The Optimum Growth Rate for Population under Critical-Level Utilitarism

by Thomas I. Renström* and Luca Spataro°

Abstract

We characterize optimal consumption, capital and population growth rates of a production economy entailed with critical-level utilitarian preferences and endogenous population size. First, we show that, under standard conditions concerning preferences and technology neither classical utilitarianism (CU) nor average utilitarianism (AU) can avoid a corner solution for the population growth rate, in that the former would prescribe that the population grows at the maximum speed (i.e. the so called "repugnant conclusion") while according to the latter such a growth rate should take the minimum value (AU). Critical level utilitarianism (CLU) does deliver an interior solution for the population growth rate provided that the critical level belongs to a positive, open interval. Second, we show that the transition to the steady state is nontrivial, in that, while consumption and capital move in the same direction, as in the standard Cass-Koopmans-Ramsey model, the optimal population growth rate and the time needed for reaching the steady state depend crucially on whether the steady state value of the optimal population growth rate is an interior or a corner solution. Finally, we perform comparative dynamics exercises on the steady state show that: a) A positive technological shock increases both capital and population growth rates, while reduces consumption; b) An increase of the critical level parameter increases consumption, leaves the capital intensity unchanged and decreases the population growth.

Keywords: Social evaluation, critical-level utilitarianism, economic growth, population

JEL Classification: D63, D90, J13

1. Introduction

It is well known that the evaluation of alternative public policies often implies the comparison of states of the world with different population. Such an evaluation becomes problematic when welfarist criteria are to be used, that is, criteria based on the well-being (utilities) of the individuals who are alive in different states of the world. Despite the relevance of this problem, the theoretical foundations of social evaluation with variable populations have received little attention in the literature. Typically, welfarist principles are adopted such as classical utilitarianism, where the objective is to sum the utilities over the population.[1]

However, these criteria cannot avoid the repugnant conclusion (Parfit (1976, 1984), Blackorby et al. (2002)), whereby any state in which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (Blackorby et al. (1995, 2002)). The implication of the repugnant conclusion is that population should grow at its maximum physically possible rate.

A strand of philosophical literature has argued that the repugnant conclusion is not a problem and that societies may avoid ending up in a situation with very large populations living just at existence-indifference level (see, for example Tännsjö (2002)). However, in economic models with classical utilitarianism those are precisely the equilibria likely to emerge. The reason is that with concave utilities (decreasing marginal utilities) for a given resource, *X,* an additive social welfare function will

---

* University of Durham (UK). Email: t.i.renstrom@durham.ac.uk.

° Dipartimento di Scienze Economiche, University of Pisa (Italy) and CHILD. Email: l.spataro@ec.unipi.it.

[1] With first-best intra-generational redistribution, the objective function becomes population size, N, times average utility, u: Nu.

take on a higher value for an $N+1$ population where everyone consumes $X/(N+1)$ than for an N size population where everyone consumes X/N. That is, for normalized utility ($u(0)=0$), we have $(N+1)u(X/(N+1)) > Nu(X/N)$[2]. Consequently, the welfare function increases as N goes to infinity (in a sense one wants an infinite population where each individual consumes zero). If the resources can grow, as we will show, the problem does not go away (such in models with capital accumulation).

There are ways of avoiding the repugnant conclusion. Some earlier literature assumed objective functions of a particular (non-welfarist) form.[3] However, such objective functions may not have an axiomatic foundation. We believe an axiomatic foundation is important, especially since we are dealing with questions regarding life (who will live and who will not).

Critical-level utilitarianism is a population principle that can avoid the repugnant conclusion. It is axiomatically founded, derived from a social preference ordering (see Blackorby et al. (1995)).[4] The critical level α can be defined as the utility level of an extra-individual i who, if added to an unaffected population N with utility distribution u, would make the two alternatives socially indifferent, i.e. (N,u) as good as $(N,u;i,\alpha)$.[5]

In our paper we rely on work by Blackorby et al. (1997) allowing for the possibility of discounting the utilities of future generations. They show that a population and utility alternative (N, u) is preferred to another alternative (N',u') if and only if

$$\sum_{i\in N}\beta\left[u_i(\cdot)-\alpha\right] \geq \sum_{i\in N'}\beta\left[u_i(\cdot')-\alpha\right].$$

Classical utilitarianism is a special case where α is set to zero. However, as we will show, with α=0, one cannot avoid the repugnant conclusion. Average utilitarianism cannot be obtained as a special case, but one has to ignore the summation over population (and only compare average utilities). The latter is close to the Samuelson (1975) formulation of optimal population growth, more recently analyzed by Jaeger and Kuhle (2009).

Several authors have criticised CLU: for example, according to Parfit (1984) CLU cannot avoid the repugnant conclusion, in that, as long as average utility is higher than α, it is always socially preferable to get larger populations with lower utility levels closer to α; and this would be "repugnant" if the critical level is too low (on the same argument see also Shiell 2008). Moreover, Broome (1992) argued that, if the critical level is set too high, then this would prevent the addition of a person whose life is worth living (i.e. with a positive utility level). In this case the same problems as those arising in presence of average utility would apply. Moreover, Ng (1986) pointed out that CLU involves counterintuitive social orderings in case the average utility is lower than α (i.e. the so called "sadistic conclusion"; see also Arrhenius 2000).

Although we believe that the philosophical discussion on the relevance of the repugnant conclusion and on CLU is far from being closed, we still adopt the critical level utilitarian criterion for two reasons: first, as stated above, it represents a logically coherent and axiomatically founded device for dealing with social evaluation of population alternatives, and, second, by departing from the

---

[2] To see this, for a, b, and $\gamma \in (0,1)$, concavity of u implies $u(\gamma a + (1-\gamma)b) > \gamma u(a) + (1-\gamma)u(b)$ by Jensen's inequality. Setting $a = X/N$ and $b = 0$ (using $u(0)=0$), and $\gamma = N/(1+N)$ gives the result.

[3] E.g. Barro and Becker (1988) and Becker and Barro (1989).

[4] Among other non utilitarian principles, seem, for example, Golosov, Jones and Tertilt (2007).

[5] Therefore, nothing precludes comparisons of alternatives where individuals have utilities below the critical level. This implies that adding a person with negative utility to a population may be preferred to an alternative where all individuals have positive utilities. This is the so called "sadistic" conclusion (see Arrhenius (2000) and Blackorby et al. (2005)).

existing literature, we introduce production and physical capital and analyze the outcomes under critical-level utilitarianism. To the best of our knowledge, this has not been done before.

Moreover, we will show that in presence of CU the system would not have an interior solution for *n*, in that in the long run it would be optimal to boost population as fast as possible. Hence, the repugnant conclusion would occur. The CU solution is avoided, i.e. an interior solution for the population growth rate emerges, only if the critical level is strictly positive and higher than a threshold level (thus, to some extent answering to the point raised by Parfit 1984.

On the other hand, we show that according to the AU view, the population should decrease as fast as physically possible. The AU solution can be avoided only if the critical level is not too high (thus to some extent addressing the point raised by Broome 1992). Finally, as for the point raised by Ng (1986), the author proposed a possible solution called the "number-dumpened critical level" (see also Hurka 1983 and 2000 on this point). Although interesting, the latter criteria violate one of the axioms on which CLU is based (namely, the "independence of the long dead"), which seems particularly undue in our context, in which we deal with the intertemporal distribution of resources and population growth.[6] For all these reasons we still decide to adopt CLU and to leave the analysis of the implications of the above mentioned alternative social ordering criteria for future research.

Finally, in a recent work Shiell (2008) argues that neither CU nor CLU can avoid (a revised version of the repugnant conclusion) under an unrestricted domain, that is, if population size and per person utility can be chosen independently. Moreover, Shiell provides examples of situations (i.e. sufficient conditions) in which the repugnant conclusion is avoided both in CU and CLU. Our work, on the one hand, is in line with Shiell (2008), because we also cannot choose per capita utility and population size independently due to the capital accumulation constraint. Moreover, in line with Shiell, we show that the modified-RC can be avoided under CLU provided that a restricted domain is imposed (i.e. critical level belonging to a positive, open interval). However, we depart from Shiell (2008) in several respects: first, among other things, we assume capital accumulation, which prevents one of the crucial assumptions by Shiell to hold (i.e. Shiell's law of conservation of matter). Moreover, in line with the tradition of the CLU literature, we assume zero neutral consumption.

One can take either a normative or a positive view on our paper. Under the normative view, we see the objective function as a social ordering and the solution states how population, consumption, and capital *should* evolve over time. Under the positive view, we take a dynastic decision maker, with critical-level utilitarianism as altruistic preferences and provide an alternative view for interpreting the differences in growth paths undertaken by developed and developing countries.

