q_2, \ldots, Q_k$ actually has to be more sophisticated in some scenarios.) Naturally, different scenarios arise according to the distribution of faulty links in $Q_1, Q_2, \ldots, Q_k$ and in dimension 1. Another complication is that the chosen Hamiltonian circuit in $Q_2$, for example, might depend upon the Hamiltonian circuit chosen in $Q_1$.

Case (i). Each $Q_i$ is such that every node is incident with at least two healthy links and no $Q_i$ contains $4n - 4$ faults. Without loss of generality (w.l.o.g.) we may assume that $Q_1$ has most faults from amongst $Q_1, Q_2, \ldots, Q_k$. Hence, each of $Q_2, Q_3, \ldots, Q_k$ has at most $2n - 2$ faults. By the induction hypothesis, $Q_1$ has a Hamiltonian circuit $C_1$. Following our basic strategy, outlined above, we wish to find a Hamiltonian circuit $C_k$ in $Q_k$ or a Hamiltonian circuit $C_2$ in $Q_2$ so that we might “join” such a Hamiltonian circuit to $C_1$ using healthy links in dimension 1. By “join” we mean replace a link $(x_1, y_1)$ of $C_1$ and the (corresponding) link $(x_2, y_2)$ of $C_2$, for example, with the links $(y_1, y_2)$ and $(y_1, x_2)$ in dimension 1. However, we must ensure that two mutually compatible links exist in $C_1$ and $C_2$ and also that the relevant dimension 1 links are healthy.

We begin by applying a counting argument to show that there exist links $(x_1, y_1)$ and $(y_1, z_1)$ of $C_1$ such that either

- $(x_2, y_2), (y_2, z_2), (x_1, x_2), (y_1, y_2)$, and $(z_1, z_2)$ are all healthy

or

- $(x_k, y_k), (y_k, z_k), (x_1, x_k), (y_1, y_k)$, and $(z_1, z_k)$ are all healthy.

Suppose that it were otherwise. Then there would exist at least $2\lfloor k^n/3 \rfloor$ faults not in $Q_1$. (Split $C_1$ into groups of three consecutive vertices and look at the links on either side in dimension 1 and in $Q_2$ and $Q_k$.) However, when $n \geq 2$ and $k \geq 4$ or when $n \geq 3$ and $k = 3$, we have that $2\lfloor k^n/3 \rfloor > 4n - 1$, which yields a contradiction. Hence, w.l.o.g. we may assume that there exist links $(x_1, y_1) and (y_1, z_1)$ of $C_1$ such that $(x_2, y_2), (y_2, z_2), (x_1, x_2), (y_1, y_2), and (z_1, z_2)$ are all healthy.
What we need to do now is to show that there is a Hamiltonian circuit $C_2$ in $Q_2$ containing either $(x_2, y_2)$ or $(y_2, z_2)$; we can then join $C_1$ and $C_2$ as described above. Suppose that it were otherwise. If necessary, mark some of the links of $Q_2$ incident with $y_2$ as faulty (that is, temporarily regard them as faulty) so that $y_2$ is incident with at most three healthy links in $Q_2$, two of which are always $(x_2, y_2)$ and $(y_2, z_2)$. Consequently, as there were originally at most $2n - 2$ faulty links in $Q_2$, there are now at most $4n - 5$ faulty links. However, in order to apply our induction hypothesis (and deduce that this amended $Q_2$ has a Hamiltonian circuit), we need that every node in (the amended) $Q_2$ is incident with at least two healthy links. Suppose that it were otherwise. Then there is a node $w_2$ incident with exactly one healthy link. This must have been because $(y_2, w_2)$ was a healthy link in the original $Q_2$ and it was subsequently marked as faulty. Amend the marking of healthy links so that $(w_2, y_2)$ is the third healthy link in the amended $Q_2$. Note that in the amended marking every node is incident with at least two healthy links (because $Q_2$ originally had at most $2n - 2$ faults). Now we can apply the induction hypothesis and deduce that $Q_2$ has a Hamiltonian circuit $C_2$ containing either $(x_2, y_2)$ or $(y_2, z_2)$ (possibly both). No matter which, we can join $C_2$ to $C_1$ (as described above) to obtain a circuit $D_2$ containing every node of $Q_1$ and $Q_2$. (Henceforth, we now treat those links of $Q_2$ which were temporarily marked as faulty as being healthy again.)

