Math. Proc. Camb. Phil. Soc. (2011), **151**, 283 © *Cambridge Philosophical Society* 2011
doi:10.1017/S030500411100020X
First published online 5 May 2011

The Walker conjecture for chains in $\mathbb{R}^d$

BY MICHAEL FARBER

Department of Mathematical Sciences,
*University of Durham, Durham DH*1 3*LE.*
e-mail: michaelsfarber@googlemail.com

JEAN–CLAUDE HAUSMANN

Mathematics section, University of Geneva,
2-4 *rue du Liévre, Geneva, Switzerland.*
e-mail: Jean-Claude.Hausmann@unige.ch

AND DIRK SCHÜTZ

Department of Mathematical Sciences,
*University of Durham, Durham DH*1 3*LE.*
e-mail: dirk.schuetz@durham.ac.uk

(*Received* 18 *March* 2010*; revised* 28 *January* 2011)

Abstract

A chain is a configuration in $\mathbb{R}^d$ of segments of length $\ell_1, \ldots, \ell_{n-1}$ consecutively joined to each other such that the resulting broken line connects two given points at a distance ℓ_n. For a fixed generic set of length parameters the space of all chains in $\mathbb{R}^d$ is a closed smooth manifold of dimension $(n-2)(d-1)-1$. In this paper we study cohomology algebras of spaces of chains. We give a complete classification of these spaces (up to equivariant diffeomorphism) in terms of linear inequalities of a special kind which are satisfied by the length parameters $\ell_1, \ldots, \ell_n$. This result is analogous to the conjecture of K. Walker which concerns the special case $d = 2$.

Introduction

For $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n_{>0}$ and d a positive integer, define the subspace $\mathcal{C}^n_d(\ell)$ of $(S^{d-1})^{n-1}$ by

$$\mathcal{C}^n_d(\ell) = \left\{ z = (z_1, \ldots, z_{n-1}) \in (S^{d-1})^{n-1} \;\middle|\; \sum_{i=1}^{n-1} \ell_i z_i = \ell_n\, e_1 \right\},$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the standard basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. An element of $\mathcal{C}^n_d(\ell)$, called a *chain*, can be visualized as a configuration of $n-1$ segments in $\mathbb{R}^d$, of length $\ell_1, \ldots, \ell_{n-1}$, joining the origin to $\ell_n e_1$; see Figure 1. The vector ℓ is called the *length vector*.

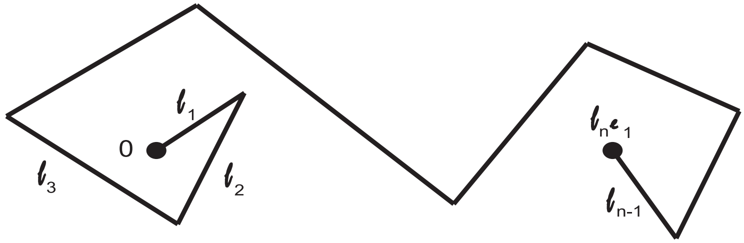

Fig. 1. A chain with length vector $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$.

The group $O(d-1)$, viewed as the subgroup of $O(d)$ stabilising the first axis, acts naturally (on the left) upon $\mathcal{C}_d^n(\ell)$. The quotient $SO(d-1)\backslash\mathcal{C}_d^n(\ell)$ is the *polygon space* $\mathcal{N}_d^n$, usually defined as

$$\mathcal{N}_d^n(\ell) = SO(d)\Bigg\backslash \left\{ z \in (S^{d-1})^n \;\middle|\; \sum_{i=1}^n \ell_i z_i = 0 \right\}.$$

When $d = 2$ the space of chains $\mathcal{C}_2^n(\ell)$ coincides with the polygon space $\mathcal{N}_2^n(\ell)$. Descriptions of several chain and polygon spaces are provided in [**8**] (see also [**7**] for a classification of $\mathcal{C}_d^4(\ell)$).

A length vector $\ell \in \mathbb{R}_{>0}^n$ is *generic* if $\mathcal{C}_1^n(\ell) = \emptyset$, that is to say there is no collinear chain. Explicitly, ℓ is generic iff $\sum_{i=1}^n \epsilon_i \ell_i \neq 0$ for all $\epsilon_i = \pm 1$. It is proven in e.g. [**7**] that, for ℓ generic, $\mathcal{C}_d^n(\ell)$ is a smooth closed manifold of dimension

$$\dim \mathcal{C}_d^n(\ell) = (n-2)(d-1) - 1.$$

Another known fact is that if $\ell, \ell' \in \mathbb{R}_{>0}^n$ satisfy

$$(\ell'_1, \ldots, \ell'_{n-1}, \ell'_n) = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(n-1)}, \ell_n)$$

for some permutation σ of $\{1, \ldots, n-1\}$, then $\mathcal{C}_d^n(\ell')$ and $\mathcal{C}_d^n(\ell)$ are $O(d-1)$-equivariantly diffeomorphic, see [**8**, 1·5].