Precisely, in the present work we characterize the steady state solution and show that neither CU nor AU can avoid a corner solution for the steady state. Second, we show that CLU preferences can avoid such a dichotomous result provided that the critical level belongs to a positive, open interval.

Moreover, we show that, along the transition path towards the steady state, capital accumulation and per capita consumption move in the same directions and that the features of the dynamic path undertaken by the economy are strongly dependent on whether the steady state value of the population growth rate is a corner or an interior solution.

Finally, by carrying out comparative dynamics exercises, we show that: a) a positive technological shock increases both optimal capital and optimal population growth rates, while reduces consumption, at the steady state; b) an increase of the critical level parameter increases optimal consumption, leaves the optimal capital intensity unchanged and decreases the optimal population growth at the steady state.

The paper is organized as follows: after presenting the model, in section 3 we characterise the steady state and, in section 4 we analyze the dynamics of the model and in section 5 we perform a comparative dynamics analysis.

---

[6] See Blackorby et al. (2005) for an extensive assessment of this and other criteria.

2. The economy

We make the simplifying assumption that each generation is alive for a period, and life-time utility is $u(c_t)$, where c_t is life-time consumption for that individual. This means that generations will not overlap. This assumption can be relaxed without changing the fundamental properties of the model. We also follow the convention that u=0 represents neutrality at individual level (i.e. if u<0 the individual prefers not to have been born), and denote the critical level as α. Furthermore, we will conduct the analysis in continuous time (it is more tractable given the nature of the problem). Then the birth-date dependent critical level utilitarian objective is

$$\max \int_{t=0}^{\infty} N_t e^{-\rho t}\left(u(c_t)-\alpha\right)dt \tag{1}$$

where $u(c_t)$ is the instantaneous utility function, increasing and concave in c_t. α is the critical level of utility, and $\rho > 0$ is the intergenerational discount rate. Since we fix neutrality consumption to zero (i.e. u(0)=0), this implies that c^{α}, satisfying $u(c^{\alpha})$=α, is strictly positive.

The population size, N_t, grows at rate n_t, i.e.

$$\frac{\dot{N}_t}{N_t} = n_t . \tag{2}$$

We assume that there are lower and upper bounds on the population growth rate: $n_t \in [\underline{n},\bar{n}]$. Realistically, there is a physical constraint at each period of time on how many children a parent can have. There is also a constraint on how low the population growth can be. First, we do not allow individuals to be eliminated from the population (there is no axiomatic foundation for that). Second, even if nobody wants to reproduce (or is prevented from doing so by the planner) there will always be accidental births.

Assuming a CRS production technology, $F(K_t,N_t)$, and capital depreciation rate δ, the capital accumulation equation is:

$$\dot{K}_t = F(K_t,N_t) - c_t N_t - \delta K_t \tag{3}$$

Clearly, from eq. (1) the problem has a solution only if $\rho > \bar{n}$, which we assume throughout our analysis.

3. The optimal solution

The problem is to maximize (1) subject to (2)-(3), and $n_t \in [\underline{n},\bar{n}]$, taking K_0 and N_0 as given. The current value Hamiltonian is:

$$H = N_t\left[u(c_t) - \alpha\right] + \lambda_t\left[F(K_t,N_t) - c_t N_t - \delta K_t\right] + \mu_t n_t N_t + \gamma\left(\bar{n} - n_t\right) + \vartheta\left(n_t - \underline{n}\right) \tag{4}$$

The first order conditions are the following:

$$\frac{\partial H}{\partial C_t} = N_t u'_t - N_t q_t = 0 \Rightarrow u'_t = q_t \tag{5}$$

$$\frac{\partial H}{\partial K_t} = \rho q_t - \dot{q}_t \Rightarrow \dot{q}_t = q_t \left(\delta + \rho - F_{K_t}\right) \tag{6}$$

$$\frac{\partial H}{\partial N_t} = \rho\lambda_t - \dot{\lambda}_t \Rightarrow \dot{\lambda}_t = (\rho - n_t)\lambda_t - u_t - \alpha - q_t\left(F_{N_t} - c_t\right) \tag{7}$$

$$\frac{\partial H}{\partial n_t} = \lambda_t N_t - \nu_t + \vartheta_t = 0 \tag{8}$$

and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t} q_t K_t = 0, \quad \lim_{t\to\infty} e^{-\rho t} \lambda_t N_t = 0. \tag{9}$$

Let us define the capital intensity $k \equiv \frac{K}{N}$, such that, by exploiting constant returns to scale in the production function we can write: $F(K,N) = Nf(k)$, $F_N(K,N) = f(k) - f'(k)k$. The capital accumulation constraint is then

$$\dot{k}_t = f(k_t) - c_t - (\delta + n_t)k_t. \tag{10}$$

Combining (5) and (6) gives the consumption Euler equation

$$\dot{c}_t = \frac{u'(c_t)}{-u''(c_t)}\left(f'(k_t) - \rho - \delta\right). \tag{11}$$

Finally, combining (7) and (5) we get:

$$\dot{\lambda}_t = (\rho - n_t)\lambda_t - u_t - \alpha - u'_t\left(F_{N_t} - c_t\right). \tag{12}$$

which, together with transversality conditions in (9), complete the set of the dynamic equations of our model. An optimal path $\{c_t, n_t, k_t\}$ has to satisfy equations (8)-(12). Before characterizing the steady state solution of the model, it is worth noting that, in principle, along the transition path n_t can be either interior or a corner solution. Hence, we turn to discuss both cases separately.

3.1 Ideal population sizes

It is clear from equation (8) that there may be corners regarding the population growth rate. E.g. if λ[7] is positive then ν is positive and the constraint $n \leq \bar{n}$ is binding, i.e. the population should reproduce

---

[7] For notational purposes in the remainder of this work we omit time subscripts whenever no ambiguity results.

itself as much as possible[8]. If λ is negative, N is too large and population growth should be at its physical minimum (i.e. only accidental births should happen). When λ is zero, the marginal value of population size is zero and consequently it constitutes an ideal size (at that instant of time any population growth rate will do, i.e. society/planner is indifferent with respect to the population growth rate). We will characterise ideal population sizes in a way suitable for the dynamic analysis.

Ideal population sizes are characterised by $\nu = \vartheta = 0$, i.e. both the multipliers associated with the constraint for n_t are zero. As mentioned above, in turn these conditions imply, from eq. (8), that λ = 0 if we were to remain at ideal size forever. Then from eq. (7) we obtain $u - \alpha = q[c - F_N]$. Hence, by using eq. (5) we get:

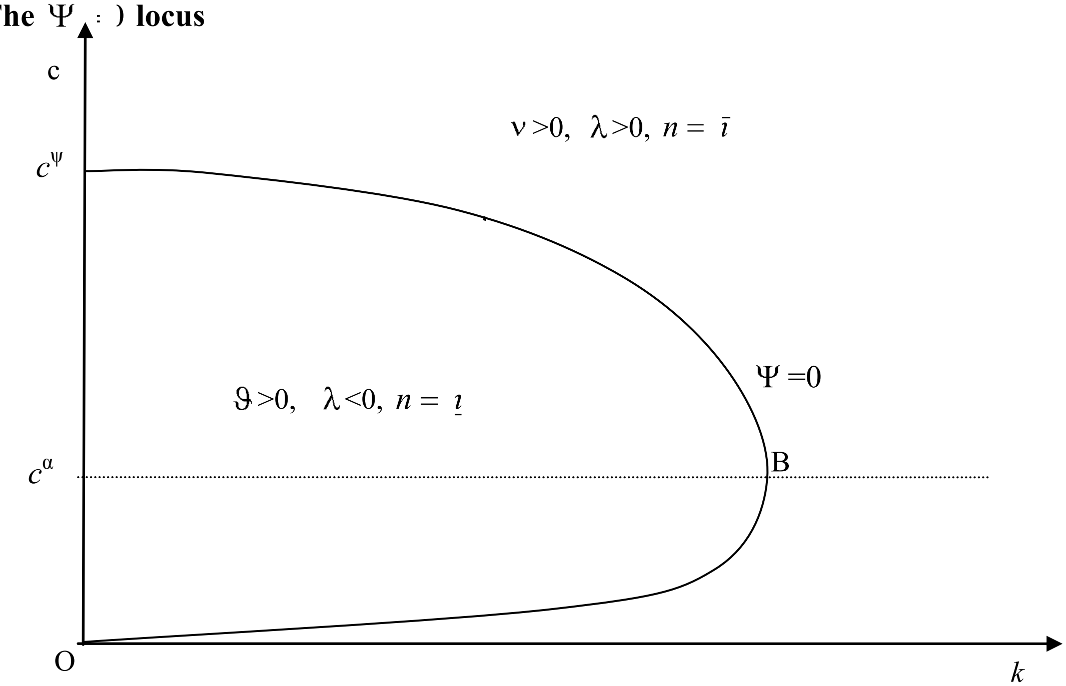
$$\Psi(c,k) \equiv \frac{u-\alpha}{u'} - c + F_N = 0 \qquad (13)$$

where $\Psi(c,k)$ relates c to k (recall that $F_N = f(k) - f'(k)k$). Equation (13) states that the addition to social welfare of increasing the population at the margin, $u - \alpha$, should equal the marginal value (in utility units) of what a newborn takes out of society, $u'(c)[c - F_N]$. What an individual takes out of society is the difference between what she consumes, c, and what she brings, F_N (the marginal value of labour). If the social value of one more person is the same as the cost, society (or planner) is indifferent in altering the population size. Notice also that (13) holds when $u<\alpha$. In that case society may be indifferent in bringing an individual with lower utility than the critical level if such an individual brings more to society than taking out, i.e. if $F_N > c$. Even lives not worth living ($u<0$) can be brought into existence just because of the resource gain.