All links of $D_2$ except for $(x_1, x_2)$ and $(y_1, y_2)$ are links in $Q_1$ or $Q_2$. Hence, there is much potential to join $D_2$, as above, to a Hamiltonian circuit in $Q_3$ or $Q_k$. Similarly to as before (by applying exactly the same counting argument), w.l.o.g. there exist two consecutive links $(u_2, v_2)$ and $(v_2, w_2)$ of $D_2 \cap Q_2$ such that the links $(u_3, v_3)$, $(v_3, w_3)$, $(u_2, v_3)$, and $(w_2, w_3)$ are healthy. Again, by arguing exactly as before, there is a Hamiltonian circuit $C_3$ in $Q_3$ containing either the link $(u_3, v_3)$ or the link $(v_3, w_3)$; and we can join $D_2$ to $C_3$ using links in dimension 1 to obtain a circuit $D_3$ containing all nodes of $Q_1$, $Q_2$, and $Q_3$. Exactly the same arguments apply so that we might extend $D_3$ to a circuit $D_4$, containing all nodes of $Q_1$, $Q_2$, $Q_3$, and $Q_4$, and so on until we obtain a Hamiltonian circuit in $Q_{k+1}$.

Case (ii). Each $Q_j$ is such that every node is incident with at least two healthy links and some $Q_j$ has exactly $4n - 4$ faults.

W.l.o.g. we may assume that $j = 1$. Suppose that there is some fault $(x_1, y_1)$ of $Q_1$ such that $(x_1, x_2)$ and $(y_1, y_2)$ are healthy. Amend $Q_1$ so that $(x_1, y_1)$ is temporarily marked as healthy. By the induction hypothesis applied to this amended $Q_1$, there is a Hamiltonian circuit $C_1$ which may or may not contain $(x_1, y_1)$; and $C_1$ is a circuit in the original $Q_1$. The circuit $C_1$ has an isomorphic copy $C_i$ in each $Q_i$ for $i = 2, 3, \ldots, k$. If $(x_1, y_1)$ is in $C_1$, the circuit $C_1$ can be joined to $C_2$ using the healthy links $(x_1, x_2)$ and $(y_1, y_2)$. Otherwise, because there are exactly three faults in dimension 1 and $[k_0/2] > 3$, there is a link $(u_1, v_1)$ of $C_1$ such that $(u_1, u_2)$ and $(v_1, v_2)$ are healthy. (Use a counting argument similar to that used before except split $C_1$ into groups of two consecutive vertices and look at the pairs of links in dimension 1 joining $Q_1$ to $Q_2$.) $C_1$ can now be joined to $C_2$ using these links to yield a circuit $D_2$ containing every node of $Q_1$ and $Q_2$. The circuit $D_2$ contains $k_0 - 1$ links of $Q_2$. As $\lfloor (k_0 - 1)/2 \rfloor > 3$, the same argument yields that there is a link $(u_2, v_2)$ of $D_2 \cap Q_2$ such that the links $(w_3, z_3)$, $(u_2, w_3)$, and $(z_2, z_3)$ are all healthy. Moreover, $(w_3, z_3)$ lies on the circuit $C_3$ of $Q_3$. Hence, we can join $D_2$ and $C_3$ to obtain a circuit $D_3$ containing every node of $Q_1$, $Q_2$, and $Q_3$. Exactly the same arguments apply so that we can extend $D_3$ to a Hamiltonian circuit of $Q_{k+1}$.

On the other hand, suppose that, for every fault $(x_1, y_1)$ of $Q_1$, at least one of
(x₁, x₂) and (y₁, y₂), and at least one of (x₁, xₖ) and (y₁, yₖ), are faulty. Let (x₁, y₁) be some fault of Q₁. As there are exactly three faults in dimension 1, it cannot be the case that two faults in Q₁ are not incident with one another. Let us now count the maximum number \( \mu \) of faults of Q₁ which could be incident with either \( x₁ \) or \( y₁ \).

Consider \( x₁ \). The number of faults incident with \( x₁ \), apart from the fault (\( x₁, y₁ \)), is at most \( 2n - 3 \). Similarly, the number of faults incident with \( y₁ \), apart from the fault (\( x₁, y₁ \)), is at most \( 2n - 3 \). Hence, \( \mu \leq (2n - 3) + (2n - 3) + 1 = 4n - 5 \). However, there are \( 4n - 4 \) faults in Q₁ and so we obtain a contradiction.