A length vector $\ell \in \mathbb{R}_{>0}^n$ is said to be *ordered* if $\ell_1 \leqslant \ell_2 \leqslant \cdots \leqslant \ell_n$. A length vector $\ell \in \mathbb{R}_{>0}^n$ is said to be *dominated* if $\ell_i \leqslant \ell_n$ for all $i = 1, \ldots, n-1$.

The goal of this paper is to show that for $d \geqslant 3$ the diffeomorphism types of spaces $\mathcal{C}_d^n(\ell)$ (for ℓ generic and dominated) are in one-to-one correspondence with some pure combinatorial objects, described below.

THEOREM 0·1. *Let $d \in \mathbb{N}$, $d \geqslant 3$. Then, the following properties of generic and dominated length vectors $\ell, \ell' \in \mathbb{R}_{>0}^n$ are equivalent*:

(a) *$\mathcal{C}_d^n(\ell)$ and $\mathcal{C}_d^n(\ell')$ are $O(d-1)$-equivariantly diffeomorphic*;
(b) *$H^*(\mathcal{C}_d^n(\ell); \mathbb{Z})$ and $H^*(\mathcal{C}_d^n(\ell'); \mathbb{Z})$ are isomorphic as graded rings*;
(c) *$H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ and $H^*(\mathcal{C}_d^n(\ell'); \mathbb{Z}_2)$ are isomorphic as graded rings*;

Moreover, if the vectors ℓ and ℓ' are ordered[1] *then the above conditions are equivalent to*:

(d) *For a subset $J \subset \{1, \ldots, n\}$, the inequality*

$$\sum_{i \in J} \ell_i < \sum_{i \notin J} \ell_i$$

[1] This can be achieved by a permutation of $\ell_1, \ldots, \ell_{n-1}$, see above.

holds if and only if

$$\sum_{i\in J}\ell'_i < \sum_{i\notin J}\ell'_i.$$

The equivalence (a) $\sim$ (d) means that the topology of the chain space $\mathcal{C}_d^n(\ell)$ determines the length vector ℓ, up to certain combinatorial equivalence.

In the case $d = 2$ we do not know if (c) $\Rightarrow$ (a) although the equivalences (a) $\sim$ (b) $\sim$ (d) are true. This is related to a conjecture of K. Walker [**11**] who suggested that planar polygon spaces are determined by their integral cohomology rings. The conjecture was proven for a large class of length vectors in [**3**] and the (difficult) remaining cases were settled in [**10**].

An analogue of the Walker Conjecture also holds for the spatial polygon spaces $\mathcal{N}_3^n$ with $n > 4$, see [**3**, theorem 3]. No such result is known for $\mathcal{N}_d^n$ when $d > 3$.

One may interpret Theorem 0·1 as follows. Consider the simplex $A^{n-1} \subset \mathbb{R}^n$ of dimension $n - 1$ given by the inequalities

$$0 < \ell_1 < \cdots < \ell_{n-1} < \ell_n = 1$$

and the hyperplanes $H_J \subset \mathbb{R}^n$ defined by the equations

$$\sum_{i\in J}\ell_i = \sum_{i\notin J}\ell_i,$$

for all possible subsets $J \subset \{1, \ldots, n\}$. The connected components of the complement $A^{n-1} - \bigcup_J H_J$ are called *chambers*. Theorem 0·1 implies that for a fixed $d \geqslant 3$ the manifolds $\mathcal{C}_d^n(\ell)$ and $\mathcal{C}_d^n(\ell')$, where $\ell, \ell' \in (A^{n-1} - \bigcup_J H_J)$, are equivariantly diffeomorphic if and only if the vectors ℓ and ℓ' lie in the same chamber. Thus we obtain a one-to-one correspondence between the chambers and the equivariant diffeomorphism types of the manifolds $\mathcal{C}_d^n(\ell)$ for generic length vectors $\ell \in A^{n-1}$.

The number c_n of chambers in A^{n-1} grows fast with the number of parameters n. It was established in [**9**] that $c_3 = 2$, $c_4 = 3$, $c_5 = 7$, $c_6 = 21$, $c_7 = 135$, $c_8 = 2470$ and $c_9 = 175428$.

We now give the scheme of the proof of Theorem 0·1. We first recall that the $O(d-1)$-diffeomorphism type of $\mathcal{C}_d^n(\ell)$ is determined by d and the sets of ℓ-short (or long) subsets, which play an important role all along this paper. A subset J of $\{1, \ldots, n\}$ is *ℓ-short*, or just *short*, if

$$\sum_{i\in J}\ell_j < \sum_{i\notin J}\ell_j.$$

The reverse inequality defines *long* (or *ℓ-long*) subsets. Observe that ℓ is generic if and only if any subset of $\{1, \ldots, n\}$ is either short or long.