As for the shape of the $\Psi(c,k)$ locus, by differentiating (12) one obtains

$$\left.\frac{dc}{dk}\right|_{\Psi=0} = -\frac{f''(k)k}{\left(\frac{-u''(u-\alpha)}{u'^2}\right)} <> 0 \text{ if } (u-\alpha) >< 0. \qquad (14)$$

Figure 1: The $\Psi(c,k)$ locus



[8] The co-state λ is the shadow value of N, consequently when λ is positive, N is too low and should be increased.

Note that, when u tends to α, eq. (14) tends to $-\infty$. Let us call c^α the level of c satisfying $u(c^\alpha) = \alpha$. In Figure 1 we depict the locus $\Psi(\cdot)=0$ for the case when $\lim_{c\to 0} u'(c) = \infty$. If $u'(0)$ is finite, the lower part of the locus will cut the horizontal axis at some k>0. This would not change our analysis. Moreover, it is easy to verify that $\frac{\partial c^\alpha}{\partial \alpha} = \frac{1}{u'} > 0$ and $\frac{\partial c^\Psi}{\partial \alpha} = -\frac{u'^2}{(u-\alpha)u''} > 0$, where c^Ψ is the intercept of the $\Psi(\cdot)=0$ on the vertical axis. Thereby, as α is reduced, the intercept c^Ψ and the critical-level consumption c^α are reduced as well. Furthermore, let us define the point (c^α, k^α) satisfying $\Psi(\cdot)=0$ (point B in Fig. 1). From eq. (13) this point is such that $c^\alpha = F_N(k^\alpha)$ and thus $\frac{\partial k}{\partial c^\alpha} = -\frac{1}{kf''} > 0$. Hence, as the critical level decreases the $\Psi(\cdot)=0$ shrinks because c^Ψ decreases and point B moves south-west (because c^α is reduced). Finally, when α goes to zero the $\Psi(\cdot)=0$ locus shrinks to the origin. To see this, substituting for $k=0, c=0$ into (13), gives $\alpha=0$. Moreover, due to concavity of u, when $\alpha=0$ we get that $\frac{u}{u'} - c \geq 0$ and, thus, $\Psi \geq 0$ for any $k>0$.

Since Ψ=0 is a relationship between per-capita consumption and capital, the equation gives all combinations of c and k such that society (or planner) is indifferent in increasing or reducing the population. This combination, however, can never coincide with an optimal consumption-capital trajectory. This implies that trajectories will go outside those combinations and possibly coincide at certain points. This means that, typically, on a trajectory towards the steady state, either population is too small, and population will grow at $\bar{n}$, or too large, and only accidental births take place.

We can now provide the following proposition showing some characteristics of optimal trajectories:

Proposition 1. For optimal trajectories remaining outside the $\Psi(\cdot)=0$ locus, the population growth rate is at its physical maximum, $n_t = \bar{n}$, while for those remaining inside the $\Psi(\cdot)=0$ locus population grows at its physical minimum (i.e. only accidental births take place), $n_t = \underline{n}$.

Proof Integrate (7) between t_0 and T to obtain

$$\lambda_T e^{-\int_{t_0}^{T}(\rho - n_s)ds} - \lambda_{t_0} = -\int_{t_0}^{T} e^{-\int_{t_0}^{s}(\rho - n_s)ds}\left[u(c_t) - \alpha - u'_t(F_{N_t} - c_t)\right]dt$$

Then, as $T \to \infty$, by exploiting the transversality condition[9] we have

$$\lambda_{t_0} = \int_{t_0}^{\infty} e^{-\int_{t_0}^{s}(\rho - n_s)ds}\left[u(c_t) - \alpha - u'_t(F_{N_t} - c_t)\right]dt \tag{15}$$

Clearly for a consumption-capital trajectory reaching the $\Psi(\cdot)=0$ locus at date t_0, and remaining there forever, $\lambda_{t_0} = 0$ and the population size is ideal. If a trajectory remains outside the $\Psi(\cdot)=0$ locus from

[9] Note that $\lim_{T\to\infty} \lambda_T e^{-\int_{t_0}^{T}(\rho - n_s)ds} = N_{t_0}^{-1} \lim_{t\to\infty} e^{-\rho t}\lambda_t N_t$, which is equal to zero if the transversality condition holds.

date t_0 and onwards, then $\lambda_{t_0} > 0 \Rightarrow \nu_{t_0} > 0 \Rightarrow n_{t_0} = \bar{n}$. The argument is reversed for trajectories remaining inside the $\Psi = 0$ locus. □

3.2 The steady state solutions

In this subsection we lay down the properties of a steady state and the conditions for its existence. It is clear from Proposition 1 that we will have corners along a transition path, and the only possibility of interior solution is at a steady state. The steady state equilibrium is given the vector (c^{ss}, k^{ss}, n^{ss}) solving the following equations:

$$f'(k^{ss}) = \delta + \rho \tag{16}$$

$$f(k^{ss}) - (\delta + n^{ss})k^{ss} = c^{ss} \tag{17}$$

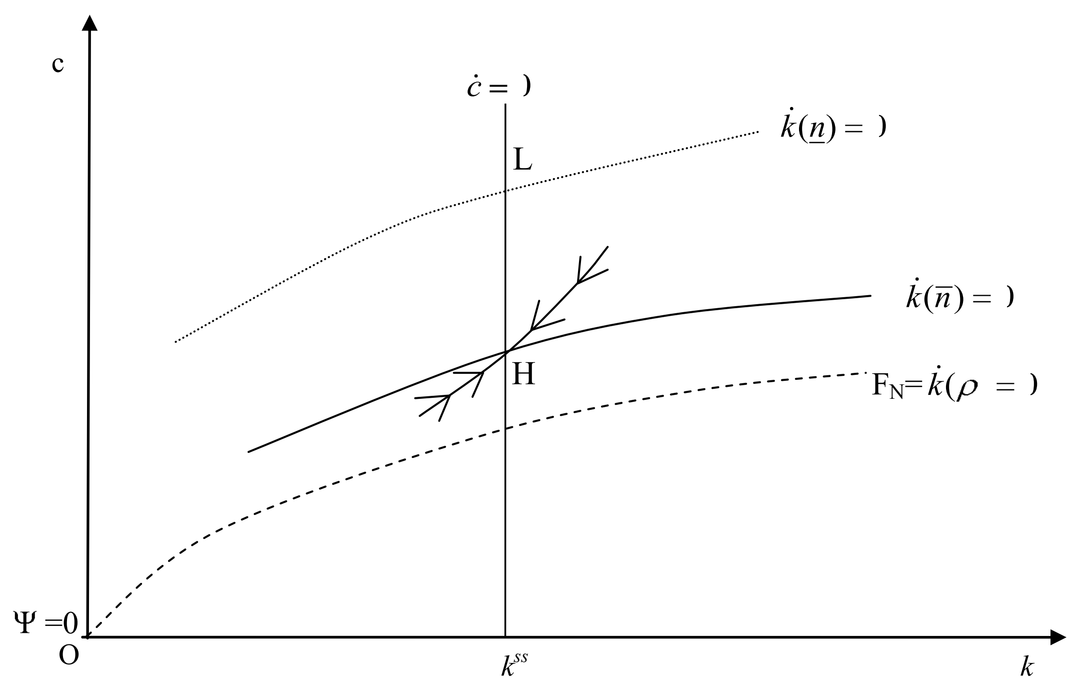
$$\frac{u(c^{ss}) - \alpha}{u'(c^{ss})} = c^{ss} - [f(k^{ss}) - f'(k^{ss})k^{ss}] \text{ if } n^{ss} \in (\underline{n}, \bar{n}) \tag{18a}$$

$$\frac{u(c^{ss}) - \alpha}{u'(c^{ss})} > (<)\, c^{ss} - [f(k^{ss}) - f'(k^{ss})k^{ss}] \text{ if } n^{ss} = \bar{n}\ (n^{ss} = \underline{n}) \tag{18b}$$

Equations (16)-(18) are obtained by setting the time derivatives to zero in (10) and (11) and (12), and by using (13).