**Case (iii).** There exists some Q₁ in which there is a node incident with exactly one healthy link in Q₁.

W.l.o.g. we may assume that the node \( x₁ \) in Q₁ is incident with exactly one healthy link, (\( x₁, y₁ \), in Q₁. As \( x₁ \) is incident with \( 2n - 1 \) faults in Q₁, each Q₁, for \( i = 2, 3, \ldots, k \), contains at most \( 2n - 3 \) faults; there is no node in any Q₁, for \( i = 2, 3, \ldots, k \), which is incident with less than three healthy links in that Q₁; and apart from \( x₁ \), there is no other node in Q₁ which is incident with less than two healthy links in Q₁. Also, as \( x₁ \) is incident with at least two healthy links in \( Qₙ₊₁ \), we may suppose that (\( x₁, x₂ \)) is healthy. Consider \( w₁ \), one of the \( 2n - 1 \) potential neighbors of \( x₁ \) in Q₁, for which the link (\( x₁, w₁ \)) is faulty. There are two scenarios.

**Case (iii)(a).** (\( w₁, w₂ \)) is a healthy link.

Mark the previously faulty link (\( x₁, w₁ \)) as temporarily healthy. By the induction hypothesis applied to this amended Q₁, there is a Hamiltonian path \( P₁ \) from \( x₁ \) to \( w₁ \). Moreover, this Hamiltonian path \( P₁ \) is a Hamiltonian path in the original Q₁ (where the links temporarily marked as faulty resume their healthy status).

Mark some of the previously healthy links in Q₂ that are incident with \( x₂ \) as temporarily faulty and mark the link (\( x₂, w₂ \)) as temporarily healthy (if necessary) so as to ensure that \( x₂ \) is incident with exactly two healthy links in this amended Q₂ (one of which is (\( x₂, w₂ \))). Note that in order to build this amended Q₂ we have introduced at most \( 2n - 2 \) temporary faults; and so this amended Q₂ has at most \( 4n - 5 \) faults and every node is incident with at least two healthy links. Hence, by the induction hypothesis, there exists a Hamiltonian path \( P₂ \) in this amended Q₂ from \( x₂ \) to \( w₂ \). Moreover, this Hamiltonian path \( P₂ \) is a Hamiltonian path in the original Q₂. Join \( P₁ \) and \( P₂ \) using the healthy links (\( x₁, x₂ \)) and (\( w₁, w₂ \)) to form a circuit \( D₂ \) which contains all nodes of Q₁ and Q₂.

Applying a counting argument similar to that used in Case (ii), along with the fact that \( \lfloor (kⁿ - 1)/2 \rfloor > 2n \) (note that the total number of faults in \( Qₙ₊₁ \) not contained in Q₁ is at most 2n), there exists a link (\( u₂, v₂ \)) of \( D₂ \cap Q₂ \) such that the links (\( u₃, v₃ \)), (\( u₂, u₃ \)), and (\( v₂, v₃ \)) are healthy. Temporarily mark healthy links in Q₃ incident with \( u₃ \) as faulty so that in this amended Q₃, \( u₃ \) is incident with exactly two healthy links, one of which is (\( u₃, v₃ \)). In order to build this amended Q₃ we have introduced at most \( 2n - 2 \) temporary faults; and so this amended Q₃ has at most \( 4n - 5 \) faults and every node is incident with at least two healthy links. By the induction hypothesis, there is a Hamiltonian circuit \( C₃ \) in the original Q₃ containing the link (\( u₃, v₃ \)). We can join \( D₂ \) and \( C₃ \), using the healthy links (\( u₂, u₃ \)) and (\( v₂, v₃ \)), to obtain a circuit \( D₃ \) containing every node of Q₁, Q₂, and Q₃. Exactly the same argument can be applied to extend \( D₃ \) to a circuit \( D₄ \) and so on until we have a Hamiltonian circuit of \( Qₙ₊₁ \).

**Case (iii)(b).** All links from every such \( w₁ \) to its corresponding node \( w₂ \) in Q₂ are faulty.

This accounts for another \( 2n - 1 \) faults in \( Qₙ₊₁ \). Also, if (\( x₁, xₖ \)) is healthy, then
by symmetry we are in Case (iii)(a) (as all but at most one link of the form \((w_1, w_k)\) is healthy). Hence, we may assume that \((x_1, x_k)\) is faulty, and this accounts for all the faults in \(Q_{n+1}^k\).