The family of subsets of $\{1, \ldots, n\}$ which are long is denoted by $\mathcal{L} = \mathcal{L}(\ell)$. Short subsets form a poset under inclusion, which we denote by $\mathcal{S} = \mathcal{S}(\ell)$. We are interested in the subposet

$$\dot{\mathcal{S}} = \dot{\mathcal{S}}(\ell) = \{J \subset \{1, \ldots, n-1\} \mid J \cup \{n\} \in \mathcal{S}\}. \tag{0·1}$$

The following lemma is proven in [**8**, lemma 1·2] (this reference uses the poset $\mathcal{S}_n(\ell) = \{J \in \mathcal{S} \mid n \in J\}$ which is determined by $\dot{\mathcal{S}}(\ell)$).

LEMMA 0·2. *Let $\ell, \ell' \in \mathbb{R}^n_{>0}$ be generic length vectors. Suppose that $\dot{\mathcal{S}}(\ell)$ and $\dot{\mathcal{S}}(\ell')$ are isomorphic as simplicial complexes. Then $\mathcal{C}^n_d(\ell)$ and $\mathcal{C}^n_d(\ell')$ are $O(d-1)$-equivariantly diffeomorphic.*

Lemma 0·2 gives the implication (d) $\Rightarrow$ (a) in Theorem 0·1. The implication (a) $\Rightarrow$ (b) is obvious and the implication (b) $\Rightarrow$ (c) follows since (under the condition that ℓ is dominated) $H^*(\mathcal{C}^n_d(\ell); \mathbb{Z})$ is torsion free, see Theorem 2·1. Remark 2·1 shows that the condition that ℓ is dominated is necessary for this conclusion; in other words, the integral homology $H_*(\mathcal{C}^n_d; \mathbb{Z})$ may have torsion if ℓ is not dominated.

Note that $H^*(\mathcal{C}^n_d; \mathbb{Z}_2) = 0$ if and only if $\mathcal{C}^n_d = \varnothing$, which happens if and only if the one-element subset $\{n\}$ is long. We can thus suppose that $\{n\}$ is short.

To prove that (c) $\Rightarrow$ (d) in Theorem 0·1, it suffices to show that the graded ring $H^*(\mathcal{C}^n_d(\ell); \mathbb{Z}_2)$ determines the simplicial complex $\dot{\mathcal{S}}(\ell)$ when the length vector ℓ is dominated.

For a finite simplicial complex Δ whose vertex set is contained in the set $\{1, 2, \dots, n-1\}$, and for an integer $d \geqslant 2$, consider the graded ring

$$\Lambda_d(\Delta) = \mathbb{Z}_2[X_1, \dots, X_{n-1}]\big/\mathcal{I}(\Delta),$$

where $\mathcal{I}(\Delta)$ is the monomial ideal of the polynomial ring $\mathbb{Z}_2[X_1, \dots, X_{n-1}]$ generated by the monomials X_i^2 and $X_{j_1} \cdots X_{j_k}$ such that $\{j_1, \dots, j_k\}$ is not a simplex of Δ; the grading of $\Lambda_d(\Delta)$ is determined by the requirement that each variable $X_1, \dots, X_{n-1}$ has degree $d-1$.

Let Δ and Δ' be two finite simplicial complexes with vertex sets contained in $\{1, \dots, n-1\}$. By a result of J. Gubeladze, any graded ring isomorphism $\Lambda_d(\Delta) \approx \Lambda_d(\Delta')$ is induced by a simplicial isomorphism $\Delta \approx \Delta'$ (see [**6**, example 3·6], and [**3**, theorem 8]). Hence, the implication (c) $\Rightarrow$ (d) of Theorem 0·1 is established once we have proven the following result:

THEOREM 0·3. *Let $\ell \in \mathbb{R}^n_{>0}$ be a generic dominated length vector. When $d \geqslant 3$, the subring $H^{*(d-1)}(\mathcal{C}^n_d(\ell); \mathbb{Z}_2)$ of $H^*(\mathcal{C}^n_d(\ell); \mathbb{Z}_2)$ is isomorphic, as a graded ring, to the ring $\Lambda_d(\dot{\mathcal{S}}(\ell))$.*

The proof of Theorem 0·3 is given in Section 3. The preceding sections are preliminaries for this goal. For instance, computation of $H^*(\mathcal{C}^n_d(\ell); \mathbb{Z})$ as a graded abelian group, is given in Theorem 2·1.