We now discuss the role of the critical level in determining the steady state solution. To start with, we consider the extreme case of $\alpha = 0$ which corresponds to Classical Utilitarianism.

Figure 2: The steady state solution: Classical Utilitarianism case



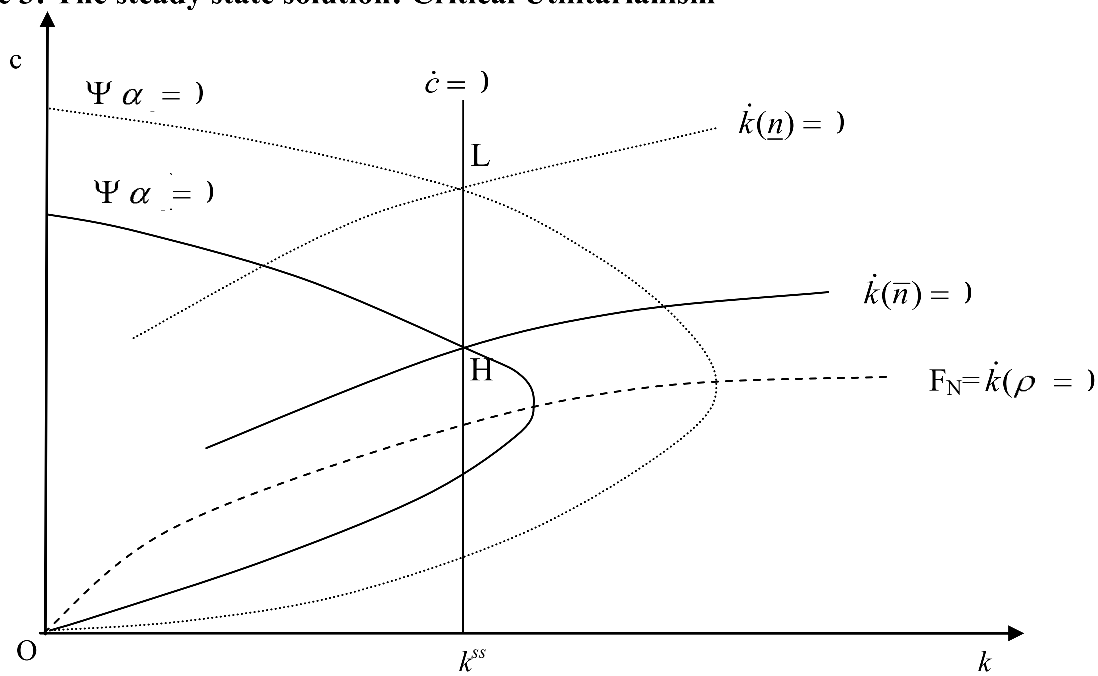
Recall that in such a case the $\Psi(\cdot)=0$ locus depicted in Figure 1, shrinks to the origin. As shown in Figure 2 there are two possible candidates for the steady state equilibrium: points L and H, corresponding to $n^{ss} = \underline{n}$ and $n^{ss} = \bar{n}$, respectively. Nevertheless, by the following Proposition we show that only point H is the optimal solution.

Proposition 2: Under Classical Utilitarianism (i.e. critical level $\alpha = 0$) population growth rate is at its maximum at all times ($n_t = \bar{n}$). Consumption and capital converge asymptotically toward a unique steady state described by (16) and (17).

Proof. By concavity of u and $u(0)=0$, we have $u \geq u'c \ \ \forall c \geq 0$. By using this in (15) and setting $\alpha = 0$, one has that $\lambda > 0 \ \ \forall k > 0$. Therefore $n = \bar{n}$ is optimal at all times. The system now behaves as a standard Cass-Koopmans-Ramsey model guided by eqs. (10) and (11). □

Let us now consider the case of positive critical levels. In Fig. 3 we depict two curves representing the $\Psi(\cdot)=0$ locus for two different values of α: $\underline{\alpha}$ and $\bar{\alpha}$, with $\underline{\alpha} < \bar{\alpha}$. The two values are chosen in such a way that the corresponding $\Psi(\cdot)=0$ loci cut the two steady points H and L.

Figure 3: The steady state solution: Critical Utilitarianism



This corresponds to impose that $\underline{\alpha}$ ($\bar{\alpha}$) is such that k and c associated with point H (L), satisfying eqs. (16) and (17), also satisfy eq. (18a). Hence, these values are[10].

$$\underline{\alpha} \equiv u(\bar{c}) - u'(\bar{c})(\rho - \bar{n})\bar{k}^{ss} \quad (19a)$$

$$\bar{\alpha} \equiv u(\underline{c}) - u'(\underline{c})(\rho - \underline{n})\underline{k}^{ss} \quad (19b)$$

where $\bar{c} \equiv F_N^{ss} + (\rho - \bar{n})\bar{k}^{ss}$ and $\underline{c} \equiv F_N^{ss} + (\rho - \underline{n})\underline{k}^{ss}$.

[10] Note that the conditions given by equations (19a) and (19b) can be interpreted as domain restrictions analogous to the analysis by Shiell (2008).

We can now state the following Proposition providing the conditions under which Critical Level Utilitarianism generates either a corner or an interior solution for the steady state population growth rate.

Proposition 3: Under Critical-Level Utilitarianism (i.e. critical level $\alpha > 0$) if $\alpha \leq \underline{\alpha}$ the steady state population growth rate is at its maximum ($n^{ss} = \bar{n}$); if $\alpha \geq \bar{\alpha}$ the steady state population growth rate is at its minimum ($n^{ss} = \underline{n}$); if $\bar{\alpha} > \alpha > \underline{\alpha}$ the steady state population growth rate is an interior solution $n^{ss} \in (\underline{n}, \bar{n})$. Steady state consumption and capital are described by eqs. (16) and (17).

Proof: Recall that by construction, if $\alpha = \underline{\alpha}$, the steady state $(\bar{c}, k^{ss}, \bar{n})$ satisfies (16) (17) and (18a) and hence, by (15) $\lambda = 0$. Now, if α from this position is lowered, Ψ in (13) turns positive and, by (15), λ turns positive as well. Consequently $\bar{n}$ is optimal and $(\bar{c}, k^{ss}, \bar{n})$ is the steady state.
On the other hand, since by construction, when $\alpha = \bar{\alpha}$, the steady state $(\underline{c}, k^{ss}, \underline{n})$ satisfies (16) (17) and (18a) and hence, by (15) $\lambda = 0$. Now, if α from this position is increased, Ψ in (13) turns negative and, by (15), λ turns negative too. As a consequence $\underline{n}$ is optimal and $(\underline{c}, k^{ss}, \underline{n})$ is the steady state. Finally, if $\bar{\alpha} > \alpha > \underline{\alpha}$ then neither $(\bar{c}, k^{ss}, \bar{n})$ nor $(\underline{c}, k^{ss}, \underline{n})$ are steady states and the solution will be (c^{ss}, k^{ss}, n^{ss}) with $c^{ss} \in (\underline{c}, \bar{c})$ and $n^{ss} \in (\underline{n}, \bar{n})$. To show this, take k^{ss} from (16) and substitute for it into (17) and (18a); then substitute (17) in (18a) to get

$$\alpha = u\left(F_N^{ss} + (\rho - n^{ss})k^{ss}\right) - u'\left(F_N^{ss} + (\rho - n^{ss})k^{ss}\right)(\rho - n^{ss})k^{ss} \qquad (20)$$

which provides all couples of α and n satisfying (16), (17) and (18a). Note that k^{ss} is independent of α. By differentiating (20) one gets:

$$\frac{dn^{ss}}{d\alpha} = \frac{1}{u''\left((\rho - n)k^{ss}\right)^2} < 0$$

that is, a negative relation between α and n^{ss}. By (19) we know that if $\alpha = \underline{\alpha}$, then $n^{ss} = \bar{n}$ and if $\alpha = \bar{\alpha}$ then $n^{ss} = \underline{n}$. Hence, for any $\alpha \in (\underline{\alpha}, \bar{\alpha})$ we have $n^{ss} \in (\underline{n}, \bar{n})$ and, therefore, by (17) we have $c^{ss} \in (\underline{c}, \bar{c})$. □

So far we have shown that when α is sufficiently low, (i.e. lower than $\alpha = \underline{\alpha}$) the steady state solution under Critical Level Utilitarianism corresponds to the one obtaining from Classical Utilitarianism. We will now show that when the critical level is set very high (i.e. higher than $\bar{\alpha}$) the steady state solution for *n* under CLU corresponds to the solution obtained under another known population criterion, namely Average Utilitarianism (AU).