Consequently, \((y_1, y_2)\) and \((y_1, y_k)\) are both healthy links. (Recall that \((x_1, y_1)\) is the only healthy link of \(Q_1\) incident with \(x_1\).) Let \(w_1\) be some potential neighbor of \(x_1\) in \(Q_1\) for which the link \((x_1, w_1)\) is faulty. Amend \(Q_1\) by marking the link \((x_1, w_1)\) as temporarily healthy. By the induction hypothesis applied to this amended \(Q_1\), there is a Hamiltonian path \(P_1\) in the original \(Q_1\) from \(x_1\) to \(w_1\). Rename the nodes of \(P_1\) as \(x_{1,1} = x_1, x_{1,2} = y_1, x_{1,3}, \ldots, x_{1,k^n} = w_1\), and note that in each \(Q_i\), \(i \geq 2\), there is a corresponding Hamiltonian path \(P_i\) which can be extended to a Hamiltonian circuit \(C_i\) of \(Q_i\) (as \((x_i, w_i)\) is healthy in \(Q_i\)). Rename the nodes of \(C_i\) as \(x_{i,1} = x_i, x_{i,2} = y_i, x_{i,3}, \ldots, x_{i,k^n} = w_i\) for each \(i \geq 2\).

For ease of notation, denote \(k^n\) by \(m\). Suppose \(k\) is even. Then the following is a Hamiltonian circuit in \(Q_{n+1}^k\):

\[
(x_{1,1}, x_{2,1}, \ldots, x_{k,1}, x_{k,2}, x_{k,3}, x_{1,3}, x_{1,4}, \ldots, x_{1,m}, x_{k,m}, x_{k-1,m}, \ldots, x_{2,m})
\]

\[
x_{2,m-1}, x_{3,m-1}, \ldots, x_{k,m-1}, x_{k,m-2}, x_{k-1,m-2}, \ldots, x_{2,m-2}, x_{2,m-3},
\]

\[
x_{3,m-3}, x_{k,m-3}, x_{k,m-4}, \ldots, x_{k,m-4}, x_{k-1,m-4}, \ldots, x_{2,2}, x_{2,3}, x_{3,3}, \ldots,
\]

\[
x_{k-1,3}, x_{k-1,2}, x_{k-1,2}, \ldots, x_{2,2}, x_{1,2}, x_{1,1}).
\]

(See Figure 2.2 where some of the healthy links between the \(Q_i\)'s are shown and bold links denote the links of the Hamiltonian circuit.) If \(k\) is odd, then the following is a Hamiltonian circuit in \(Q_{n+1}^k\):

\[
(x_{1,1}, x_{2,1}, \ldots, x_{k,1}, x_{k,2}, x_{k-1,2}, \ldots, x_{2,2}, x_{2,3}, x_{3,3}, \ldots, x_{k,3}, x_{k,4}, x_{k-1,4}, \ldots,
\]

\[
x_{2,4}, x_{2,5}, \ldots, x_{2,m}, x_{3,m}, \ldots, x_{2,m}, x_{k,m}, \ldots, x_{1,m}, x_{1,m-1}, \ldots, x_{1,2}, x_{1,1}).
\]

(See Figure 2.3.)

**Case (iv).** There exists some \(Q_i\) in which there is a node incident with no healthy links in \(Q_i\).

W.l.o.g. we may assume that \(x_1\) is incident with no healthy links in \(Q_1\). As \(x_1\) is incident with at least two healthy links in \(Q_{n+1}^k\), the links \((x_1, x_2)\) and \((x_1, x_k)\) must be healthy. There are at least 2 faults in \(Q_1\), and so there must be at most \(2n - 4\) faults distributed amongst \(Q_2, Q_3, \ldots, Q_k\). Hence, apart from \(x_1\), there are no nodes which are incident with less than four healthy links in their respective copy of \(Q_i\).

The node \(x_1\) has \(2n\) potential neighbors in \(Q_1\). Each of these potential neighbors is incident with a potential dimension 1 link to \(Q_1\) and a potential dimension 1 link to \(Q_k\). (These dimension 1 links might be faulty.) As there are at most \(2n - 4\) faults in dimension 1, there must exist potential neighbors \(y_1\) and \(z_1\) of \(x_1\) such that the links \((y_1, y_2)\) and \((z_1, z_k)\) are healthy. (Partition the potential neighbors into \(n\) pairs \(\{y_1, z_1\}\) and look at the pairs of dimension 1 links \(\{(y_1, y_2), (z_1, z_k)\}\) and \(\{(y_1, y_k), (z_1, z_2)\}\).)

Mark the faulty links \((x_1, y_1)\) and \((x_1, z_1)\) as temporarily healthy in \(Q_1\). Applying the induction hypothesis to this amended \(Q_1\), we obtain a path \(P_1\) in the original \(Q_1\) from \(y_1\) to \(z_1\) upon which every node of \(Q_1\) appears exactly once, except for \(x_1\) which does not appear at all.