1. *Robot arms in* $\mathbb{R}^d$

Let $\mathbb{S} = \mathbb{S}^n_d = \{\rho\colon \{1, \dots, n\} \to S^{d-1}\} \approx (S^{d-1})^n$. Points of $\mathbb{S}^n_d$ can be viewed as labeled configurations of n points lying on the sphere S^{d-1}. By post-composition, the orthogonal group $O(d)$ acts smoothly on the left upon $\mathbb{S}$. In [**5**, section 4–5], the quotient $W = SO(2)\backslash\mathbb{S}^n_2 \approx (S^1)^{n-1}$ is used to get cohomological informations about $\mathcal{C}^n_2$. In this section, we extend these results for $d > 2$. The quotient $SO(d)\backslash\mathbb{S}^n_d$ is no longer a convenient object to work with, so we replace it by the fundamental domain for the $O(d)$-action given by the submanifold

$$Z = Z^n_d = \{\rho \in \mathbb{S} \mid \rho(n) = -e_1\} \approx (S^{d-1})^{n-1}.$$

Observe that Z inherits an action of $O(d-1)$.

For a length vector $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{R}^n_{>0}$, the *$\ell$-robot arm* is the smooth map $\tilde{F}_\ell\colon \mathbb{S} \to \mathbb{R}^d$ defined by $\tilde{F}_\ell(\rho) = \sum_{i=1}^n \ell_i \rho(i)$.

Consider a smooth function

$$f : Z \longrightarrow \mathbb{R} \quad \text{given by} \quad f(\rho) = -|F_\ell(\rho)|^2.$$

Observe that

$$\mathcal{C} = \mathcal{C}_d^n(\ell) = f^{-1}(0) \subset Z$$

and denote $Z' = Z - \mathcal{C}$. It is well known that the set of critical points $Crit(f)$ of the function $f: Z \to \mathbb{R}$ consists of the critical submanifold $\mathcal{C}$ and finitely many Morse critical points $\rho \in Z'$ corresponding to collinear configurations, i.e. such that $\rho(i) = \pm e_1$ for $i \in \{1, 2, \ldots, n-1\}$.

From now on we will assume that the length vector ℓ is generic. We will label the critical points of f by long subsets $J \in \mathcal{L}$ as follows. For each $J \in \mathcal{L}$ with $n \notin J$ let $\rho_J \in Z'$ be given by

$$\rho_J(i) = \begin{cases} e_1, & \text{if } i \in J, \\ -e_1, & \text{if } i \notin J. \end{cases}$$

However, for $J \in \mathcal{L}$ with $n \in J$ let $\rho_J \in Z'$ be given by

$$\rho_J(i) = \begin{cases} -e_1, & \text{if } i \in J, \\ e_1, & \text{if } i \notin J. \end{cases}$$

LEMMA 1·1. *The Morse index of ρ_J, as a critical point of f, is equal to $(d-1)(n-|J|)$.*

Proof. The non-degeneracy of ρ_J and the computation of its index are classical in topological robotics; they are based on arguments described in [**7**, proof of theorem 3·2].

Consider the axial involution $\hat{\tau}$ on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ defined by $\hat{\tau}(x, y) = (x, -y)$. It induces an involution τ on $\mathbb{S}$ and on Z. The map f is τ-invariant. Moreover, the critical set of $f|Z' : Z' \to (-\infty, 0)$ coincides with the fixed point set Z^τ. By Lemma 1·1 and [**5**, theorem 4], this proves the following

LEMMA 1·2. *The Morse function $f' = f|Z': Z' \to (-\infty, 0)$ is $\mathbb{Z}$-perfect, in the sense that $H_i(Z')$ is free abelian of rank equal to the number of critical points of f' of index i. Moreover, the induced map τ_*: $H_i(Z') \to H_i(Z')$ is multiplication by $(-1)^i$.*

[**5**, Theorem 4] is stated for a Morse function f: $M \to \mathbb{R}$ where M is a compact manifold with boundary. To use it in the proof of Lemma 1·2, just replace Z' by $Z - N$ where N is a small open tubular neighbourhood of $\mathcal{C}$.)

We now represent homology bases for Z and Z' by convenient closed submanifolds. For a subset $J \subset \{1, \ldots, n\}$ we define

$$W_J = \{\rho \in Z; \rho(i) = \rho(j) \quad \text{for all} \quad i, j \in J\}.$$

It is clear that W_J is diffeomorphic to a product of $(n - |J|)$ copies of the sphere S^{d-1}. We also observe that $\rho_J \in W_J$.

We denote by $[W_J] \in H_{(d-1)(n-|J|)}(Z; \mathbb{Z})$ the class represented by $W_J \subset Z$ (for some chosen orientation of W_J).

If J is long, then W_J is contained in Z' and we denote by $[W_J]$ the class represented by the submanifold $W_J \subset Z'$ in $H_{(d-1)(n-|J|)}(Z'; \mathbb{Z})$.

LEMMA 1·3.