In fact, in case the latter criterion is adopted, one seeks to maximize average utility, that is

$$\max \int_{t=0}^{\infty} e^{-\rho t} u(c_t)\, dt .$$

Since only per capita consumption enters the objective function, the appropriate constraint on this problem is eq. (10). We see that in eq. (10) *n* enters as a "cost" and therefore it should take on the lowest possible value, that is, $n^{ss} = \underline{n}$, thus corresponding to CLU solution when $\alpha > \underline{\alpha}$. This is point L in Figures 2 and 3.

Hence, we can conclude that neither CU nor AU can avoid corner solutions for the population growth rate; on the contrary, CLU can generate an interior solution if the critical level is in an open, positive interval, i.e. $\alpha \in (\underline{\alpha}, \bar{\alpha})$.

4. Transitional dynamics

In this section we discuss the dynamic properties of the system by distinguishing two cases, according to whether the steady state solution for *n* is interior or a corner.

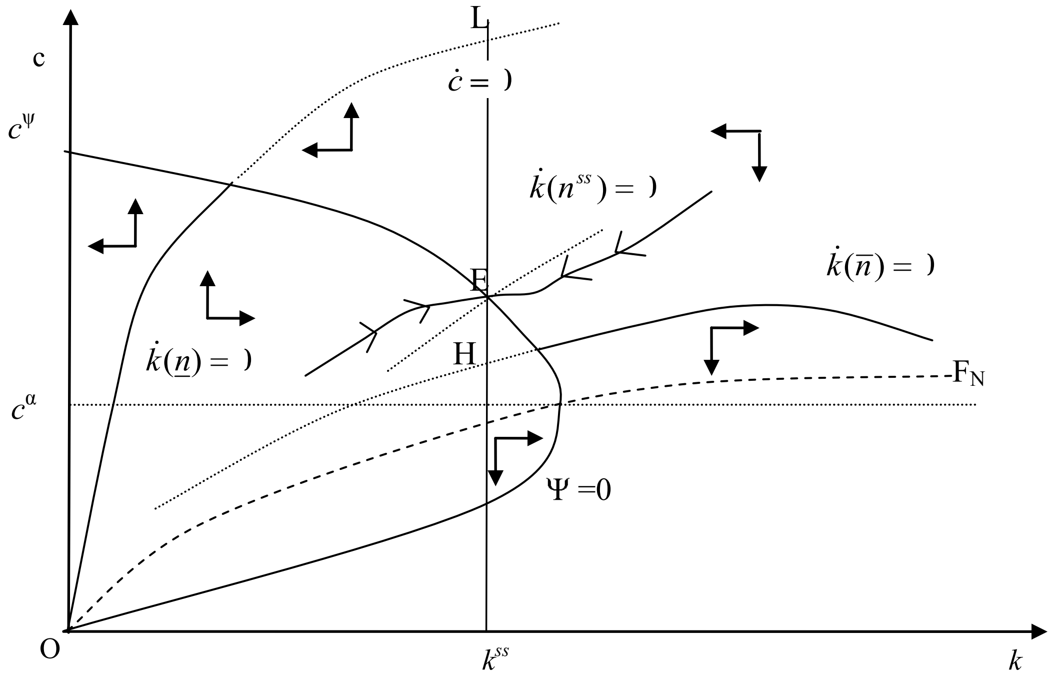
4.1 The case of interior solution for the rate of growth of population

Suppose that $\alpha \in (\underline{\alpha}, \bar{\alpha})$, i.e. an interior steady-state solution for *n* exists. To analyze the transition dynamics we need to keep track of two systems. System (I) applies to trajectories that remain inside the $\Psi = 0$ locus (i.e. when λ<0). In this dynamical system we have $\dot{k} = f(k_t) - (\delta + \underline{n})k_t - c_t$. The $\dot{k} = 0$ locus for system (I) is drawn in Figure 4, with a solid line where it applies (and dotted where it does not apply). It crosses the $\dot{c} = 0$ locus in point L. Similarly, system (II) applies to trajectories that are outside the $\Psi = 0$ locus (i.e. when λ > 0). In this system $\dot{k} = f(k_t) - (\delta + \bar{n})k_t - c_t = 0$. The $\dot{k} = 0$ locus here is depicted with a solid line where it applies and a dotted where it does not (crossing the $\dot{c} = 0$ locus in point H). Inside and outside the $\Psi = 0$ locus, λ is changing value over time, but is not affecting the dynamical system of per-capita capital and per-capita consumption (equations (10) and (11)), as *n* is constantly at a corner. Therefore, the 2x2 per-capita consumption and capital system can be analysed independently (just keeping track of the sign of λ, to determine at which corner *n* is). So a 3x3 system is not needed.

The optimal trajectory is shown in Figure 4, leading to point E. If the initial per-capita capital stock is lower than its steady state value, k^{ss}, it is optimal to move along an unstable trajectory in the system (I) (i.e. $n = \underline{n}$). This trajectory moves north east away from the steady state in the system (I). Eventually, in finite time, it reaches the intersection of the ψ=0 and $\dot{c} = 0$ loci, and the population growth rate jumps from $\underline{n}$ to n^{ss}, and the economy remains in this point forever. Consequently, the steady state is reached in finite time. Similarly, if the initial per-capita capital stock is greater than the steady state value, it is optimal to take an unstable trajectory in system (II) (where $n = \bar{n}$), moving south west, away from the would-be steady state in that system. In finite time the $\Psi = 0$ and $\dot{c} = 0$ intersection is reached, and the $\bar{n}$ falls to n^{ss}. Again the steady state is reached in finite time. In the light of the analysis above, we can summarize our findings in the following Proposition:

Proposition 4. For critical level $\alpha \in (\underline{\alpha}, \bar{\alpha})$ the unique steady state (with n^{ss} interior: $n^{ss} \in (\underline{n}, \bar{n})$) is stable and is reached in finite time. If $k_0 < k^{ss}$, then during the transition path, $n = \underline{n}$, and consumption and capital increase over time. At the time when k^{ss} is reached *n* is raised to n^{ss}. If $k_0 > k^{ss}$, then during the transition path, $n = \bar{n}$, and consumption and capital decrease over time. At the time when k^{ss} is reached *n* is lowered to n^{ss}.

Figure 4: The phase diagram of the model



Proof: As seen in Proposition 3 when $\alpha \in (\underline{\alpha}, \bar{\alpha})$ there exists only one steady state, and in this steady state $n^{ss} \in (\underline{n}, \bar{n})$. In constructing the proof we follow Koopmans (1965).
When $k_0 < k^{ss}$, an increasing consumption path is optimal by equation (11). The steady state must be approached from below. Any trajectory approaching and reaching the steady state will be inside the $\Psi = 0$ locus, i.e. where $\lambda<0$ (follows from (15), since the integral is over negative values). Consequently the constraint is binding and by (8), $n = \underline{n}$. The dynamics is then guided by the system $\dot{k} = f(k_t) - (\delta + \underline{n})k_t - c_t$ and (11). The stable trajectory in this system cannot be taken (this trajectory ends in the steady state where $\dot{k} = 0$ for $n = \underline{n}$, -point L in Fig. 4- and there $\lambda>0$, prescribing $n = \bar{n}$). Instead there is a trajectory with lower initial consumption moving away from the otherwise stable trajectory reaching the point [k^{ss}, c^{ss}] at, say, time t_1. When this point is reached, the control previously being kept at $n = \underline{n}$ switches to n^{ss}, yielding $\dot{k} = 0$, (follows from (10) and (17)). At this point $\dot{c} = 0$ by (11) and (16). Since capital and consumption remain on the $\Psi = 0$ locus forever, the integral in (15) is zero (follows by (18a)), so $\lambda(t_1) = 0$, (and n^{ss} satisfies the optimality condition). Population size is ideal. This is the only trajectory satisfying the optimality conditions (8)-(12). Any other trajectory will cross over the $\Psi = 0$ locus, and either diverge to the left (reaching k=0 in finite time, making consumption dropping to zero, and violating equation (11)), or diverging to the right (and in finite time reaching a point $k = \bar{k} : f(\bar{k}) - (\delta + \bar{n})\bar{k} = 0$, where consumption hits zero and violates equation (11)).