By marking previously healthy links in \(Q_2\) that are incident with \(x_2\) as temporarily faulty, and by marking the link \((x_2, y_2)\) as temporarily healthy (if necessary), ensure that \(x_2\) is incident with exactly two healthy links in this amended \(Q_2\), one of which is \((x_2, y_2)\). This involves introducing at most \(2n - 2\) temporary faults into \(Q_2\); and so the amended \(Q_2\) has at most \(4n - 6\) faults and every node is incident with at least
two healthy links. The induction hypothesis yields that there is a Hamiltonian path from $x_2$ to $y_2$ in the original $Q_2$. Likewise, there is a Hamiltonian path from $x_k$ to $z_k$ in $Q_k$. Hence, let $D_2$ be the circuit obtained by joining $P_1$, $P_2$, and $P_k$ using the healthy links $(x_1, x_2)$, $(y_1, y_2)$, $(x_1, x_k)$, and $(z_1, z_k)$.

Applying a counting argument similar to that used in Case (ii), along with the fact that $\left\lfloor \left(\frac{k^3 - 1}{2}\right) \right\rfloor > 2n - 1$ (note that the total number of faults in $Q_{n+1}^k$ not contained in $Q_1$ is at most $2n - 1$), there exists a link $(u_2, v_2)$ of $D_2 \cap Q_2$ such that the links $(u_3, v_3)$, $(u_2, u_3)$, and $(v_2, v_3)$ are healthy. Temporarily mark healthy links in $Q_3$ incident with $u_3$ as faulty so that in this amended $Q_3$, $u_3$ is incident with exactly two healthy links, one of which is $(u_3, v_3)$. In order to build this amended $Q_3$ we have introduced at most $2n - 2$ temporary faults; and so this amended $Q_3$ has at most $4n - 6$ faults and every node is incident with at least two healthy links. By the induction hypothesis, there is a Hamiltonian circuit $C_3$ in the original $Q_3$ containing the link $(u_3, v_3)$. We can join $D_2$ and $C_3$, using the healthy links $(u_2, u_3)$ and $(v_2, v_3)$, to obtain a circuit $D_3$ containing every node of $Q_k$, $Q_1$, $Q_2$, and $Q_3$. Exactly the same argument can be applied to extend $D_3$ to a circuit $D_4$ and so on until we have a Hamiltonian circuit of $Q_{n+1}^k$.

It remains to show that the result holds for the base cases of the induction, namely, when $n = 2$ and $k \geq 4$, and when $n = 3$ and $k = 3$.

**Lemma 2.2.** If $Q_2^k$, where $k \geq 4$, has three faulty links and is such that every node is incident with at least two healthy links, then $Q_2^k$ has a Hamiltonian circuit.

**Proof.** There exists some dimension, say dimension 1, that contains at least two healthy links. The induction hypothesis yields that there is a Hamiltonian path from $x_2$ to $y_2$ in the original $Q_2$. Likewise, there is a Hamiltonian path from $x_k$ to $z_k$ in $Q_k$. Hence, let $D_2$ be the circuit obtained by joining $P_1$, $P_2$, and $P_k$ using the healthy links $(x_1, x_2)$, $(y_1, y_2)$, $(x_1, x_k)$, and $(z_1, z_k)$.

Applying a counting argument similar to that used in Case (ii), along with the fact that $\left\lfloor \left(\frac{k^3 - 1}{2}\right) \right\rfloor > 2n - 1$ (note that the total number of faults in $Q_{n+1}^k$ not contained in $Q_1$ is at most $2n - 1$), there exists a link $(u_2, v_2)$ of $D_2 \cap Q_2$ such that the links $(u_3, v_3)$, $(u_2, u_3)$, and $(v_2, v_3)$ are healthy. Temporarily mark healthy links in $Q_3$ incident with $u_3$ as faulty so that in this amended $Q_3$, $u_3$ is incident with exactly two healthy links, one of which is $(u_3, v_3)$. In order to build this amended $Q_3$ we have introduced at most $2n - 2$ temporary faults; and so this amended $Q_3$ has at most $4n - 6$ faults and every node is incident with at least two healthy links. By the induction hypothesis, there is a Hamiltonian circuit $C_3$ in the original $Q_3$ containing the link $(u_3, v_3)$. We can join $D_2$ and $C_3$, using the healthy links $(u_2, u_3)$ and $(v_2, v_3)$, to obtain a circuit $D_3$ containing every node of $Q_k$, $Q_1$, $Q_2$, and $Q_3$. Exactly the same argument can be applied to extend $D_3$ to a circuit $D_4$ and so on until we have a Hamiltonian circuit of $Q_{n+1}^k$.