(a) *The integral homology group $H_*(Z';\mathbb{Z})$ is freely generated by the classes $[W_J]$ where J runs over all subsets $J \subset \{1,\dots,n\}$ which are long with respect to ℓ.*

(b) *The homology group $H_*(Z;\mathbb{Z})$ is freely generated by the classes $[W_J]$ for all $J \subset \{1,\dots,n\}$ with $n \in J$.*

Proof. For (a), we invoke [**5**, theorem 5]. Indeed, the the collection of τ-invariant manifolds $\{W_J \mid J \in \mathcal{L}\}$ satisfies all the hypotheses of this theorem (see also [**5**, lemma 8]). The statement (b) follows directly from the Künneth formula.

Let $J, J' \subset \{1,\dots,n\}$ be such that $|J|+|J'| = n+1$. Then one has $\dim W_J + \dim W_{J'} = \dim Z = \dim Z'$ and the intersection number $[W_J]\cdot[W_{J'}] \in \mathbb{Z}$ is defined (the intersection number in Z). We shall use the following formulae, compare [**5**]:

LEMMA 1·4. *Let $J, J' \subset \{1,\dots,n\}$ be subsets with $|J|+|J'| = n+1$. Then*

$$[W_J]\cdot[W_{J'}] = \begin{cases} \pm 1 & \text{if } |J\cap J'| = 1,\\ 0 & \text{if } |J\cap J'| > 1 \text{ and } n \in J\cup J'. \end{cases}$$

Proof. Suppose that $J\cap J' = \{q\}$. Then $|J\cup J'| = |J|+|J'|-|J\cap J'| = n$. Hence, $n \in J\cup J'$ and $W_J \cap W_{J'}$ consists of the single point $\rho_{J\cup J'}$ (satisfying $\rho_{J\cup J'}(i) = -e_1$ for all $i \in \{1,\dots,n\}$). It is not hard to check that the intersection is transversal (see [**5**, proof of (34)]), so that $[W_J]\cdot[W_{J'}] = \pm 1$.

In the case $|J\cap J'| > 1$, there exists $q \in J\cap J'$ with $q \neq n$. Let α be a rotation of $\mathbb{R}^d$ such that $\alpha(e_1) \neq e_1$. Let $h\colon Z \to Z$ be the diffeomorphism such that $h(\rho)(k) = \rho(k)$ if $k \neq q$ and $h(\rho)(q) = \alpha\circ\rho(q)$. We now use that $n \in J\cup J'$, say $n \in J'$. Then, $\rho(q) = -e_1$ for all $\rho \in W_{J'}$. Therefore, $h(W_J)\cap W_{J'} = \emptyset$. As h is isotopic to the identity of Z, this implies that $[W_J]\cdot[W_{J'}] = 0$.

Remark 1·5. In Lemma 1·4, the hypothesis $n \in J\cup J'$ is not necessary if d is even, by the above proof, since there exists a diffeomorphism of S^{d-1} isotopic to the identity and without fixed point. But, for example, if $n = d = 3$, one checks that $[W_J]\cdot[W_{J'}] = \pm 2$ for $J = J' = \{1,2\}$.

In the case $n \in J\cap J'$ and $|J|+|J'| = n+1$, Lemma 1·4 takes the following form:

$$[W_J]\cdot[W_{J'}] = \begin{cases} \pm 1 & \text{if } J\cap J' = \{n\},\\ 0 & \text{otherwise.} \end{cases} \tag{1·1}$$

Therefore, the basis $\{[W_J] \mid |J| = n-k, n \in J\}$ of $H_{k(d-1)}(Z;\mathbb{Z})$ has a dual basis (up to sign) $\{[W_J]^\sharp \in H_{(n-k)(d-1)}(Z;\mathbb{Z}) \mid |J| = n-k, n \in J\}$ for the intersection form, defined by $[W_J]^\sharp = [W_K]$, where $K = \bar{J}\cup\{n\}$ (with $\bar{J}$ denoting the complement of J in $\{1,\dots,n\}$).

We are now in position to express the homomorphism

$$\phi_r : H_{r(d-1)}(Z';\mathbb{Z}) \longrightarrow H_{r(d-1)}(Z;\mathbb{Z}) \tag{1·2}$$

induced by the inclusion $Z' \subset Z$. By Lemma 1·3, one has a direct sum decomposition

$$H_{r(d-1)}(Z';\mathbb{Z}) = A_r \oplus B_r,$$

where

(i) A_r is the free abelian group generated by $[W_J]$ with $J \subset \{1,\dots,n\}$ long, $|J| = n-r$, and $n \in J$.

(ii) B_r is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ long, $|J| = n-r$ and $n \notin J$.

Part (b) of Lemma 1·3 gives a direct sum decomposition

$$H_{r(d-1)}(Z; \mathbb{Z}) = A_r \oplus C_r,$$

where

(i) A_r is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ with $n \in J$, $|J| = n - r$, and J long.

(ii) C_r is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ with $n \in J$, $|J| = n - r$, and J short.

LEMMA 1·6.