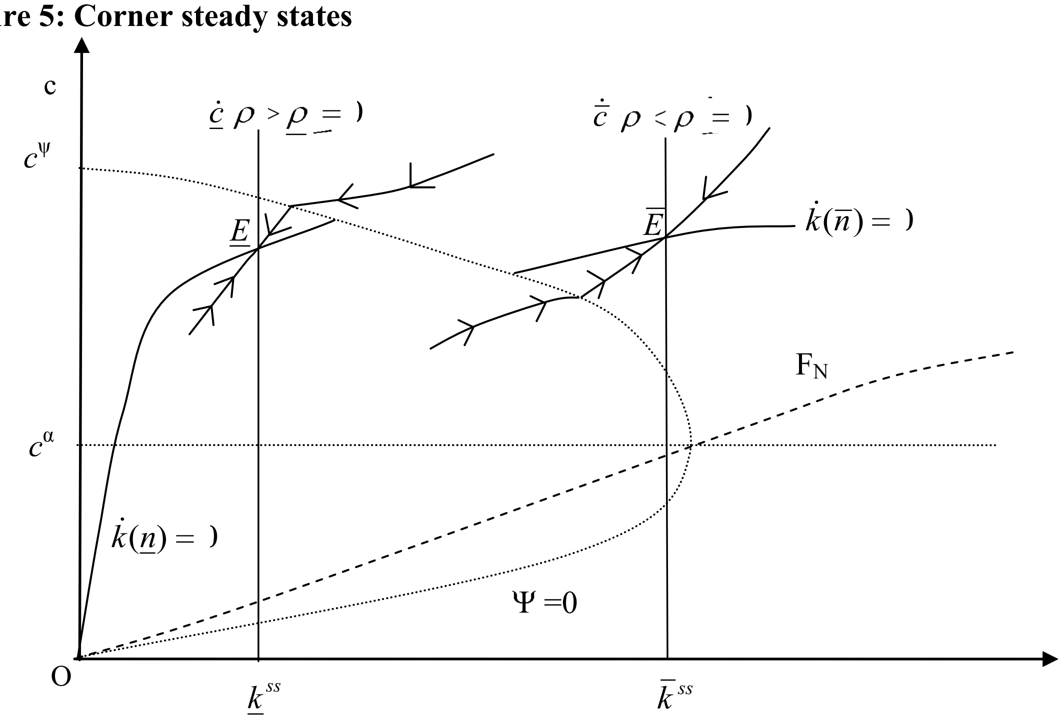
When $k_0 > k^{ss}$, the equation (11) prescribes a decreasing consumption path. Any trajectory moving down and left, toward the steady state, will be in the region where $\lambda > 0$, i.e. outside the $\Psi = 0$ locus (follows from (15)). Consequently $n = \bar{n}$. The dynamics is guided by $\dot{k} = f(k_t) - (\delta + \bar{n})k_t - c_t = 0$ and (11). Again, there is only one trajectory leading to the steady state. When reached, say, at time t_2, the control n is lowered from $n = \bar{n}$ to n^{ss}, and consequently $\dot{k} = 0 = \dot{c} = 0$. Remaining on the $\Psi = 0$ locus, yields $\lambda(t_2) = 0$ (by (15) and (18a)). □

Let us now comment on some particular features of the solution. First, from our analysis it may well be the case that it is optimal to choose the lower limit of population growth even if individuals'utility is higher than the critical level (i.e. $u > \chi$). However, such a result does not depend on the hypothesis of capital accumulation, since it would obtain also in the "static model" with given resources. The reason is that in this case the social value of one extra person is lower than his/her cost (i.e. $\frac{u-\alpha}{u'} < (1 - f - f'k)$), which may well happen also in the static case. Second, it may appear somehow counterintuitive that the solution involves a discrete jump of the rate of growth at the steady state value, in finite time. However, this feature does not represent a problem because all equations (5)-(9) are well defined in the presence of such a discrete change (in that the population rate of growth is a control variable and λ is continuous, approaching zero on the $\Psi = 0$ locus).

4.2 The case of non-interior solution for the rate of growth of population

We saw in section 3.2 that if α is too low or too high in relation to threshold values of α we have corner solutions for *n*. These threshold values are functions of ρ, $\underline{n}$ and $\bar{n}$ (equation 19), as well as other primitives. Here, we limit ourselves to variations in the discount rate, ρ. Keeping α constant we can find threshold values of ρ, for having an interior solution. We define one threshold value as ρ: $\alpha = \chi$ in equation (19a), and $\underline{\rho}$: $\alpha = \chi$ in equation (19b). Consequently, for $\rho \in (\bar{\rho}, \underline{\rho})$ we have an interior solution, and for discount rates outside this interval we have corners. Two corners are depicted in Figure 5.

Figure 5: Corner steady states



In either cases both consumption and capital intensity will behave exactly as they would do in a classical Cass-Koopmans-Ramsey growth model (given that the population would be exogenous).

We now characterize and discuss the dynamic properties of our model in the presence of corner solutions.

First, it can be shown that either equilibria are saddle path stable.

Take eqs. (10), (11) and (12) and let us linearize them around the steady state. We get that

$$\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{\lambda} \end{pmatrix} = J \begin{pmatrix} c - c^{ss} \\ k - k^{ss} \\ \lambda - \lambda^{ss} \end{pmatrix} \text{ where J=} \begin{pmatrix} 0 & -\frac{u'}{u''} f'' & 0 \\ -1 & f' - \delta - n & 0 \\ -u''(F_N - c) & u' k f'' & \rho - n \end{pmatrix}$$

The eigenvalues ω_i of the matrix H are $\omega_1 = \rho - n > 0$, $\omega_{2,3} = \frac{\rho - n}{2} \pm \sqrt{\left(\frac{\rho - n}{2}\right)^2 + \frac{u'}{u''} f''}$ the latter two with opposite signs; since two eigevalues are positive and one is negative, we can conclude that the steady state point is saddle point.

As for the dynamics, consider the case when $\rho > \underline{\rho}$. By looking at Figure 5 we can see that if $k_0 < \underline{k}^{ss}$, it is optimal to move along the stable trajectory in the system with $n = \underline{n}$, approaching $\underline{E}$ from below. If $k_0 >> \underline{k}^{ss}$ society should move along an unstable trajectory in the system with $n = \bar{n}$ and reach the stable trajectory in the system with $n = \underline{n}$ exactly on the $\Psi(\cdot)$ locus. At that point the population growth rate falls from $\bar{n}$ to $\underline{n}$, and follows the stable trajectory in the system with $n = \underline{n}$ approaching $\underline{E}$ from above. Here, in either case, it takes infinite time to reach the steady state. Similarly, consider the case $\rho < \bar{\rho}$. If $k_0 << \bar{k}^{ss}$, it is optimal to follow an unstable trajectory in the system with $n = \underline{n}$ and when reaching the $\Psi(\cdot)$ locus to increase n from $\underline{n}$ to $\bar{n}$, and continue on the stable trajectory in the system with $n = \bar{n}$, approaching $\bar{E}$ from below. Finally if $k_0 > \bar{k}^{ss}$, society should pick the stable trajectory in the system with $\bar{n}$. Again, the economy will reach the steady state in infinite time.

We can summarize our findings through the following proposition:

Proposition 5: The corner steady state is characterized by the values of c^{ss}, k^{ss}, n^{ss} that are the solutions of eqs. (16), (17) and (18b) and satisfies the condition $u > \alpha$. Along the transition path the economy will undertake a stable trajectory and i) *n* can jump from the upper (lower) value to the lower (upper) steady state value if k_0 is sufficiently low (high): ii) capital intensity and per-capita consumption move in the same direction along the transition path, and precisely they increase (decrease) if $k_t < k^{ss}$ ($k_t > k^{ss}$). Finally, the economy will reach the steady state in infinite time.

5. Comparative dynamics

Let us now investigate the effects of the change of some relevant parameters of our model on both the steady state and on the dynamics of the model. More precisely, in the next two subsections we focus on a technological shock affecting total productivity, and on a change of the critical level, respectively. For the sake of brevity we will focus on the interior solution case for *n*, corresponding to a critical level $\alpha \in (\underline{\alpha}, \bar{\alpha})$.

5.1 The effects of a technological shock

In order to analyze the effects on population growth of technological shock we introduce a total factor productivity parameter A, and replace the previous production function *f=Ag(k)*. The steady state solution is now described by the following equations:

$$Ag'(k^{ss}) = \delta + \rho \tag{16'}$$

$$Ag(k^{ss}) - \left(\delta + n^{ss}\right)k^{ss} = c^{ss} \tag{17'}$$

$$\frac{u(c^{ss})-\alpha}{u'(c^{ss})} = c^{ss} - \left[Ag(k^{ss}) - Ag'(k^{ss})k^{ss}\right] \tag{18a'}$$

Let us point out the consequences on the steady state of an unexpected permanent increase in *A* occurring in period t_1. We will also discuss the implications for the dynamics of the model of this technological shock.