It remains to show that the result holds for the base cases of the induction, namely, when $n = 2$ and $k \geq 4$, and when $n = 3$ and $k = 3$.
faults. Partition $Q^k_2$ over dimension 1 to obtain $k$ copies of $Q^k_1$, namely $Q_1, Q_2, \ldots, Q_k$.

Case (i). All faults are in dimension 1.

Consider the circuit $Q_1$ of length $k$. As there are three faults in dimension 1, w.l.o.g. there exists an edge $(x_1, y_1)$ of $Q_1$ such that the links $(x_1, x_2)$ and $(y_1, y_2)$ are both healthy. (Apply our usual counting argument.) Join $Q_1$ and $Q_2$ using these links to obtain a circuit $D_2$ containing every node of $Q_1$ and $Q_2$. By proceeding as we have done throughout, the same argument can be used to extend $D_2$ to (w.l.o.g.) a circuit $D_3$ and so on until we obtain a Hamiltonian circuit of $Q^k_2$.

Case (ii). Dimension 1 has exactly two faults.

W.l.o.g. the only fault not in dimension 1 may be assumed to be $(x_1, y_1)$ in $Q_1$. If the links $(x_1, x_2)$ and $(y_1, y_2)$ are both healthy or the links $(x_1, x_k)$ and $(y_1, y_k)$ are both healthy, then we can join $Q_1$ with $Q_2$ or $Q_k$, respectively, as in Case (i), and extend this circuit to a Hamiltonian circuit of $Q^k_2$.

Hence, w.l.o.g. we may assume that the links $(x_1, x_2)$ and $(y_1, y_k)$ are both faulty. If $k$ is even, then there exists a Hamiltonian circuit in $Q^k_2$ as pictured in Figure 2.2. (In that picture, $x_{1,k}, x_{1,2}, x_{2,3},$ and $x_{k,2}$ play the roles of $x_1, y_1, x_2,$ and $y_k,$ respectively.) If $k$ is odd, then there exists a Hamiltonian circuit in $Q^k_2$ as pictured in Figure 2.3. (In that picture, $x_{1,m}, x_{1,1}, x_{2,m},$ and $x_{k,1}$ play the roles of $x_1, y_1, x_2,$ and $y_k,$ respectively.)

**Lemma 2.3.** If $Q^k_3$ has three faulty links and is such that every node is incident with at least two healthy links, then $Q^k_3$ has a Hamiltonian circuit unless these three faulty links form a circuit of length 3.
Proof. There exists some dimension, say dimension 1, that contains at least two faults. Partition $Q_3^1$ over dimension 1 to obtain three copies of $Q_3^1$, namely $Q_1$, $Q_2$, and $Q_3$. We may assume that either $Q_1$ contains one fault or all faults are in dimension 1. Denote the nodes of $Q_1$ by $x_i$, $y_i$, and $z_i$ for $i = 1, 2, 3$.

Case (i). $Q_1$ contains one fault.

W.l.o.g. we may assume that the fault in $Q_1$ is $(x_1, y_1)$.

Case (i)(a). The links $(x_1, x_2)$ and $(y_1, y_2)$ are healthy.

Form the circuit $C = (x_1, z_1, y_1, y_2, z_2, x_2, x_1)$ in $Q_3^2$. There are two possibilities: either one of the sets of pairs

$$\{(x_1, x_3), (z_1, z_3)\}, \{(y_1, y_3), (z_1, z_3)\}, \{(x_2, x_3), (z_2, z_3)\}, \{(y_2, y_3), (z_2, z_3)\}$$

consists of two healthy links or the faulty links in dimension 1 are $(z_1, z_3)$ and $(z_2, z_3)$. In the former case, the circuit $C$ can be joined to the circuit $(x_3, y_3, z_3, x_3)$ using the pair of healthy links to obtain a Hamiltonian circuit in $Q_3^2$. In the latter case, we can define our Hamiltonian circuit in $Q_3^2$ to be $(x_1, z_1, z_2, y_2, y_1, y_3, z_3, x_3, x_2, x_1)$.

Case (i)(b). At least one of the links $(x_1, x_2)$ and $(y_1, y_2)$ is faulty.

By symmetry, we may also assume that at least one of $(x_1, x_3)$ and $(y_1, y_3)$ is faulty (as otherwise we are in Case (i)(a)); so this accounts for all faults in $Q_3^2$. The only configuration possible, up to isomorphism, is that in Figure 2.4(a), and so there is a Hamiltonian circuit as depicted in that figure. (In Figure 2.4(a), the nodes $x_1$, $y_1$, and $z_1$ of $Q_1$ form the central column, with the other two columns similarly depicting the nodes of $Q_2$ and $Q_3$. Faults are denoted by missing links, and links of the Hamiltonian circuit are drawn in bold.)