(a) *The homomorphism ϕ_r restricted to A_r coincides with the identity of A_r.*

(b) *Suppose that the length vector ℓ is dominated. Then the image of ϕ_r coincides with A_r. In particular, $\phi_r(B_r) \subset A_r$.*

Proof. The claim (a) is obvious. For (b), let $[W_J] \in B_{(n-|J|)}$ and

$$\phi_*[W_J] = \sum_{n \in K} \alpha_K \cdot [W_K],$$

where $|K| = |J|$ and $n \in K$. We claim that $\alpha_K = 0$ for K short. Indeed,

$$\alpha_K = \pm[W_J] \cdot [W_K]^\sharp = \pm[W_J] \cdot [W_{K'}]$$

where $K' = \bar{K} \cup \{n\}$. By Lemma 1·4, $\alpha_K \neq 0$ implies that

$$J \cap (\bar{K} \cup \{n\}) = J - K = \{i\}$$

with $i < n$. As $|K| = |J|$, this is equivalent to $K = (J - \{i\}) \cup \{n\}$, i.e. K is obtained from J by adding n and deleting i. This gives a contradiction since J is long, K is short and $\ell_n \geqslant \ell_j$.

2. *The Betti numbers of the chain space*

Let $\ell = (\ell_1, \ldots, \ell_n)$ be a generic dominated length vector. Let $a_k = a_k(\ell)$ be the number of short subsets J containing n with $|J| = k + 1$, as introduced in [**5**]. Alternatively, a_k is the number of sets $I \in \dot{\mathcal{S}}$ with $|I| = k$.

THEOREM 2·1. *Let $\ell = (\ell_1, \ldots, \ell_n)$ be a generic dominated length vector. If $d \geqslant 3$ then $H^k(\mathcal{C}_d^n(\ell); \mathbb{Z})$ is free abelian of rank*

$$\operatorname{rk} H^k(\mathcal{C}_d^n(\ell); \mathbb{Z}) = \begin{cases} a_s & \text{if } k = s(d-1), \quad s = 0, 1, \ldots, \quad n-3, \\ a_{n-s-2} & \text{if } k = s(d-1) - 1, \quad s = 0, 1, \ldots, \quad n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let N be a closed tubular neighbourhood of $\mathcal{C} = \mathcal{C}_d^n(\ell)$ in $Z = Z_d^n$. By Poincaré-Lefschetz duality and excision, one has the isomorphisms on integral homology

$$H^k(\mathcal{C}) \approx H^k(N) \approx H_{(n-1)(d-1)-k}(N, \partial N) \approx H_{(n-1)(d-1)-k}(Z, Z').$$

By Lemma 1·3 we know that the homology of Z and Z' are concentrated in degrees which are multiples of $(d-1)$. The possibly non-vanishing part of $H^k(\mathcal{C})$ sits in the exact sequence

$$0 \longrightarrow H^{(n-r-1)(d-1)-1}(\mathcal{C}) \longrightarrow H_{r(d-1)}(Z') \xrightarrow{\phi_r} H_{r(d-1)}(Z) \longrightarrow H^{(n-r-1)(d-1)}(\mathcal{C}) \longrightarrow 0$$

which is obtained from the exact sequence of (Z, Z') via the isomorphisms mentioned in the preceding paragraph. Using Lemma 1·6 we obtain that the kernel and cokernel of ϕ_r (see (1·2)) are free abelian of rank rk B_r and rk C_r correspondingly. It follows that integral cohomology of $\mathcal{C}$ is free abelian and

$$\operatorname{rk} H^{(n-r-1)(d-1)-1}(\mathcal{C}) = \operatorname{rk} B_r = a_{r-1},$$
$$\operatorname{rk} H^{(n-r-1)(d-1)}(\mathcal{C}) = \operatorname{rk} C_r = a_{n-r-1}.$$

Besides, the homology groups of $\mathcal{C}$ vanish in all other dimensions. This completes the proof.

Remark 2·1. Theorem 2·1 is false if ℓ is not dominated. For example, let $\ell = (1, 1, 1, \varepsilon)$ with $\varepsilon < 1$. Then, $\mathcal{C}^4_d(\ell)$ is diffeomorphic to the unit tangent bundle T^1S^{d-1} of S^{d-1}: a map g: $\mathcal{C}^4_d(\ell) \to T^1S^{d-1}$ is given by $g(\rho) = (\rho(1), \hat{\rho}(2))$, where the latter is obtained from $(\rho(1), \rho(2))$ by the Gram–Schmidt orthonormalization process. The map g is clearly a diffeomorphism for $\varepsilon = 0$ and the robot arm $F_{(1,1,1)}$: $\mathbb{S}^3_d \to \mathbb{R}^d$ of Section 1 has no critical value in the disk $\{|x| < 1\} \subset \mathbb{R}^d$. In particular, $\mathcal{C}^4_3(\ell)$ is diffeomorphic to $SO(3)$, and thus $H^2(\mathcal{C}^4_3(\ell); \mathbb{Z}) = \mathbb{Z}_2$. What goes wrong is Point (b) of Lemma 1·6: for instance $A_2 = 0$, $B_2 = H_2(Z, Z') \approx H^2(Z) = C_2 \approx \mathbb{Z}^3$ and, by the proof of Theorem 2·1, ϕ: $H^2(Z') \to H^2(Z)$ must be injective with cokernel $\mathbb{Z}_2$. To obtain this fine result with our technique would require to control the signs in Lemma 1·4.