All equations are affected. First, as far as steady state capital is concerned, when *A* increases the new steady state capital will be higher: differentiating (16'), since *g* is concave, we get $\frac{dk^{ss}}{dA} = -\frac{g'}{Ag''} > 0$ and the $\dot{c} = 0$ line shifts right. As for the steady state per capita consumption, by differentiating (18a') with respect to *A* and realizing that k^{ss} is a function of *A* (from (16')), we have $\frac{dc^{ss}}{dA} = \frac{g}{u''}\frac{u'^2}{u-\alpha} < 0$.

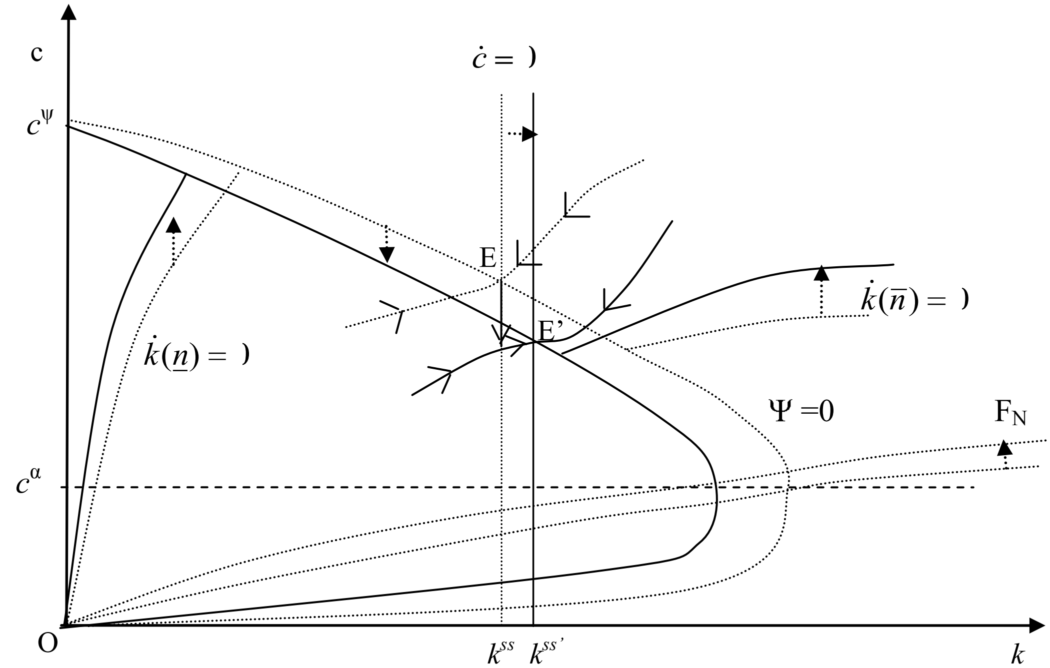
Finally, by totally differentiating (17') with respect to *A* and using the derivatives for k^{ss} and c^{ss}, we get: $\frac{dn^{ss}}{dA} = \frac{1}{k}\left[g - (\rho - n)\frac{g'}{Ag''} - \frac{g}{u''}\frac{u'^2}{u-\alpha}\right] > 0$.

As for the locus $\Psi(\cdot) = 0$, recall that it is defined by $\frac{u-\alpha}{u'} - c + Ag - kAg' = 0$, providing all combinations of *c* and *k* satisfying (18a). To see how such a locus changes with *A*, we must see how *c* associated with each *k* varies. Thus we take the derivative with respect to *A*, keeping *k* constant, which yields: $\frac{dc}{dA} = \frac{u'^2}{u''}\frac{g-kg'}{u-\alpha} \gtrless 0 \Leftrightarrow u \lessgtr \alpha$; hence, we can conclude that the locus $\Psi(\cdot) = 0$ shifts inwards when *A* increases, apart from the intercepts on the vertical axis, where *k* is equal to zero.

All this considered, as for the steady state, when *A* increases, c^{ss} decreases while k^{ss} and n^{ss} increase as well. The new steady state moves from point E to point E' in Figure 6.

Consumption must jump instantaneously on to the new stable trajectory leading to E'. Otherwise, if *c* remains where it is it would take a path north-west and reach the *k=0* axis in finite time, forcing eventually consumption to drop to zero and violating eq. (11). The drop in consumption is depicted in Figure 6. The stable trajectory leading to the new steady state is inside the $\Psi(\cdot) = 0$ locus and consequently λ is negative, by (15), and the population growth rate is at its minimum, $\underline{n}$. Along this trajectory, per capita consumption and capital are increasing over time. When the economy reaches E' at say time t_2 the population growth rate jumps to its new interior steady state level which is higher than the previous steady state level as shown above.

Figure 6: The effects of a positive technological shock



Let us provide an economic intuition of the dynamics of the model initiated by the increase in *A*. First note that at time t_1 consumption falls down (see Fig. 7a): in fact, since the new steady state capital intensity is higher, it is optimal for the society to boost capital accumulation in order to exploit the higher productivity of both capital and labour. Thus, the society finds it convenient to increase capita accumulation by reducing the amount of resources consumed in period t_1, in such a way that the economy jumps on the new saddle path at time t_1. In the following periods, both consumption and capital intensity grow steadily towards the new steady state. Note that the increase of both variables during the transition is favoured by the reduction of the population growth rate, which remains at the lower boundary level throughout (see Fig. 7b).

We can summarize our findings with the following proposition:

Proposition 6: An increase in *A* increases k^{ss}, reduces c^{ss} and increases n^{ss}. Along the transition path, the economy experiences a demographic deceleration, with *n* sticking at the lower limit $\underline{n}$, while per-capita consumption experiences an undershooting movement, in that after falling down, it progressively increases towards the new steady state value. Finally, per capita capital constantly increases along the transition.

Figure 7a: Dynamic path of consumption caused by an increase of total productivity

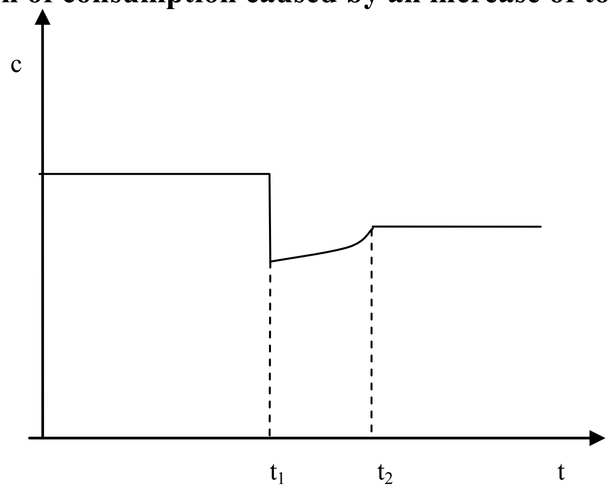
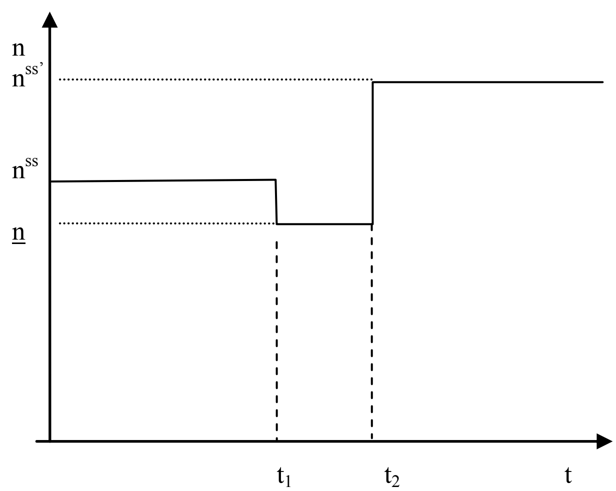


Figure 7b: Dynamic path of population growth rate caused by an increase of total productivity