Case (ii). All faults are in dimension 1.

Up to isomorphism, there are six different configurations possible, shown in Figure 2.4(b)–(g), with Hamiltonian circuits as depicted except for Figure 2.4(g) where no such Hamiltonian circuit exists. (In Figure 2.4(g), w.l.o.g. the bold links are necessarily in any Hamiltonian circuit, if there were to exist one; and one can immediately see that there is no extension of these bold links to a Hamiltonian circuit.)

**Lemma 2.4.** If $Q_3^1$ has seven faulty links and is such that every node is incident with at least two healthy links, then $Q_3^2$ has a Hamiltonian circuit.

Proof. Case (i). $Q_3^1$ contains faults forming a circuit $C$ of length 3.

All of the faults in $C$ must appear in the same dimension, say dimension 1. Partition $Q_3^1$ across dimension 1 to obtain three copies of $Q_3^1$, namely $Q_1$, $Q_2$, and $Q_3$, and let the faulty links in $C$ be $(x_1, x_2)$, $(x_2, x_3)$, and $(x_3, x_1)$. We may assume that $Q_1$ contains the most faults amongst these copies, then $Q_2$, and then $Q_3$.

Case (i)(a). $Q_1$ contains faults forming a circuit $D$ of length 3.

Let $y_1$ and $z_1$ be nodes of $D$ different from $x_1$ ($x_1$ may or may not be on $D$) so that the number of faults incident with $y_1$ is no greater than the number of faults incident with any node of $D$ different from $x_1$. (Note that $x_1$ is incident with at most two faults in $Q_1$.) If $y_1$ is incident with one healthy link in $Q_1$, then every other node of $Q_1$ is incident with at least two healthy links in $Q_1$. (As $Q_3^1$ has seven faults, $y_1$ must be incident with at least one healthy link in $Q_1$.) In this case, temporarily mark the link $(y_1, z_1)$ as healthy so that there are at most three faults in the amended $Q_1$. (And these faults do not form a circuit.) Lemma 2.3 yields that there is a Hamiltonian path in the original $Q_1$ from $y_1$ to $z_1$.

If $y_1$ is incident with two healthy links in $Q_1$, then every node in $Q_1$ is incident with at least two healthy links in $Q_1$. Mark the link $(y_1, z_1)$ as temporarily healthy and a healthy link of $Q_1$ incident with $y_1$ as temporarily faulty. Every node in the
amended $Q_1$ is incident with at least two healthy links, and there are at most three faults. (And these faults do not form a circuit.) Lemma 2.3 yields that there is a Hamiltonian path in the original $Q_1$ from $y_1$ to $z_1$.

Whichever of the above scenarios applies, denote the Hamiltonian path in $Q_1$ from $y_1$ to $z_1$ by $P_1$. The faults in $Q_1$ and the faults $(x_1, x_2)$, $(x_2, x_3)$, and $(x_3, x_1)$ account for at least six of the seven faults in $Q_3$. Hence, w.l.o.g. we may assume that the links $(x_1, x_2)$ and $(y_1, y_2)$ are healthy. There is at most one fault in $Q_2$. By marking healthy links of $Q_2$ as temporarily faulty (if necessary), ensure that $(y_2, z_2)$ is healthy and $y_2$ is incident with exactly two healthy links. Applying Lemma 2.3 to this amended $Q_2$ yields that there is a Hamiltonian circuit $C_2$ (that is also a Hamiltonian circuit in the original $Q_2$) including the link $(y_2, z_2)$. Join $P_1$ and $C_2$ using the healthy links $(y_1, y_2)$ and $(z_1, z_2)$ to obtain a circuit $D_2$ containing every node of $Q_1$ and $Q_2$.

$Q_3$ has an isomorphic copy $C_3$ of $C_2$, and there are no faults in $Q_3$. As $C_3$ has length 9 and there are at most four faults in dimension 1, by applying our counting argument as we have done throughout, we can join $D_2$ and $C_3$ using appropriate dimension 1 links to obtain a Hamiltonian circuit in $Q_3$.

Case (i)(b). $Q_1$ does not contain faults forming a circuit $D$ of length 3.

Note that the proofs of Cases (i), (ii), (iii), and (iv) of the main theorem hold for
such that, throughout, instead of appealing to an inductive hypothesis, we use Lemma 2.3: in Case (i), we assume that dimension 1 contains at most five faults; and in Case (iii)(a), when amending $Q_2$ we must ensure that we do not introduce a circuit of faults of length 3. (This can be done as $Q_2$ has at most 1 fault.) Consequently, we are left with one scenario to consider: the subcase of Case (i) when each $Q_i$ is such that every node is incident with at least two healthy links and when dimension 1 contains six or seven faults.