3. *Proof of Theorem* 0·3

In this section all homology and cohomology groups are understood with $\mathbb{Z}_2$ coefficients. We assume that the length vector ℓ is generic and dominated and $d > 2$.

Recall that $H^*(Z)$ is an exterior algebra on generators $X_1, \ldots, X_{n-1}$ where the class $X_j \in H^{d-1}(Z)$ is induced by the projection $\pi_j: Z \to S^{d-1}$ given by $\pi_j(\rho) = \rho(j)$; here $\rho \in Z$ and $j = 1, \ldots, n-1$.

Consider the inclusion $i: \mathcal{C} \to Z$ and the induced homomorphism on cohomology with $\mathbb{Z}_2$ coefficients $i^*: H^*(Z) \to H^*(\mathcal{C})$. We claim that *for any $s = 0, 1, \ldots, n-1$ the homomorphism*

$$i^* : H^{s(d-1)}(Z) \longrightarrow H^{s(d-1)}(\mathcal{C}) \qquad (3·1)$$

is an epimorphism and its kernel is additively generated by the monomials $X_{i_1}X_{i_2}\ldots X_{i_s}$ such that $i_1 < i_2 < \cdots < i_s < n$ and the set $\{i_1, i_2, \ldots, i_s, n\}$ is long with respect to the length vector ℓ. Indeed, one has the following commutative diagram

$$\begin{array}{ccccc} H^{s(d-1)}(Z) & \xrightarrow{i^*} & H^{s(d-1)}(\mathcal{C}) & & \\ \downarrow\simeq & & \downarrow\simeq & & \\ H_{r(d-1)}(Z') \xrightarrow{\phi_r} & H_{r(d-1)}(Z) & \longrightarrow & H_{r(d-1)}(Z, Z') & \longrightarrow 0. \end{array}$$

Here $r = n - 1 - s$ and the lower horizontal row is a part of the exact sequence of the pair (Z, Z'); its $\mathbb{Z}$-coefficient version was discussed in detail in the previous section. The vertical isomorphism on the left is the Poincaré duality map. The vertical isomorphism on the right is as in the previous section (a combination of Poincaré–Lefschetz duality and excision). The diagram commutes as shown in [**1**, theorem I·2·2]. From this diagram one sees that $i^* : H^{s(d-1)}(Z) \to H^{s(d-1)}(\mathcal{C})$ is an epimorphism and its kernel coincides with the Poincaré dual of the image of ϕ_r. From Lemma 1·6 we know that the image of ϕ_r coincides with the subgroup $A_r \subset H_{r(d-1)}(Z)$, i.e. it is freely generated by the classes $[W_J]$ with $J \subset \{1, \dots, n\}$ being long, $n \in J$ and $|J| = n - r = s + 1$.

Next we note that the Poincaré dual of the homology class $[W_J] \in H_{r(d-1)}(Z)$ (assuming that $n \in J$) coincides with the monomial

$$X_I = X_{i_1} X_{i_2} \cdots X_{i_s} \in H^{s(d-1)}(Z)$$

where $J = \{i_1 < i_2 < \cdots < i_s < n\}$ and $I = \{i_1 < i_2 < \cdots < i_s\}$. This follows from the formula

$$[W_J] \cdot [W_K] = \langle X_I, [W_K] \rangle \in \mathbb{Z}_2 \tag{3·2}$$

which holds for any subset $K \subset \{1, \dots, n\}$ with $n \in K$ and $|K| = r + 1$. In (3·2) the brackets $\langle\, , \rangle$ denote the evaluation of the cohomology class on homology class. Formula (3·2) is a consequence of (1·1).

Hence we see that the kernel of the epimorphism $i^*: H^{*(d-1)}(Z) \to H^{*(d-1)}(\mathcal{C})$ is the ideal $\mathcal{I}$ consisting of linear combinations of all monomials of the form X_I where the subset $I \subset \{1, \dots, n-1\}$ is such that $I \cup \{n\}$ is long with respect to ℓ. Thus, the subalgebra $H^{*(d-1)}(\mathcal{C})$ is isomorphic to the quotient of the $\mathbb{Z}_2$-coefficient exterior algebra on $X_1, \dots, X_{n-1}$ with respect to the ideal $\mathcal{I}$. This completes the proof of Theorem 0·3.