5.2. Increase of the critical level utility

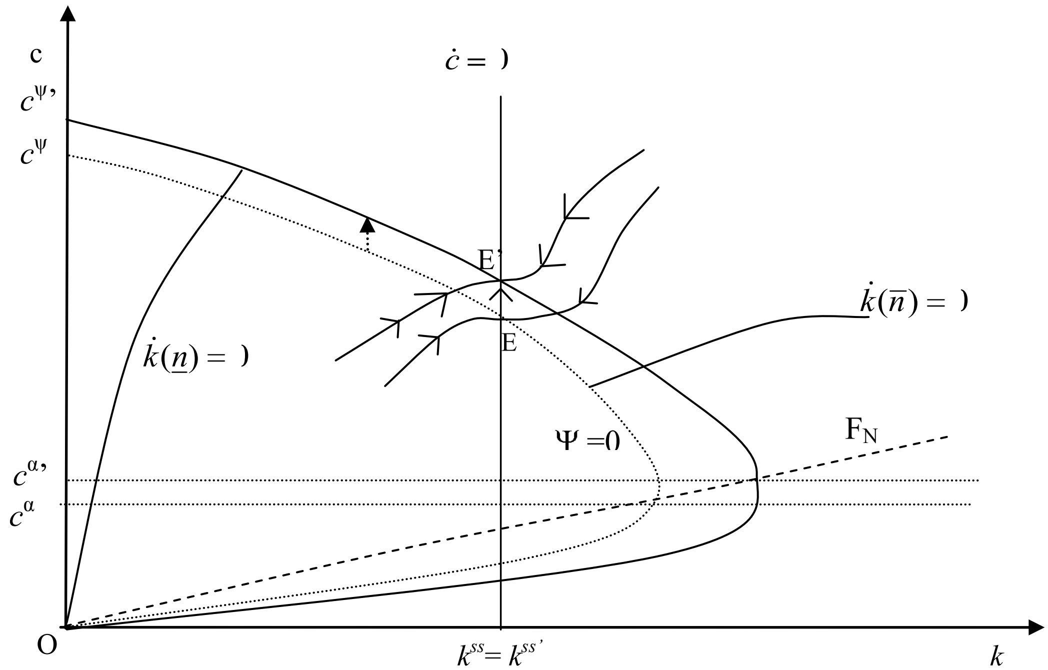
Suppose now that at a certain time t_1 the critical level α is increased. In fact, the only locus affected by the change of α is $\Psi(\cdot)$: $\frac{(u-\alpha)}{u'} = -\tilde{r}_N + \cdot$. More precisely, since for any given level of *k* it turns out that $\frac{dc}{d\alpha}\Big| = -\frac{u'}{u''(u-\alpha)} > (<) 0$ iff $u > (<) \chi$, then the locus shifts outwards (see Fig. 8).

Note that given that k^{ss} remains constant, the outwards shift of the $\Psi(\cdot)$ implies an increase of c^{ss}. Moreover, from eq. (16)-(18a) it turns out that $\frac{dn^{ss}}{d\alpha} = \frac{1}{k^{ss}}\frac{dc^{ss}}{d\alpha}$<0. As for the adjustment towards the equilibrium, it is easy to realize that there is no dynamics, in that the economy jumps instantaneously on the new steady state, characterized by higher consumption and lower population growth rate. Finally, when the critical level takes values which are outside the $(\alpha, \bar{\alpha})$, further changes of it produce no effect on the steady state values for c^{ss}, k^{ss}, n^{ss} .

We summarize our findings in the following proposition:

Proposition 7: An increase of the critical level increases c^{ss}, leaves k^{ss} unchanged, and decreases n^{ss}. The economy achieves the new steady state instantaneously.

Figure 8: The effects on an increase of the critical level



6. Conclusions

In the present work we characterize the steady state equilibrium and the dynamic properties of an intertemporal economy with production and endogenous fertility, under a critical-level utilitarian welfare criterion (CLU). We show that the steady state solution emerging under the CLU approach would coincide with the one generated by Classical Utilitarianism (i.e. population should grow at the maximum speed –the so-called "the repugnant conclusion") only if the critical level adopted is too low, while the solution would coincide with the one obtaining in the Average Utilitarianism approach (i.e. population should grow at the maximum speed) only if the adopted critical level is too high. Such a result has two straightforward implications: from a normative standpoint, the CLU approach can provide results which differ from the traditional ones if the critical level to be chosen belongs to a positive, closed set. From a positive perspective, the results stemming from the CLU approach may represent an alternative interpretation of the observed different population growth paths undertaken by developed and underdeveloped countries (that is, in terms of different critical levels). In fact, according to our results it may well be the case that underdeveloped countries find it optimal to grow as fast as possible given that they are entailed with a lower critical level, while developed countries have reduced their fertility rates due to a higher critical level. Clearly, the conjecture that the critical level can vary according to the stage of development an economy is experiencing seems plausible and is left for future research (preliminary results on such an issue confirm that this extension can strengthen the robustness and the explicative power of the CLU approach).

As for the optimal transition towards the equilibrium, we show that while consumption and capital move in the same directions, the optimal rate of population growth is a corner solution, either at the lower or at the upper limit, depending on whether the initial capital intensity is higher or lower than the steady state level, respectively. Moreover, we point out that the economy reaches the steady state in either finite or infinite time, depending on whether the steady state solution for the population growth rate is either interior or corner solution. We also perform a comparative dynamics analysis, in which we show that, at the steady state: a) a positive technological shock increases both optimal capital and optimal population growth rates, while reduces consumption; b) An increase of the critical level parameter increases optimal consumption, leaves the optimal capital intensity unchanged and decreases the optimal population growth rate. While in the former case the dynamics of the economy implies an initial reduction of per-capita consumption and then a parallel increase of both consumption and capital along the transition (with the rate of growth at the minimum level), in the latter the economy achieves the new steady state equilibrium instantaneously.

Acknowledgments: We are grateful to two anonymous referees and the Editor for suggestions that greatly improved the paper. We also thank participants at the seminars held at the University of Pisa (Italy) and Durham (UK) and at the MMF 2010 Annual Conference (Cyprus) for their helpful suggestions and comments We are also grateful to Elvio Arcinelli for his useful comments and suggestions. All remaining errors are our own responsibility.

References

Arrhenius, G. (2000). "An impossibility theorem for welfarist axiologies", *Economics and Philosophy*, 16, 247-66;

Barro, R.J., Becker, G.S., (1989): "A Reformulation of the Economic Theory of Fertility", *Quarterly Journal of Economics*, 103, 1-25;

Becker G.S., Barro R.J. (1988): "Fertility Choice in a Model of Economic Growth", *Econometrica*, 57, 481-501;

Blackorby, C., Bossert, W., and Donaldson, D. (1995). "Intertemporal population ethics: critical-level utilitarian principles", *Econometrica* 63, 1303-20;

Blackorby, C., Bossert, W., and Donaldson, D. (1997). "Birth-date dependent population ethics: critical-level principles", *Journal of Economic Theory, 77*, 260-84;

Blackorby, C., Bossert, W., and Donaldson, D. (2002). "Critical –lelvel population principles and the repugnant conclusion", *Department of Economics, University of British Columbia Discussion paper* n.: 02-16;

Blackorby, C., Bossert, W., and Donaldson, D. (2005). *Population Issues in Social Choice Theory, Welfare Economics, and Ethics*, Cambridge University Press;

Broome, J. (1992). *"Counting the Cost of Global Warming"*, London, White Horse Press;

Cass, D. (1965). "Optimum Growth in an Aggregative Model of Capital Accumulation", *Review of Economic Studies*, Vol. 32, pp.233-40.

Dasgupta, P.S. (1969). "On the Concept of Optimum Population," *The Review of Economic Studies*, Vol. 36, pp. 295-318;

Golosov, M., Jones, L.E., and Tertilt M. (2007). "Efficiency with Endogenous Population Growth", *Econometrica*, 75, 1039-71;

Hurka, T. (1983). "Value and population size", *Ethics,* 93, 496-507;

Hurka, T. (2000). "Comment on 'Population Principles with Number-Dependent Critical levels'", unpublished manuscript, Department of Philosophy, University of Calgary;

Jaeger, K., and Kuhle, W. (2009). "The Optimum Growth Rate for Population Reconsidered", *Journal of Population Economics*, 22, 23-41;

Koopmans, T.C. (1965). “On the Concept of Optimal Economic Growth”, *Pontificiae Academiae Scientiarum Scripta Varia*, Vol. 28, pp. 225-287. Reprinted as Cowles Foundation Paper 238 (1966);
Ng, Y-K. (1986). “Social Criteria for Evaluating Population Change: an Alternative to the Blackorby-Donaldson Criterion”, *Journal of Public Economics,* 29, 375-381;
Parfit, D. (1976). “On Doing the Best for Our Children”, in *Ethics and Populations*, ed. By M. Bayles. Cambridge: Schenkman;
Parfit, D. (1984). “*Reasons and Persons*”, Oxford/New York: Oxford University Press;
Ramsey, F.P. (1928). “A Mathematical Theory of Saving”, *Economic Journal*, 38, 543-559;
Samuelson, P.A. (1975). “The Optimum Growth Rate for Population”, *International Economic Review*, 16, 539-544;
Shiell, L. (2008). “The Repugnant Conclusion and Utilitarianism under Domain Restrictions”, *Journal of Public Economic Theory*, 10 (6), 1011-1031;
Tännsjö T. (2002). “Why we ought to accept the repugnant conclusion”, *Utilitas*, 14, 339-59.