Let (a new) 3-ary 2-cube $Q_3^3$ be such that there is a fault $(x, y)$ in $Q_3^3$ if and only if there is a fault $(x_i, y_i)$ in $Q_i$ for some $i \in \{1, 2, 3\}$. Then $Q_3^3$ has at most two faults and, by Lemma 2.3, it has a Hamiltonian circuit $C$. For each $i \in \{1, 2, 3\}$, let $C_i$ be the isomorphic copy of $C$ in $Q_i$. (Note that each $C_i$ consists entirely of healthy links.) Even if dimension 1 (of our original $Q_3^3$) contains seven faults, our usual counting argument yields that there exists a pair of healthy links $\{(u_1, u_2), (v_1, v_2)\}$ or $\{(u_1, u_3), (v_1, v_3)\}$, where $(u_1, v_1)$ is a link of $C_1$: w.l.o.g. we may assume that these healthy links are $(u_1, u_2)$ and $(v_1, v_2)$. We can join $C_1$ and $C_2$ using these healthy links and then proceed similarly to join the resulting circuit to $C_3$ and obtain a Hamiltonian circuit of $Q_3^3$.

Case (ii). $Q_3^3$ does not contain faults forming a circuit of length 3.

There exists a dimension, say dimension 1, containing at least three faults. Partition $Q_3^3$ across dimension 1 to obtain three copies of $Q_2^3$, namely $Q_1$, $Q_2$, and $Q_3$. Let $Q_1$ contain the most faults amongst these copies, then $Q_2$, and then $Q_3$. Proceeding as in Case (i)(b) yields the result.

The main theorem now follows by induction.

The result in Theorem 2.1 is optimal in the following sense. Let $a$, $b$, $c$, and $d$ be four nodes in $Q_n^k$, where $k \geq 4$ and $n \geq 2$, or $k = 3$ and $n \geq 3$, such that there are links $(a, b)$, $(b, c)$, $(c, d)$, and $(d, a)$. Let the faults of $Q_n^k$ consist of those links incident with $a$ that are different from $(a, b)$ and $(a, d)$, and those links incident with $c$ that are different from $(b, c)$ and $(c, d)$. In particular, $Q_n^k$ has $4n - 4$ faults and every node is incident with at least two healthy links; but this faulty $Q_n^k$ does not contain a Hamiltonian circuit, as any Hamiltonian circuit necessarily contains the links $(a, b)$ and $(a, d)$, and also the links $(c, b)$ and $(c, d)$, which yields a contradiction.

3. Conclusions. We have proven that every $k$-ary $n$-cube $Q_n^k$ which has at most $4n - 5$ faulty links and is such that every node is incident with at least two healthy links has a Hamiltonian circuit. As mentioned earlier, an analogous result for hypercubes was proven by Chan and Lee [8]. In [8], it was also shown that the problem of deciding whether a faulty binary $n$-cube has a Hamiltonian circuit is NP-complete. Their complexity-theoretic reduction (from the 3-satisfiability problem) can easily be adapted to show that the problem of deciding whether a faulty $k$-ary $n$-cube has a Hamiltonian circuit is also NP-complete. (We leave the proof of this as a simple exercise.)

As open problems relating to the research in this paper, we propose the following. The construction of our Hamiltonian circuits in our faulty $k$-ary $n$-cubes does not yield efficient parallel distributed algorithms for actually building the Hamiltonian circuits. For example, suppose one had a parallel computer whose underlying interconnection network was a $k$-ary $n$-cube and each node, i.e., processor, had local (or even global) knowledge of the faulty links. How could we develop an efficient message-passing algorithm so that, upon termination, every node knew its successor and predecessor on a Hamiltonian circuit (without necessarily knowing the Hamiltonian circuit in its entirety)? Such an algorithm would be extremely useful. Also, whilst we provide a
precise result as to the threshold value on the number of faulty links occurring in a $k$-ary $n$-cube so that there still exists a Hamiltonian circuit (under the assumption that every node is incident with at least two healthy links) and we also remark that the general decision problem is NP-complete, it would be useful if “safe patterns” of faults could be established so that even though there were more than $4n - 5$ faulty links present, one could still be sure of the existence of a Hamiltonian circuit because these faults were arranged in some specific formation. Finally, we have addressed only the problem of finding longest circuits in $k$-ary $n$-cubes in the presence of faulty links. It would be interesting to do likewise in the presence of faulty nodes, or even faulty nodes and links.

REFERENCES