4. *Comments*

1. The Betti numbers alone do not distinguish the chain spaces up to diffeomorphism (therefore, it is necessary to use the cohomology algebras as we do in this paper). The first example occurs for $n = 6$ with $\ell = (1, 1, 1, 2, 3, 3)$ and $\ell' = (\varepsilon, 1, 1, 1, 2, 2)$, where $0 < \varepsilon < 1$. (The chamber of ℓ is $\langle 632, 64 \rangle$ and that of ℓ' is $\langle 641 \rangle$, see [**8**, table C].) Then, $\tilde{\mathcal{S}}(\ell)$ and $\tilde{\mathcal{S}}(\ell')$ are both graphs with 4 vertices and 3 edges. Therefore, $a_s(\ell) = a_s(\ell')$ for all s which, by Theorem 2·1, implies that $\mathcal{C}_d^6(\ell)$ and $\mathcal{C}_d^6(\ell')$ have the same Betti numbers. However, $\tilde{\mathcal{S}}(\ell)$ and $\tilde{\mathcal{S}}(\ell')$ are not poset isomorphic: the former is not connected while the latter is.

2. It would be interesting to know if, in Theorem 0·1, the ring $\mathbb{Z}_2$ could be replaced by any other coefficient ring. In the corresponding result for spatial polygon spaces $\mathcal{N}_3^n(\ell)$, which are distinguished by their $\mathbb{Z}_2$-cohomology rings if $n > 4$ [**3**, theorem 3], the ring $\mathbb{Z}_2$ cannot be replaced by $\mathbb{R}$. Indeed, $\mathcal{N}_3^5(\varepsilon, 1, 1, 1, 2) \approx \mathbb{C}P^2 \sharp \bar{\mathbb{C}}P^2$ while $\mathcal{N}_3^5(\varepsilon, \varepsilon, 1, 1, 1) \approx S^2 \times S^2$ (ε small; see [**8**, table B]). These two manifolds have non-isomorphic $\mathbb{Z}_2$-cohomology rings but isomorphic real cohomology rings. One can of course replace $\mathbb{Z}_2$ by $\mathbb{Z}$ in Theorem 0·1 since, by Theorem 2·1, $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z})$ determines $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ when ℓ is dominated.

3. We do not know if Theorem 0·1 is true for generic length vectors which are not dominated. We believe that the techniques developed in [**2**] may be useful to study this more general case.

4. Let $\ell \in \mathbb{R}^n_{>0}$ be a length vector. In [**7**, **8**], a smooth manifold $V_d(\ell)$ is introduced whose boundary is $\mathcal{C}_d^n(\ell)$. When ℓ is generic and dominated, it can be shown that homomorphism $H^*(V_d(\ell)) \to H^*(\mathcal{C}_d^n(\ell))$ is injective with image equal to the subring $H^{(d-1)*}(\mathcal{C}_d^n(\ell); \mathbb{Z})$. By Theorem 0·3, this implies that the graded $H^*(V_d(\ell))$ is isomorphic to $\Lambda(\dot{\mathcal{S}}(\ell))$ if $q \geqslant 3$. Details may be found in an earlier version of this paper [**4**].

Acknowledgment: The authors thank the referee who suggested a simplified proof of Theorem 0·3.

REFERENCES

[**1**] W. BROWDER. *Surgery on Simply Connected Manifolds*. (Springer-Verlag, 1972).

[**2**] M. FARBER and V. FROMM, Homology of planar telescopic linkages. *Algebr. Geom. Topol.* **10** (2010), 1063–1087.

[**3**] M. FARBER, J–CL. HAUSMANN and D. SCHÜTZ. On the conjecture of Kevin Walker. *Journal of Topology and Analysis* **1** (2009), 65–86.

[**4**] M. FARBER, J–CL. HAUSMANN and D. SCHÜTZ. On the cohomology ring of chains in $\mathbb{R}^d$ ArXiv:0903.0472v2 [math.AT].

[**5**] M. FARBER and D. SCHÜTZ. Homology of planar polygon spaces. *Geom. Dedicata* **125** (2007), 75–92.

[**6**] J. GUBELADZE. The isomorphism problem for commutative monoid rings. *J. Pure Appl. Algebra* **129** (1998), 35–65.

[**7**] J.–C. HAUSMANN. Sur la topologie des bras articulé. in *Algebraic Topology Poznań* 1989, Springer Lectures Notes **1474** (1989), 146–159.

[**8**] J.–C. HAUSMANN. Geometric descriptions of polygon and chain spaces. In *Topology and Robotics* Contemp. Math. Amer. Math. Soc. **438** (2007), 47–57.

[**9**] J.–C. HAUSMANN and E. RODRIGUEZ. The space of clouds in an Euclidean space. *Experiment. Math.* **13** (2004), 31–47.

[**10**] D. SCHÜTZ. The isomorphism problem for planar polygon spaces. *Journal of Topology* **3** (2010), 713–742.

[**11**] K. WALKER. Configuration spaces of linkages. Bachelor's thesis. Princeton (1985).